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978-1-107-64025-2 - Theory of Differential Equations: Part II: Ordinary Equations,
not Linear: Vol. II

Andrew Russell Forsyth

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THEORY
OF
DIFFERENTIAL EQUATIONS.

PART II.
ORDINARY EQUATIONS, NOT LINEAR.

BY
ANDREW RUSSELL FORSYTH,
Sc.D. (CAMB.), HON. Sc.D. (DUBL.), F.R.S.,
SADLERIAN PROFESSOR OF PURE MATHEMATICS,
FELLOW OF TRINITY COLLEGE, CAMBRIDGE.

VOL. II.

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PREFACE.

THE two volumes, now published as Part II, are my second contribution towards the fulfilment of an old promise. They deal almost entirely with the functional character of the solutions of ordinary differential equations. At one time, I hoped to discuss the whole of this theory in the present Part; the extent of the subject has prevented me from realising this hope. Accordingly, I have reserved the theory of linear differential equations for another Part.

The revision of the proof-sheets has been made lighter for me by the assistance of three friends. **Mr. E. T. Whittaker**, M.A., Fellow of Trinity College, Cambridge, has read both the volumes. **Prof. W. Burnside**, M.A., F.R.S., Honorary Fellow of Pembroke College, Cambridge, read a large part of the first volume. **Mr. R. W. H. T. Hudson**, B.A., Scholar of St. John's College, Cambridge, has read the whole of the second

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PREFACE

volume. I wish to make a grateful acknowledgement of the help given me by these gentlemen.

I wish also to express my thanks to the Staff of the University Press, for the care and trouble they have taken, and the uniform consideration they have shewn me, during the progress of the printing.

A. R. FORSYTH.

TRINITY COLLEGE, CAMBRIDGE,
16 *December*, 1899.

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