CHAPTER I.

INTRODUCTORY*.

1. THE theory of the solution of differential equations has been developed along several distinct lines of research. One of the many problems of the subject is the determination of those classes of differential equations which possess solutions expressible in terms of the functions already known in analysis. The most notable example of such a class is that of linear differential equations with constant coefficients: these can be solved by means of the exponential function. Another problem is the determination of those classes of differential equations which can be integrated by quadratures, that is, can be transformed so as to depend on the integration of equations of the form

$$\frac{dy}{dz} = Z,$$

where Z is a function of z. The solution of such equations involves only those transcendents which occur in the Integral Calculus. Examples of this kind are equations of the first order and the first degree in which the variables can be separated, and equations of the first order from which one of the variables is explicitly absent.

Equations of such classes were at one time the chief object of study in the theory of differential equations. They are somewhat limited in character and range. Many of the simpler results which

* In regard to the contents of this chapter, the following works may be consulted:—

Jordan, *Cours d'Analyse*, t. III, ch. I;

Königsberger, *Lehrbuch der Theorie der Differentialgleichungen*, Kap. I, Abschn. I, II.

have been obtained are given in introductory text-books on differential equations and therefore will not be developed here.

In the modern general theory, the problem of solution is considered from a different standpoint. It is proved that, within a suitably chosen region, a converging series of powers of the independent variable can be found which satisfies the differential equation. When, as is often the case, the function thus obtained is not included among the functions previously known in analysis, it is regarded as defined by the equation, and its properties are deduced as far as possible from the characteristic properties of the equation. Thus it may be possible to determine from the equation whether its integral is a uniform function or a multiform function; what are the places and the nature of its zeros, its singularities, its branch-points: and so on. In this way, new classes of functions are introduced to analysis, and the classes of differential equations, which can be solved by means of them, can be constructed.

It is to the consideration of this aspect of the theory of ordinary differential equations that nearly the whole of the present Part of this work is devoted. It will be seen that many of the investigations have regard to existence-theorems and are concerned with the character of the integral function in the vicinity of singularities. When all the singularities are known and the general character of the integral function in the immediate vicinity is determined, the further explicit determination of the integral is frequently taken for granted. The reason of this omission is that, as the respective investigations shew the kind of analytical expression which the integral acquires in the domain of any point, the actual substitution of an appropriately constructed expression and the determination of the coefficients, so as to make the equation identically satisfied, are matters of direct algebra. Such a process may be laborious but is not intrinsically difficult, and therefore only a few instances of it are carried through to their end; in several cases, it is omitted because its details are sufficiently obvious.

Moreover, the investigations will be restricted mainly to the analytical character of the solution of the equation. Some incidental illustrations may be given; but theories, that are concerned with descriptive and other properties of the equations considered, will be omitted.

2. Simple considerations shew how little can be regarded at the outset as established knowledge and indicate that practically all the accepted propositions of a general character require to be reviewed so that their meaning and range may be clear and determinate. For instance, the complete solution of an ordinary equation of the first order is known to contain an arbitrary constant; and it is customary to declare that, in order to satisfy the conditions of a special problem associated with the equation, the value of the constant can be determined by any assigned relation. On this basis, a solution of the equation

$$\frac{dy}{dx} = -\frac{y^2}{x}$$

might be required that would make y vanish when x vanishes. The complete solution is

$$\frac{1}{y} = A + \log x,$$

where A is left arbitrary by the equation; the datum is insufficient to determine A in the absence of information as to the mode in which x and y vanish. A precise solution cannot in this case be obtained; and a question is suggested as to possible limitations on data that serve to determine solutions. Further, the difficulty indicated has arisen after the general solution of the equation has been obtained; at least as grave a doubt must occur in the case of equations of which an explicit solution cannot be written down. In consequence, it is necessary to consider the fundamental question as to whether an integral exists; when the existence is established, some investigation must obtain the conditions by which it is limited and must deduce the characteristics of the function in the vicinity of ordinary and of critical values within and upon the boundary of its region of existence.

The existence-theorem for a system of equations

$$\frac{dw_i}{dz} = \phi_i(w_1, \ldots, w_n, z), \qquad (i = 1, 2, \ldots, n),$$

requires:

(i) the establishment of integrals in the vicinity of values for which the functions ϕ_i are regular*: the range of existence of the integrals must also be considered:

* The term *regular* is applied, in accordance with Weierstrass's definition, *Ges. Werke*, t. II, p. 154, to a uniform function, (or to a uniform branch of a function),

(ii) a proof of the uniqueness within the range of existence; this gives rise to various questions connected with the appropriate determining conditions, and also leads to a discussion of some classes of singularities:

(iii) (connected partially with (ii) in fact, though substantially a quite independent investigation), a discussion of integrals in the vicinity of values for which the functions ϕ_i cease to be regular.

There is another class of investigations, distinct from those just indicated: they are suggested by the corresponding class of questions that arise in connection with ordinary linear equations. When the path of the variable between z and some initial value z_0 is deformed in the plane, how is a particular set of simultaneous solutions of the system affected? Or when the variable returns to the initial point z_0 after describing any circuit in the plane, what is the effect on the composite integral?

The various investigations thus suggested will, as far as possible, be taken into successive account: the last class is, however, discussed only briefly, for reasons adduced later.

3. All the variables that occur are supposed to be complex quantities with initially unlimited variation. As is usual, a separate region is associated with each of them so that the variation can be represented geometrically; the region of any variable is generally a plane and, being so, is referred to as the plane of the variable.

In most of the succeeding investigations, there is only a single independent variable. The system of equations determining the dependent variables is regarded as being constituted of simultaneous equations, which are independent of one another in the sense that no one of them can be derived from the others by any combination of algebraical and analytical processes. The number of equations in such a system is the same as the number of dependent variables. The dependent variables are generally denoted by u, v, w, ..., the independent variable by z.

in a region of the variables at every point of which it can be represented in the form of a converging power-series: it is finite and continuous for all values of the variables included in such a region.

The equations may contain differential coefficients of any orders; but a transformation can be effected after which the only differential coefficients that occur are of the first order, it being sufficient for this purpose to associate appropriate equations of the type

$$\frac{dw}{dz} = w_1, \quad \frac{dw_1}{dz} = w_2, \ldots$$

with the system, which in its changed form will still be composed of as many equations as there are dependent variables. All the equations discussed will be algebraical in each of the derivatives of the dependent variables, and they will usually be algebraical also in these variables themselves, any deviation being indicated when it is of importance; but no such limitation as to functional occurrence is imposed as regards the occurrence of the independent variable.

It may happen that some equations of the system are free from derivatives: or it may be possible to construct such an equation from the system without integration or any equivalent process. Let such an equation be

$$g(u, v, w, \ldots, z) = 0,$$

so that

$$\frac{\partial g}{\partial u}\frac{du}{dz} + \frac{\partial g}{\partial v}\frac{dv}{dz} + \frac{\partial g}{\partial w}\frac{dw}{dz} + \ldots + \frac{\partial g}{\partial z} = 0.$$

By means of the latter, some one of the derivatives can be eliminated from all the equations of the system; by means of $g = 0$, the corresponding variable can be eliminated from each of the modified equations in turn. In this form, the number of equations is greater by unity than the number of dependent variables, so that one equation is satisfied in virtue of the remainder; this equation is therefore superfluous and should be removed. The original system is thus replaced by another, containing one dependent variable less and one equation less; the solution of the original system is compounded of the solution of the new system and of the equation $g = 0$. It would thus be sufficient to obtain the solution of the modified system; all the further processes necessary to solve the original system are algebraical in character.

For example, in the system

$$\left.\begin{aligned} P_1\frac{du}{dz}+P_2\frac{dv}{dz}+P_3\frac{dw}{dz}=P_4\\ Q_1\frac{du}{dz}+Q_2\frac{dv}{dz}+Q_3\frac{dw}{dz}=Q_4\\ R_1\frac{du}{dz}+R_2\frac{dv}{dz}+R_3\frac{dw}{dz}=R_4\end{aligned}\right\},$$

the coefficients P, Q, R, supposed to be functions of the variables, may be such that the determinant $(P_1Q_2R_3)$, say Δ, vanishes identically. In order that the derivatives of u, v, w, may not have infinite values only, it is necessary that the equations

$$(P_2Q_3R_4)=0,\quad (P_1Q_3R_4)=0,\quad (P_1Q_2R_4)=0$$

be satisfied. They cannot all be identities, for the original system would then contain only two independent equations; properties of determinants shew that, as Δ vanishes identically, they are satisfied in virtue of a single new equation, say $S=0$. This equation $S=0$ would be used to transform the system into one involving only two dependent variables.

Systems of equations which can be transformed so as to yield, merely by processes of algebraical elimination, one or more equations free from derivatives, are called reducible; systems which do not admit of such a transformation are called irreducible. The process of modifying a reducible system, so that ultimately an irreducible system in a smaller number of variables shall remain, has been indicated; the properties of reducible systems and the tests of reducibility must be sought elsewhere. For the present purpose, it will be assumed that all the systems of equations under consideration are irreducible; and manifestly there is no loss of generality in assuming that each equation in a system is rationally irresoluble.

Construction and Preparation of Normal Forms.

4. Before undertaking the discussion of the integral equivalent of a system of equations, it is desirable to select some typical form for the equations as one in which any given system can be expressed.

When a system is composed of only a single equation and when therefore only one dependent variable is involved, it is of the form

$$f\left(\frac{dw}{dz}, w, z\right)=0,$$

where (after preceding explanations) f is rational and integral in dw/dz, is usually rational as regards w, and is unlimited as regards z.

When a system is composed of two equations and when therefore two dependent variables are involved, it will initially in the most general case have the form

$$f\left(\frac{du}{dz}, \frac{dv}{dz}, u, v, z\right)=0, \qquad g\left(\frac{du}{dz}, \frac{dv}{dz}, u, v, z\right)=0,$$

where both f and g are rational and integral in du/dz and in dv/dz; so that the two equations may be regarded temporarily as algebraical equations expressing du/dz and dv/dz in terms of u, v, z. To select a typical form of reference, the simultaneous roots of the two equations are to be found; and, for this purpose, Sylvester's dialytic process of elimination can be used. If f be of degree m_1 and g of degree m_2 in du/dz, then m_1+m_2 equations are constructed, being in fact

$$\left(\frac{du}{dz}\right)^n f=0, \qquad \left(\frac{du}{dz}\right)^{n'} g=0,$$

for $n=0, 1, \ldots, m_2-1$ and $n'=0, 1, \ldots, m_1-1$. When all the m_1+m_2-1 powers of du/dz are eliminated, the result is an equation

$$F\left(\frac{dv}{dz}, u, v, z\right)=0$$

which is rational and integral in dv/dz; and any m_1+m_2-1 of the equations*, solved linearly for du/dz, lead to an equation of the form

$$\frac{du}{dz}=G\left(\frac{dv}{dz}, u, v, z\right),$$

where G is algebraical and generally fractional† in $\frac{dv}{dz}$. Moreover,

* This is true in a case of complete generality; but nugatory forms may arise for particular cases. A full discussion of alternatives would require much of the algebra associated with the discussion of the intersections of algebraical plane curves.

† It can easily be made integral as follows. Let V denote dv/dz and suppose that the fractional form of G is

$$\frac{p(V, u, v, z)}{q(V, u, v, z)},$$

so that for any root, say V_1, of $F=0$, we have

$$\frac{du}{dz}=\frac{p(V_1, u, v, z)}{q(V_1, u, v, z)}.$$

G is rational in $\frac{dv}{dz}$ when the first power of $\frac{du}{dz}$ can be deduced; it is a root of a rational function in $\frac{dv}{dz}$ when not the first power of $\frac{du}{dz}$ but some higher power is directly deduced from the $m_1 + m_2 - 1$ equations. The original system is thus equivalent to

$$F\left(\frac{dv}{dz}, u, v, z\right) = 0, \qquad \frac{du}{dz} = G\left(\frac{dv}{dz}, u, v, z\right),$$

which can be taken as a typical form for a system determining two dependent variables.

When a system is composed of three equations and when therefore three dependent variables are involved, it will initially in the most general case have the form

$$f\left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}, u, v, w, z\right) = 0,$$

$$g\left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}, u, v, w, z\right) = 0,$$

$$h\left(\frac{du}{dz}, \frac{dv}{dz}, \frac{dw}{dz}, u, v, w, z\right) = 0,$$

where f, g, and h are rational and integral in each of the three derivatives. To obtain the modified form for this system, the process of dialytic elimination for several simultaneous equations is used*: it is similar in kind to the modification in the preceding case, but the details are more complicated. The result is that an equation of the form

$$F\left(\frac{dw}{dz}, u, v, w, z\right) = 0$$

Let the other roots of $F=0$ be $V_2, V_3, \ldots$; then

$$\frac{du}{dz} = \frac{p(V_1, u, v, z)\, q(V_2, u, v, z)\, q(V_3, u, v, z) \ldots}{q(V_1, u, v, z)\, q(V_2, u, v, z)\, q(V_3, u, v, z) \ldots}.$$

The denominator is a symmetric function of $V_1, V_2, V_3, \ldots$ and therefore, by means of $F=0$, can be made a function of u, v, z; the numerator is a symmetric function of $V_2, V_3, \ldots$ which, by means of the same equation, can be made a function of V_1, u, v, z which is integral in V_1; hence the new form of $\frac{du}{dz}$ in terms of $\frac{dv}{dz}$ is integral and no longer fractional. Moreover, by means of $F=0$, its degree in $\frac{dv}{dz}$ can be made lower than that of $F=0$.

* Salmon's *Higher Algebra*, 3rd edn., §§ 91—94; Faà de Bruno, *Théorie générale de l'élimination*, 3me partie, chap. II, § iv; Cayley, *Coll. Math. Papers*, vol. I, pp. 370—374.

subsists, obtained as the eliminant of a number of equations linearly involving powers of du/dz, dv/dz and their combinations; and when all but one of these equations are treated simultaneously as giving the powers and products of du/dz and dv/dz, they generally lead to equations of the form

$$\frac{du}{dz} = G\left(\frac{dw}{dz}, u, v, w, z\right),$$

$$\frac{dv}{dz} = H\left(\frac{dw}{dz}, u, v, w, z\right),$$

where G and H are algebraical and fractional* in $\frac{dw}{dz}$. The two latter, with $F=0$, can be taken as a typical form for a system determining three dependent variables.

When a system is composed of n equations and when therefore n dependent variables are involved, it will initially in the most general case have the form

$$f_s\left(\frac{du_1}{dz}, \frac{du_2}{dz}, \ldots, \frac{du_n}{dz}, u_1, u_2, \ldots, u_n, z\right) = 0$$

for $s = 1, 2, \ldots, n$. The application of a similar dialytic process leads to the typical form for the system, as constituted by the equations

$$F_n\left(\frac{du_n}{dz}, u_1, u_2, \ldots, u_n, z\right) = 0,$$

$$\frac{du_s}{dz} = G_s\left(\frac{du_n}{dz}, u_1, u_2, \ldots, u_n, z\right),$$

for $s = 1, 2, \ldots, n-1$; the equation $F_n = 0$ is rational and integral in du_n/dz, and all the expressions† G_s are algebraical and generally fractional (which can be made integral) in du_n/dz.

A very special example of such a system occurs in the case of

* Both G and H can be made integral in $\frac{dw}{dz}$ by the same process as in the preceding instance.

† It may be remarked that these expressions are usually rational; they cease to be rational only when the algebraical solution of the simultaneous equations used in the dialytic process leads to a nugatory result for du_s/dz, and then some finite integral power of the expression G_s is rational. The same remark applies to the expression G in the system with two variables, and to the expressions G, H in the system with three variables.

the ordinary linear differential equation of order n in a single variable. When it is

$$\frac{d^n y}{dx^n} + P_1 \frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_{n-1}\frac{dy}{dx} + P_n y = 0,$$

the equivalent system of the preceding type is

$$\frac{dy}{dx} = y_1, \quad \frac{dy_1}{dx} = y_2, \ldots, \frac{dy_{n-2}}{dx} = y_{n-1},$$

$$\frac{dy_{n-1}}{dx} + P_1 y_{n-1} + \ldots + P_{n-1} y_1 + P_n y = 0.$$

In the transformation of the system involving n variables, there is no special reason for retaining du_n/dz as a derivative of reference in the typical form; any other of the derivatives might similarly have been retained. If each be retained in turn, there would be an equation

$$F_r\left(\frac{du_r}{dz}, u_1, \ldots, u_n, z\right) = 0,$$

$r = 1, \ldots, n$; these n equations would be distributed through the n reduced typical systems.

The aggregate of these n equations $F_r = 0$ is sometimes regarded as the normal form for the original system. It undoubtedly includes the original system; but it includes more. For a simultaneous solution of the aggregate would be given by combining any root of $F_1 = 0$, with any root of $F_2 = 0$, with any root of $F_3 = 0$, and so on; but only a limited number of such combinations would satisfy the original system*. The aggregate therefore cannot be regarded as a proper equivalent of an original system in which each equation involves all the derivatives.

It has been assumed that, in the original system of equations, all the derivatives occur in each equation or at least so many occur as to make the purely algebraical transformation possible. But sets of equations, that are less general, may be propounded. For example, in the system

$$f\left(z, u, v, w, \frac{du}{dz}\right) = 0,$$

$$g\left(z, u, v, w, \frac{dv}{dz}, \frac{dw}{dz}\right) = 0, \qquad h\left(z, u, v, w, \frac{dv}{dz}, \frac{dw}{dz}\right) = 0,$$

* An analogous case would arise in finding the coordinates of the intersections of two curves, if they were determined from the x-eliminant and the y-eliminant alone.