A Window Into Zeta and Modular Physics
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Introduction

Some exciting, bold new cooperative explorations of various interconnections between traditional domains of "pure" mathematics and exotic new developments in theoretical physics have continued to emerge in recent years. The beautiful interlacing of theory and application, and cross-discipline interaction has led, as is usual, to notable, fruitful, and bonus outcomes.

These interconnections range from topology, algebraic geometry, modular forms, Eisenstein series, zeta functions, vertex operators, and knot theory to gauge theory, strings and branes, quantum fields, cosmology, general relativity, and Bose–Einstein condensation. They are broad enough in scope to present the average reader with not only a measure of enchantment but with some mild bewilderment as well. A new journal, *Communications in Number Theory and Physics*, has recently been launched to follow and facilitate interactions and dynamics between these two disciplines, for example. Various books that are now available, in addition to an array of conference and workshop activity, accent this fortunate merger of mathematics and physical theory and assist greatly in bridging the divide, although in some cases the themes are pitched at a level more suitable for advanced readers and researchers.

In an attempt to further bridge the divide, at least in some modest way for students and non-experts, and to provide a window into this adventurous arena of intertwining ideas, a graduate workshop entitled "A Window into Zeta and Modular Physics" was presented at MSRI during the period from 16 to 27 June 2008. The workshop consisted of daily expository lectures, speakers' seminar lectures (where the material was more technical and represented their research, but which at the same time did connect to and enlarge on the daily lectures), and four special student lectures. Given the excellent preparation and presentation of the lectures, it was proposed that it would be of further benefit to the students (especially given their enthusiastic response), and to the mathematical physics community in general, if the lectures were eventually molded in some form as a book. Thus the present volume, with the workshop title and similar style, evolved. Besides the editors, there were three workshop speakers: Geoffrey Mason, Audrey Terras, and Michael Tuite. The student speakers were Jennie

D'Ambroise, Shabnam Beheshti, Savan Kharel, and Paul Nelson. Among these four, Jennie and Shabnam (who have since earned their PhDs) accepted the invitation to have their lectures appear here.

The volume consists of two parts. Part I contains basic, expository lectures, except that it was found convenient to also include some seminar material in the combined presentations of Geoff and Michael, and thus to make those one long set of lectures. Part II consists of speakers' seminar lectures, where we have included the lectures of Jennie and Shabnam.

A more specific description of Part I is as follows. Lectures of Floyd Williams cover topics such as the Riemann zeta function (proof of its meromorphic continuation and functional equation), Euler products (in particular, of Hecke and Dirichlet L-functions), holomorphic modular forms, Dedekind's eta function, the quasimodular form G_2, the Rademacher–Zuckerman formula for the Fourier coefficients of forms of negative weight, and non-holomorphic Eisenstein series, with some physical applications including gravity in extra dimensions, finite temperature zeta functions, the zeta regularization of Casimir energy, a determinant (or path integral) computation in Bosonic string theory, and an abstract Cardy formula for black hole entropy, based on the asymptotic behavior of Fourier coefficients of modular forms of zero weight. Several appendices are included. These provide a proof of the Poisson summation formula, the Jacobi inversion formula, the Fourier expansion of a holomorphic periodic function, etc., and thus they render further completeness of the material. The lectures, in general, provide some background material for some of the other lectures, with some overlap.

Further applications of zeta functions — those of Hurwitz, Barnes, and Epstein, and also spectral zeta functions — are presented in the lectures of Klaus Kirsten. Integral representations of these functions are established and their association with eigenvalue problems for partial differential operators is discussed by way of concrete computational examples, where various types of boundary conditions are imposed. For the atomic Schrödinger operator corresponding to a harmonic oscillator potential in three dimensions, for example, the spectral zeta function is shown to be given in terms of the Barnes zeta function.

Applications of zeta functions to other areas such as the Casimir effect and Bose–Einstein condensation are also discussed in the Kirsten lectures, along with some provocative, motivating questions: *Can one hear the shape of a drum? What does the Casimir effect know about a boundary? What does a Bose gas know about its container?* Regarding the first of these questions — which was posed in 1966 by M. Kac, but may even go back to H. Weyl — some heat kernels and their small-time asymptotics and Minakshisundaram–Pleijel coefficients are presented by way of examples, and by way of a general result for Laplace-type differential operators acting on smooth sections of a vector bundle over a smooth compact Riemannian manifold with or without boundary. Bose–Einstein condensation is the final topic treated, where zeta functions continue to play a relevant role. A Bose–Einstein condensate is the ground state (lowest energy state) assumed by interacting bosons under the influence of some external trapping potential. This condensation phenomenon, which was predicted in 1924 by S. Bose and A. Einstein, is an example of phase transition at the quantum level. It took some 70 years, however, for its first experimental verification, in some 1995 vapor studies of rubidium and sodium.

The subject matter of zeta functions (of Riemann, Selberg, Ruelle, and Ihara) and quantum chaos is taken up in the lectures of Audrey Terras, where emphasis is laid on the Ihara zeta function of a finite graph, and its connections to chaos and random matrix theory. The Selberg zeta function is a zeta function attached to a compact Riemann surface (say of genus at least two), and gives rise to a duality between lengths of closed geodesics and the spectrum of the Laplace–Beltrami operator — closed geodesics being regarded as Selberg “primes”. This zeta function can be attached, more generally, to compact quotients of hyperbolic spaces, or in fact to the quotient of a rank 1, noncompact symmetric space modulo a cofinite volume discrete group of isometries. The Ruelle zeta function is a dynamical systems zeta function, whose motivation goes back to work of M. Artin and B. Mazur on the zeta function of a projective algebraic variety over a finite field, where the Frobenius map is replaced by a suitable diffeomorphism of a smooth compact manifold. It is shown in the lectures that the Ihara zeta function is generalized by the Ruelle zeta function, the former function being the graph version of the Selberg zeta function where the duality is replaced by that between closed paths and the spectrum of an *adjacency matrix*.

Some particular topics include edge and path zeta functions, Ihara determinant formulas, a graph prime number theorem, a Riemann hypothesis regarding the poles of the Ihara zeta function (which for a regular connected graph is true precisely when the graph is *Ramanujan* — a condition on the eigenvalues of the adjacency matrix, roughly speaking), the Alon conjectures for regular and irregular graphs (which refer to Riemann hypotheses for these graphs), and some closing material with a focus on the quantum chaos question that concerns a connection between the poles of Ihara zeta and eigenvalues of a random matrix. Here a comparison is made between the spacing of poles and the spacing of the eigenvalues of a certain (non-symmetric) edge adjacency matrix.

Quantum chaos, which is not always precisely defined, is described sometimes as the statistics of eigenvalues (or energy levels) of particular non-classical systems. For example, E. Wigner, in the 1950s, first introduced the notion of the statistical distribution of energy levels of heavy atomic nuclei. From this work there evolved the *Wigner surmise* (which Audrey also discusses) for the probability density for finding two adjacent eigenvalues with a given spacing — a density that differs markedly from a Poisson density.

Part I concludes with an extended set of lectures by Geoff Mason and Michael Tuite that cover basic, introductory material on vertex operator algebras (VOAs) and modular forms, as well as some research material indicative of their presentations during the workshop speakers' seminar.

VOAs, also known as chiral algebras to physicists, special cases of which include W-algebras, were formally defined by R. Borcherds. They figure prominently in many areas such as finite group theory (in particular in regards to "monstrous moonshine"), representations of affine Kac–Moody Lie algebras, knot theory, quantum groups, geometric Langlands theory, etc., and in fact they provide for a mathematical, axiomatic formulation of two-dimensional conformal field theory (CFT). Various CFT manipulations, that sometimes are in want of rigor, are neatly handled, algebraically, in the VOA world. In the so-called operator product expansion for *primary* fields, for example (as postulated by A. Belavin, A. Polyakov, and A. Zamolodchikov), the fundamental requirement of associativity encodes neatly in VOA structure — which in the lectures amounts to the Jacobi identity (page 188). This identity also embodies the key notion of locality. Locality relaxes the requirement of commutativity of fields, which if imposed would be too strong a physical condition.

String theory for bosons can be regarded as a CFT, and it therefore has natural ties to VOAs. For example, to an even, positive definite lattice L is attached a VOA (pages 221–224), which corresponds to a bosonic string compactified on the torus defined by L. Other naturally constructed VOAs are Heisenberg VOAs (that correspond to a free boson), Virasoro VOAs, and VOAs attached to a Lie algebra g equipped with a symmetric, non-degenerate, invariant bilinear form, from whence is attached to g an affine Kac–Moody Lie algebra. The moonshine module also has a VOA structure (due to I. Frenkel, J. Lepowsky, and J. Meurman), which is discussed on page 233.

Modular forms and elliptic functions arise naturally in VOA theory by way of VOA characters (or partition functions) and, more generally, correlation functions. For *nonrational* VOAs, however, these functions might only be quasi-modular. But the character of the (nonrational) Heisenberg VOA, for example, is a modular form (of weight $-\frac{1}{2}$) as it is the reciprocal of the Dedekind eta function. A characteristic property of rational VOAs is that they have only finitely many inequivalent, irreducible representations. In particular, a distinguished class of rational VOAs (called Virasoro *minimal models*) is provided by certain Virasoro quotient modules of zero conformal weight and a central charge $c = 1 - 6(p-q)^2/pq$ parametrized by a pair of coprime integers p, q greater than 1. The (finite) number of irreducible representations here is $(p-1)(q-1)$. $p=4$ and $q=3$, for example, give rise to a central charge $\frac{1}{2}$, which at the physical level corresponds to the Ising model in statistical mechanics, whereas $p=5$, $q=4$ corresponds to the tricritical Ising model with central charge $\frac{7}{10}$, and $p=6$, $q=5$ corresponds to the three-states Potts model with central charge $\frac{4}{5}$.

In addition to one-point correlation functions, Geoff and Michael also consider two-point correlation functions that have relevance to their research on higher genus CFT. These functions are shown to be elliptic. Zhu recursion formulas (for correlation functions), Zhu's finiteness condition (referred to as C_2-cofiniteness in the lectures), the important issue of modular invariance, and other important matters are given careful attention in the latter part of the paper, with a discussion of current research areas. A deep, initial result of Y. Zhu is that for rational VOAs subject to his finiteness condition, the complete (finite) set of inequivalent, irreducible characters enjoys the beautiful modular property of being holomorphic functions on the upper half-plane that span a subspace invariant under the action of the modular group $\mathrm{SL}(2,\mathbb{Z})$. At the Lie-algebra representation theory level, apart from VOA theory, modular properties of characters have profound implications — which is the case for Weyl–Kac characters of affine SU(2), for example, where modularity is exploited in the study of BTZ black hole entropy corrections.

Seminar lectures comprise Part II of the volume (with the exception of those of Geoff and Michael, which, as mentioned, appear in Part I). We describe them briefly. Jennie's lecture deals with elliptic and theta function solutions of Einstein's gravitational field equations in the FLRW model. A soliton–black hole connection in two-dimensional gravity is considered in Shabnam's lecture. Floyd discusses the role of the Patterson–Selberg zeta function in three-dimensional gravity. Klaus points out how contour integration methods lead to closed formulas for functional determinants in low and high dimensions.

The editors express sincere thanks and appreciation to their workshop colleagues for their valued contributions, and to MSRI Director Dr. Robert Bryant and Associate Director Dr. Kathleen O'Hara for the invitation to present the workshop and the kind hospitality we were accorded. We are also appreciative of the gifted workshop students that we were fortunate to interact with. Our sincere thanks extend moreover to the MSRI Book Series Editor, Dr. Silvio Levy, for his helpful guidance during the preparation of the volume.

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Lectures on zeta functions, L-functions and modular forms with some physical applications

FLOYD L. WILLIAMS

Introduction

We present nine lectures that are introductory and foundational in nature. The basic inspiration comes from the Riemann zeta function, which is the starting point. Along the way there are sprinkled some connections of the material to physics. The asymptotics of Fourier coefficients of zero weight modular forms, for example, are considered in regards to black hole entropy. Thus we have some interests also connected with Einstein's general relativity. References are listed that cover much more material, of course, than what is attempted here.

Although his papers were few in number during his brief life, which was cut short by tuberculosis, Georg Friedrich Bernhard Riemann (1826–1866) ranks prominently among the most outstanding mathematicians of the nineteenth century. In particular, Riemann published only one paper on number theory [32]: "Über die Anzahl der Primzahlen unter einer gegebenen Grösse", that is, "On the number of primes less than a given magnitude". In this short paper prepared for Riemann's election to the Berlin Academy of Sciences, he presented a study of the distribution of primes based on complex variables methods. There the now famous *Riemann zeta function*

$$\zeta(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{0.1}$$

defined for $\operatorname{Re} s > 1$, appears along with its analytic continuation to the full complex plane $\mathbb{C}$, and a proof of a functional equation (FE) that relates the values $\zeta(s)$ and $\zeta(1-s)$. The FE in fact was conjectured by Leonhard Euler, who also obtained in 1737 (over 120 years before Riemann) an *Euler product* representation

$$\zeta(s) = \prod_{p>0} \frac{1}{1 - p^{-s}} \qquad (\operatorname{Re} s > 1) \tag{0.2}$$

of $\zeta(s)$ where the product is taken over the primes p. Moreover, Riemann introduced in that seminal paper a query, now called the *Riemann Hypothesis* (RH), which to date has defied resolution by the best mathematical minds. Namely, as we shall see, $\zeta(s)$ vanishes at the values $s = -2n$, where $n = 1, 2, 3, \ldots$; these are called the *trivial* zeros of $\zeta(s)$. The RH is the (yet unproved) statement that if s is a zero of ζ that is *not* trivial, the real part of s must have the value $\frac{1}{2}$!

Regarding Riemann's analytic approach to the study of the distribution of primes, we mention that his main goal was to set up a framework to facilitate a proof of the *prime number theorem* (which was also conjectured by Gauss) which states that if $\pi(x)$ is the number of primes $\leq x$, for $x \in \mathbb{R}$ a real number, then $\pi(x)$ behaves asymptotically (as $x \to \infty$) as $x/\log x$. That is, one has (precisely) that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1, \tag{0.3}$$

which was independently proved by Jacques Hadamard and Charles de la Vallée-Poussin in 1896. A key role in the proof of the monumental result (0.3) is the fact that at least all nontrivial zeros of $\zeta(s)$ reside in the interior of the *critical strip* $0 \leq \operatorname{Re} s \leq 1$.

Riemann's deep contributions extend to the realm of physics as well - Riemannian geometry, for example, being the perfect vehicle for the formulation of Einstein's gravitational field equations of general relativity. Inspired by the definition (0.1), or by the Euler product in (0.2), one can construct various other zeta functions (as is done in this volume) with a range of applications to physics. A particular zeta function that we shall consider later will bear a particular relation to a particular solution of the Einstein field equations — namely a *black hole* solution; see my Speaker's Lecture.

There are quite many ways nowadays to find the analytic continuation and FE of $\zeta(s)$. We shall basically follow Riemann's method. For the reader's benefit, we collect some standard background material in various appendices. Thus, to a large extent, we shall attempt to provide details and completeness of the material, although at some points (later for example, in the lecture on modular forms) the goal will be to present a general picture of results, with some (but not all) proofs.

Special thanks are extended to Jennie D'Ambroise for her competent and thoughtful preparation of all my lectures presented in this volume.

CONTENTS

Lecture 1. Analytic continuation and functional equation of the Riemann zeta function

Since $|1/n^s| = 1/n^{\mathrm{Re}\, s}$, the series in (0.1) converges absolutely for $\mathrm{Re}\, s > 1$. Moreover, by the Weierstrass M-test, for any $\delta > 0$ one has uniform convergence of that series on the strip

$$S_\delta \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 1 + \delta\},$$

since $|1/n^s| = 1/n^{\mathrm{Re}\, s} < 1/n^{1+\delta}$ on S_δ, with

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

Since any compact subset of the domain $S_0 \stackrel{\text{def}}{=} \{s \in \mathbb{C} \mid \mathrm{Re}\, s > 1\}$ is contained in some S_δ, the series, in particular, converges absolutely and uniformly on compact subsets of S_0. By Weierstrass's general theorem we can conclude that the Riemann zeta function $\zeta(s)$ in (0.1) is holomorphic on S_0 (since the terms $1/n^s$ are holomorphic in s) and that termwise differentiation is permitted: for $\mathrm{Re}\, s > 1$

$$\zeta'(s) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s}. \tag{1.1}$$

We wish to analytically continue $\zeta(s)$ to the full complex plane. For that purpose, we begin by considering the world's simplest *theta function* $\theta(t)$, defined for $t > 0$:

$$\theta(t) \stackrel{\text{def}}{=} \sum_{n \in \mathbb{Z}} e^{-\pi n^2 t} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2 t} \tag{1.2}$$

where $\mathbb{Z}$ denotes the ring of integers. It enjoys the remarkable property that its values at t and t inverse (i.e. $1/t$) are related:

$$\theta(t) = \frac{\theta(1/t)}{\sqrt{t}}. \tag{1.3}$$

The very simple formula (1.3), which however requires some work to prove, is called the *Jacobi inversion formula*. We set up a proof of it in Appendix C, based on the *Poisson Summation Formula* proved in Appendix C. One can of course define more complicated theta functions, even in the context of higher-dimensional spaces, and prove analogous Jacobi inversion formulas.

For $s \in \mathbb{C}$ define

$$J(s) \stackrel{\text{def}}{=} \int_1^{\infty} \frac{\theta(t) - 1}{2} t^s \, dt. \tag{1.4}$$

By Appendix A, $J(s)$ is an entire function of s, whose derivative can be obtained, in fact, by differentiation under the integral sign. One can obtain both the analytic continuation and the functional equation of $\zeta(s)$ by introducing the sum

$$I(s) \stackrel{\text{def}}{=} \sum_{n=1}^{\infty} \int_0^{\infty} (\pi n^2)^{-s} e^{-t} t^{s-1} \, dt, \tag{1.5}$$

which we will see is well-defined for $\operatorname{Re} s > \frac{1}{2}$, and by computing it in different ways, based on the inversion formula (1.3). Recalling that the gamma function $\Gamma(s)$ is given for $\operatorname{Re} s > 0$ by

$$\Gamma(s) \stackrel{\text{def}}{=} \int_0^{\infty} e^{-t} t^{s-1} \, dt \tag{1.6}$$

we clearly have

$$I(s) \stackrel{\text{def}}{=} \pi^{-s} \left(\sum_{n=1}^{\infty} \frac{1}{n^{2s}} \right) \Gamma(s) = \pi^{-s} \zeta(2s) \Gamma(s), \tag{1.7}$$

so that $I(s)$ is well-defined for $\operatorname{Re} 2s > 1$: $\operatorname{Re} s > \frac{1}{2}$. On the other hand, by the change of variables $u = t/\pi n^2$ we transform the integral in (1.5) to obtain

$$I(s) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-\pi n^2 t} t^{s-1} \, dt.$$