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Introduction to algebraic stacks

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Abstract

These are lecture notes based on a short course on stacks given at the Isaac Newton Institute in Cambridge in January 2011. They form a self-contained introduction to some of the basic ideas of stack theory.

Contents

Moduli Spaces, eds. L. Brambila-Paz, O. García-Prada, P. Newstead and R. Thomas. Published by Cambridge University Press.

Introduction

Stacks and algebraic stacks were invented by the Grothendieck school of algebraic geometry in the 1960s. One purpose (see [11]) was to give geometric meaning to higher cohomology classes. The other (see [9] and [2]) was to develop a more general framework for studying moduli problems. It is the latter aspect that interests us in this chapter. Since the 1980s, stacks have become an increasingly important tool in geometry, topology and theoretical physics.

Stack theory examines how mathematical objects can vary in families. For our examples, the mathematical objects will be the triangles, familiar from Euclidean geometry, and closely related concepts. At least to begin with, we will let these vary in continuous families, parametrized by topological spaces.

A surprising number of stacky phenomena can be seen in such simple cases. (In fact, one of the founders of the theory of algebraic stacks, M. Artin, is famously reputed to have said that one need only understand the stack of triangles to understand stacks.)

This chapter is divided into three parts, Sections 1.1, 1.2, and 1.3. Section 1.1 is a very leisurely and elementary introduction to stacks, introducing the main ideas by considering a few elementary examples of topological stacks. The only prerequisites for this section are basic undergraduate courses in abstract algebra (groups and group actions) and topology (topological spaces, covering spaces, the fundamental group).

Section 1.2 introduces the basic formalism of stacks. The prerequisites are the same, although this section is more demanding than the preceding one.

Section 1.3 introduces algebraic stacks, culminating in the Riemann–Roch theorem for stacky curves. The prerequisite here is some basic scheme theory.

We do not cover much of the "algebraic geometry" of algebraic stacks, but we hope that these notes will prepare the reader for the study of more advanced texts, such as [16] or the forthcoming book.[1]

The following outline uses terminology that will be explained in the body of the text.

The first fundamental notion is that of a *symmetry groupoid of a family of objects*. This is introduced first for discrete and then for continuous families of triangles.

In Sections 1.1.1–1.1.3, we consider Euclidean triangles up to similarity (the stack of such triangles is called $\mathfrak{M}$). We define what a fine moduli space is, and show how the symmetries of the isosceles triangles and the equilateral triangle prevent a fine moduli space from existing. We study the coarse moduli space of triangles, and discover that it parametrizes a *modular family*, even though this family is, of course, not universal.

Sections 1.1.4–1.1.6, introduce other examples of moduli problems. In Section 1.1.4, we encounter a fine moduli space (the fine moduli space of scalene triangles); in Section 1.1.5, where we restrict attention to isosceles triangles, we encounter a coarse moduli space supporting several non-isomorphic modular families. Restricting attention entirely to the equilateral triangle, in Section 1.1.6, we come across a coarse moduli space that parametrizes a modular family which is versal, but not universal.

In Section 1.1.7, we finally exhibit an example of a coarse moduli space which does not admit any modular family at all. We start studying *oriented triangles*. We will eventually prefer working with oriented triangles, because they are more closely related to algebraic geometry. The stack of oriented triangles is called $\widetilde{\mathfrak{M}}$.

In Section 1.1.8, we first make a few general and informal remarks about stacks and their role in the study of moduli problems.

The second fundamental concept is that of *versal family*. Versal families replace universal families, where the latter do not exist. Stacks that admit versal families are called *geometric*, which means *topological* in Sections 1.1 and 1.2, but will mean *algebraic* in Section 1.3.

We introduce versal families in Section 1.1.9, and give several examples. We explain how a stack which admits a versal family is essentially equal to the stack of 'generalized moduli maps' (or torsors, in more advanced terminology).

In Section 1.1.10, we start including degenerate triangles in our examinations: triangles whose three vertices are collinear. The main reason we do this

[1] Contact Martin Olsson, www.math.berkeley.edu/~molsson.

is to provide examples of compactifications of moduli stacks. There are several different natural ways to compactify the stack of triangles. There is a naïve point of view, which we dismiss rather quickly. We then explain a more interesting and natural, but also more complicated, point of view: in this, the stack of degenerate triangles turns out to be the quotient stack of a bipyramid modulo its symmetries, which form a group of order 12. This stack of degenerate triangles is called $\overline{\mathfrak{M}}$.

We encounter a very useful construction along the way: the construction of a stack by *stackification*, which means first describing families only locally, then constructing a versal family, and then giving the stack as the stack of generalized moduli maps to the universal family (or torsors for the symmetry groupoid of the versal family).

We then consider oriented degenerate triangles and introduce the *Legendre family* of triangles which is parametrized by the Riemann sphere. It exhibits the stack of oriented degenerate triangles as the quotient stack of the Riemann sphere by the action of the dihedral group with six elements. (In particular, it endows the stack of oriented degenerate triangles with the structure of an algebraic, not just topological, stack.) We call this stack $\mathfrak{L}$, and refer to it as the *Legendre compactification* of the stack of oriented triangles $\widetilde{\mathfrak{M}}$.

The Legendre family provides the following illustration of the concept of *generalized moduli map* (or groupoid torsor). We try to characterize, i.e. completely describe, the similarity type of an (oriented, maybe degenerate) triangle, by specifying the complex cross-ratio of its three vertices together with the point at infinity. However, the cross-ratio is not a single-valued invariant, but rather a multi-valued one: the six possible values of the cross-ratio are acted upon by the group S_3. Thus the stack $\mathfrak{L}$ of (oriented, maybe degenerate) triangles is the quotient stack of the Riemann sphere divided by S_3.

In Section 1.1.11, we explain how to relate different versal families for the same stack with one another, and how to recognize two stacks as being essentially the same, by exhibiting a bitorsor for the respective symmetry groupoids of respective versal families. We apply this both ways: we exhibit two different versal families for "non-pinched" triangles and show how a bitorsor intertwines them. Then we construct a bitorsor intertwining two potentially different moduli problems, namely two potentially different ways to treat families containing "pinched" triangles, thus showing that the two moduli problems are equivalent.

In Section 1.1.12, we introduce another compactification of the moduli stack of oriented triangles, which we call the *Weierstrass compactification*, because we construct it from the family of degree 3 polynomials in Weierstrass normal form. We denote this stack by $\mathfrak{W}$. We encounter our first example of a

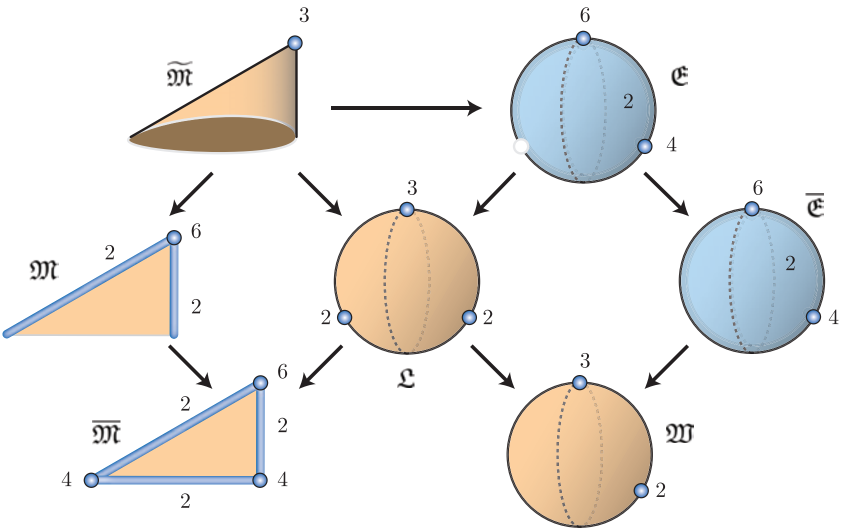


Figure 1.1. Some of the stacks we encounter in these notes, and the morphisms between them. Stacky points (coloured blue) are labeled with the order of their isotropy groups.

non-trivial morphism of stacks, namely the natural morphism $\mathfrak{L} \to \mathfrak{W}$. We also introduce a holomorphic coordinate on the coarse moduli space of oriented triangles known as the j-invariant.

In Section 1.2, we introduce the formalism of stacks. This will allow us to discuss topological stacks in general, without reference to specific objects such as triangles.

In Sections 1.2.1–1.2.4 we discuss the standard notions. We start with categories fibered in groupoids, which formalize what a moduli problem is. Then come the prestacks, which have well-behaved isomorphism spaces, and allow for the general definition of versal family. After a brief discussion of stacks, we define topological stacks to be stacks that admit a versal family. We discuss the basic fact that every topological stack is isomorphic to the stack of torsors for the symmetry groupoid of a versal family. This also formalizes our approach to stackification: start with a prestack, find a versal family, and then replace the given prestack by the stack of torsors for the symmetry groupoid of the versal family.

In Section 1.2.5 we discuss a new idea: symmetry groupoids of versal families should be considered as gluing data for topological stacks, in analogy to atlases for topological manifolds. This also leads to the requirement that the parameter space of a versal family should reflect the local topological structure of a stack faithfully, and, conversely, that a topological stack should locally behave in a manner controlled by the parameter space of a versal family, in order that we can "do geometry" on the stack.

This idea leads to the introduction of *étale versal families*, and the associated stacks, which we call *Deligne–Mumford topological stacks*, in analogy with the algebraic case. We prove a structure theorem that says that every separated Deligne–Mumford topological stack has an open cover by finite group quotient stacks.

This shows that all "well-behaved" moduli problems with discrete symmetry groups are locally described by finite group quotients. Therefore, the seemingly simple examples we start out with in fact turn out to be quite typical of the general case.

We also encounter examples of moduli problems without symmetries, which nevertheless do not admit fine moduli spaces. For sufficiently badly behaved equivalence relations (when the quotient map does not admit local sections), the quotient space is not a fine moduli space.

In Section 1.2.6, we continue our series of examples of moduli problems related to triangles by considering lattices up to homothety. This leads to the stack of elliptic curves, which we call $\mathfrak{E}$, and its compactification, $\overline{\mathfrak{E}}$. We see another example of a morphism of stacks, namely $\mathfrak{E} \to \mathfrak{W}$, which maps a lattice to the triangle of values of the Weierstrass $\wp$-function at the half periods. This is an example of a $\mathbb{Z}_2$-gerbe.

As an illustration of some simple "topology with stacks," we introduce the fundamental group of a topological stack in Section 1.2.7, and compute it for some of our examples.

Section 1.3 is a brief introduction to algebraic stacks. The algebraic theory requires more background than the topological one: we need, for example, the theory of cohomology and base change. We will therefore assume that the reader has a certain familiarity with scheme theory as covered in [15].

We limit our attention to algebraic stacks with affine diagonal. This avoids the need for algebraic spaces as a prerequisite. For many applications, this is not a serious limitation. As typical examples, we discuss the stack of elliptic curves $\mathfrak{E}$ and its compactification $\overline{\mathfrak{E}}$, as well as the stack of vector bundles on a curve.

Our definition of algebraic stack avoids reference to Grothendieck topologies, algebraic spaces, and descent theory. Essentially, a category fibered in groupoids is an algebraic stack if it is equivalent to the stack of torsors for an algebraic groupoid. Sometimes, for example for $\overline{\mathfrak{E}}$, we can verify this condition directly. We discuss a useful theorem, which reduces the verification that a given groupoid fibration is an algebraic stack to the existence of a versal family, with sufficiently well-behaved symmetry groupoid, and the gluing property in the étale topology.

We include a discussion of the coarse moduli space in the algebraic context: the theory is much more involved than in the topological case. We introduce algebraic spaces as algebraic stacks "without stackiness." We sketch the proof that separated Deligne–Mumford stacks admit coarse moduli spaces, which are separated algebraic stacks. As a by-product, we show that separated Deligne–Mumford stacks are locally, in the étale topology of the coarse moduli space, finite group quotients.

We then define what vector bundles and coherent sheaves on stacks are, giving the bundle of modular forms on $\overline{\mathfrak{E}}$ as an example. In a final Section 1.3.6, we study stacky curves, and as an example of some algebraic geometry over stacks we prove the Riemann–Roch theorem for orbifold curves. As an illustration, we compute the well-known dimensions of the spaces of modular forms.

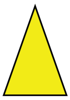
1.1 Topological stacks: triangles

This section is directed at the student of mathematics who has taken an introduction to topology (covering spaces and the fundamental group) and an introduction to abstract algebra (group actions). Most of the formal mathematics has been relegated to exercises, which can be skipped by the reader who lacks the requisite background. The end of these exercises is marked with the symbol "□."

We are interested in two ideas, *symmetry* and *form*, and their role in *classification*.

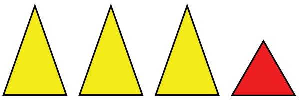
1.1.1 Families and their symmetry groupoids

Consider a mathematical concept, for example *triangle*, together with a notion of isomorphism, for example *similarity*. This leads to the idea of *symmetry*. Given an object (for example, an isosceles triangle)



a *symmetry* is an isomorphism of the object with itself (for example, the reflection across the "axis of symmetry"). All the symmetries of an object form a *group*, the *symmetry group* of the object. (The symmetry group of our isosceles triangle is {id, refl}.)

To capture the essence of *form*, in particular how form may vary, we consider *families* of objects rather than single objects (for example, the family of four triangles



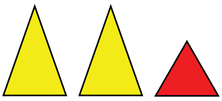
(1.1)

consisting of three congruent isosceles triangles and one equilateral triangle).

Definition 1.1. A **symmetry** of a family of objects is an isomorphism of one member of the family with another member of the family.

Example 1.2. The family (1.1) of four triangles has 24 symmetries: there are two symmetries from each of the isosceles triangles to every other (including itself), adding up to 18, plus six symmetries of the equilateral triangle.

If we restrict the family to contain only the latter two isosceles triangles and the equilateral triangle,

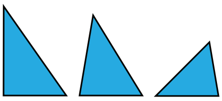


the family has 14 symmetries.

Various types of symmetry groupoids

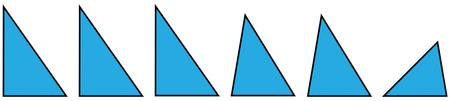
Definition 1.3. The collection of all symmetries of a given family is called the **symmetry groupoid** of the family.

Example 1.4. (Set) The symmetry groupoid of a family of non-isomorphic asymmetric objects



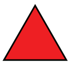
consists of only the trivial symmetries, one for each object. Such a groupoid is essentially the same thing as the set of objects in the family (or, more precisely, the indexing set of the family).

Example 1.5. (Equivalence relation) The symmetry groupoid of a family of asymmetric objects



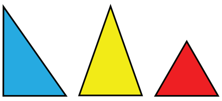
is *rigid*. From any object to another there is at most one symmetry. A rigid groupoid is essentially the same thing as an equivalence relation on the set of objects (or the indexing set of the family).

Example 1.6. (Group) The symmetry groupoid of a single object



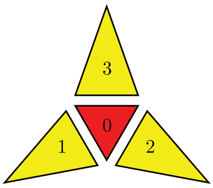
is a group.

Example 1.7. (Family of groups) The symmetry groupoid of a family of non-isomorphic objects



is a family of groups.

Example 1.8. (Transformation groupoid) Consider again the family of triangles (1.1) above, but now rearranged like this:



This figure has dihedral symmetry, and so the dihedral group with six elements, i.e. the symmetric group on three letters S_3, acts on this figure. Each element of S_3 defines four symmetries of the family, because it defines a symmetry originating at each of the four triangles.

For example, the rotation by $\frac{2\pi}{3}$ (or the permutation $1 \longmapsto 3$, $3 \longmapsto 2$, $2 \longmapsto 1$), gives rise to the $\frac{2\pi}{3}$-rotational symmetry of the equilateral triangle in the center of the figure, as well as three isomorphisms, each from one isosceles triangle to another.