PART I

Universal constructions

Our focus in the first part of this book is on the construction of certain universal objects that are crucial to the algebraic approach to the study of the asymptotic behavior of dynamical systems (flows). For the purposes of this exposition a flow is a pair (X, T), where X is a compact Hausdorff space, and T is a group which acts on X (on the right). A homomorphism of flows is a continuous mapping which preserves the actions. When the orbit closure of some point $x_0 \in X$ is all of X, that is $\overline{x_0 T} = X$, we say that the flow (X, T) is point transitive. If $\overline{xT} = X$ for all $x \in X$ we say that (X, T) is minimal. The collection of point transitive flows has a universal object, $(\beta T, T)$, in the sense that every point transitive flow is a homomorphic image of $(\beta T, T)$ (see **2.5**). The action of T on βT extends in a natural way to a semigroup structure on βT which plays an important role in the study of flows. In section 1, we give an exposition of the structure of βT, relegating its construction via ultrafilters on T to an appendix.

We exploit the properties of βT in section 2 to give a treatment of the enveloping semigroup $E(X, T)$ of a flow (X, T). In Section 4 βT and $E(X, T)$ are used to introduce many of the fundamental notions which will be studied throughout the book. Of particular importance is the structure of the minimal ideals in $E(X, T)$ discussed in section 3. The fact that βT is its own enveloping semigroup allows us to apply these ideas to a minimal right ideal $M \subset \beta T$. On the other hand for such a minimal ideal, the flow (M, T) is a universal object for the collection of all minimal flows (see **3.16**). Our approach to the study of minimal flows involves exploiting the structure of M and the group of automorphisms of M to gain an understanding of the structure of the icers (closed invariant equivalence relations) on M. These ideas are pursued further in section 7 of Part II.

Another construction which will play a significant role in our exposition is that of a quasi-factor of the flow (X, T); this is by definition a subflow of the flow $(2^X, T)$. Here by 2^X we mean the space whose elements are closed non-empty subsets of X. In the appendix to section 5, we give an outline of the construction of the Vietoris topology, a compact Hausdorff topology on 2^X. In the body of section 5 we develop some of the properties of the flow $(2^X, T)$, including the extension of the natural action of T on 2^X to an action of βT on 2^X given by the so-called circle operator.

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The Stone-cech compactification βT

The Stone-Cech compactification βT, is a compact Hausdorff space containing the discrete group T as a dense subset. Of course one can construct the Stone-Cech compactification of any discrete set; a construction via ultrafilters is outlined in the appendix to this section. On the other hand βT is characterized by certain properties which we take as its definition for the purposes of this section. When T is a group there is a natural semigroup structure on βT, for which left multiplication by all elements, and right multiplication by elements of T are continuous. This semigroup structure plays a fundamental role in our study. In proposition **1.3** we deduce this structure as a consequence of the characterizing properties of βT; in the appendix the semigroup structure is defined directly in terms of ultrafilters.

Definition 1.1 Let T be a set with the discrete topology. The *Stone-Cech compactification* βT *of* T is determined up to homeomorphism by the following properties:

(i) $T \subset \beta T$ with $\overline{T} = \beta T$,
(ii) βT is a compact Hausdorff space, and
(iii) if X is a compact Hausdorff space and $f{:}T \to X$, then there exists a unique continuous extension $\hat{f}: \beta T \to X$.

The uniqueness of the extension in (iii) above is crucial. For instance it has as a consequence the fact that βT is unique up to homeomorphism. Indeed if Y is any space satisfying (i), (ii), and (iii), then the inclusions $T \subset Y$ and $T \subset \beta T$ extend to continuous maps $\varphi: \beta T \to Y$ and $\psi: Y \to \beta T$. The composition $\varphi \circ \psi$ is thus a continuous extension of the inclusion $T \subset Y$ to Y, and hence by uniqueness must be the identity. Similarly $\psi \circ \varphi$ is the identity on βT, and therefore φ is a homeomorphism with inverse ψ. This shows that as a topological space βT is completely determined by the conditions in **1.1**.

The following theorem confirms this by exhibiting a base for the topology on βT. It is interesting to note that this base consists of sets which are both open and closed in βT.

Theorem 1.2 Let:

(i) T be a set with the discrete topology and βT be its Stone-Cech compactification,
(ii) $A \subset T$, and
(iii) $V \subset \beta T$ be an open set.

Then:

(a) $\beta T = \overline{A} \cup \overline{T \setminus A}$ is a disjoint union, and thus $\overline{A}$ is both open and closed (clopen) in βT,
(b) $\overline{V} = \overline{V \cap T}$, and hence $\overline{V}$ is both open and closed, and
(c) $\{\overline{A} \mid A \subset T\}$ is a base for the topology on βT.

PROOF: (a) 1. Let $\emptyset \neq A \subset T$.
2. Let $\chi_A : T \to \{0, 1\}$ be defined by $\chi_A(t) = \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{otherwise} \end{cases}$.
3. There exists a continuous extension $\hat{\chi}_A : \beta T \to \{0, 1\}$. (by 2, **1.1**(iii))
4. $\hat{\chi}_A^{-1}(1)$ and $\hat{\chi}_A^{-1}(0)$ are clopen with $\overline{A} \subset \hat{\chi}_A^{-1}(1)$ and $\overline{T \setminus A} \subset \hat{\chi}_A^{-1}(0)$. (by 2, 3)
5. Let $p \in \hat{\chi}_A^{-1}(1)$ and $W \subset \beta T$ be open with $p \in W$.
6. There exists $t \in T$ with $t \in W \cap \hat{\chi}_A^{-1}(1)$. (by 4, 5, **1.1**(i))
7. $t \in A \cap W$. (by 2, 3, 6)
8. $p \in \overline{A}$. (by 5, 7)
9. $\hat{\chi}_A^{-1}(1) \subset \overline{A}$. (by 5, 8)
10. $\hat{\chi}_A^{-1}(0) \subset \overline{T \setminus A}$. (similar argument)
11. $\hat{\chi}_A^{-1}(1) = \overline{A}$ and $\hat{\chi}_A^{-1}(0) = \overline{T \setminus A}$. (by 4, 9, 10)
(b) 1. Clearly $\overline{V \cap T} \subset \overline{V}$.
2. Let $W \subset \beta T$ be open and $p \in \overline{V} \cap W$.
3. There exists $t \in T$ with $t \in V \cap W$. (by 2, **1.1**(i))
4. $t \in (V \cap T) \cap W$. (by 3)
5. $p \in \overline{V \cap T}$. (by 2, 4)
(c) 1. Let $\emptyset \neq V \subset \beta T$ be open and $p \in V$.
2. There exists W open with $p \in W \subset \overline{W} \subset V$. ($\beta T$ is compact Hausdorff)
3. $p \in \overline{W} = \overline{W \cap T} \subset V$. (by 2, part (b))
4. $\{\overline{A} \mid A \subset T\}$ is a base for the topology on βT. (by 1, 3)

We will be most interested in the space βT when T is a group. In this case, and in fact whenever T is a semigroup, the semigroup structure on T induces a semigroup structure on βT. Once again the uniqueness of the extension in **1.1** (iii) is crucial. The following proposition details the construction.

Proposition 1.3 Let T be a semigroup, so that T is provided with an associative binary operation:

$$\begin{aligned} T \times T &\to T \\ (s,t) &\to st. \end{aligned}$$

Then the semigroup structure on T extends to one on βT,

$$\begin{aligned} \beta T \times \beta T &\to \beta T \\ (p,q) &\to pq \end{aligned}$$

such that:

(a) the right multiplication map $\begin{aligned} R_t : \beta T &\to \beta T \\ p &\to pt \end{aligned}$ is continuous for all $t \in T$,

and

(b) the left multiplication map $\begin{aligned} L_p : \beta T &\to \beta T \\ q &\to pq \end{aligned}$ is continuous for all $p \in \beta T$.

PROOF: 1. Let $m^t(s) = st$ for all $s, t \in T$.
2. There exists a continuous extension $R_t : \beta T \to \beta T$ of m^t for every $t \in T$. (by (iii) of **1.1**)
3. There exists a continuous extension $L_p : \beta T \to \beta T$ of the map

$$\begin{aligned} T &\to \beta T \\ t &\to R_t(p) \end{aligned}. \qquad \text{(by (iii) of } \mathbf{1.1}\text{)}$$

4. For $p, q \in \beta T$ we define $pq \equiv L_p(q)$.
5. Let $t, s \in T$. Then the maps $\begin{aligned} \beta T &\to \beta T \\ p &\to (ps)t \end{aligned}$ and $\begin{aligned} \beta T &\to \beta T \\ p &\to p(st) \end{aligned}$ are both continuous extensions of the map

$$\begin{aligned} T &\to \beta T \\ t' &\to (t's)t = t'(st). \end{aligned}$$

6. $p(st) = (ps)t$ for all $p \in \beta T$ and $s, t \in T$. (by 5 and uniqueness in (iii) of **1.1**)
7. Let $p, q \in \beta T$. Then the maps $\begin{aligned} \beta T &\to \beta T \\ q &\to (pq)t \end{aligned}$ and $\begin{aligned} \beta T &\to \beta T \\ q &\to p(qt) \end{aligned}$ are both continuous extensions of the map

$$\begin{aligned} T &\to \beta T \\ s &\to (ps)t = p(st). \end{aligned} \qquad \text{(by 3, 6)}$$

8. $p(qt) = (pq)t$ for all $p \in \beta T$ and $t \in T$.
(by 7 and uniqueness in (iii) of **1.1**)

9. The maps $\begin{array}{rcl}\beta T & \to & \beta T \\ r & \to & (pq)r\end{array}$ and $\begin{array}{rcl}\beta T & \to & \beta T \\ r & \to & p(qr)\end{array}$ are both continuous extensions of the map

$$\begin{array}{l} T \to \beta T \\ t \to (pq)t = p(qt). \end{array} \qquad \text{(by 3, 8)}$$

10. $p(qr) = (pq)r$ for all $p, q, r \in \beta T$. (by 9 and uniqueness in (iii) of **1.1**)

The space βT can be provided with a (different) semigroup structure in which left multiplication is continuous for all $t \in T$, and right multiplication is continuous for all $p \in \beta T$. Merely mimic the proof of **1.2** starting with the map $\begin{array}{rcl} m_t : T & \to & T \\ s & \to & ts \end{array}$. We will most often be interested in **right** actions of a **group** T.

Henceforth we will always assume unless explicitly indicated otherwise that T is a group, and that βT is provided with the semigroup structure of **1.2**. In the upcoming sections we will make extensive use of this semigroup structure and in particular the fact that it makes $(\beta T, T)$ into a flow. It is important to note that the assumption that T is a group, so that every element of T has an inverse does **not** guarantee that the elements of βT have inverses. In fact βT is a group only if T is finite and $\beta T = T$. In general, the only elements of βT which have inverses are the elements of T. This follows immediately from the fact that $p, q \in \beta T$ with $pq \in T$ implies that $p, q \in T$. Indeed if $pq = t \in T$, then $t \in L_p(\beta T) = L_p(\overline{T}) = \overline{L_p(T)}$ since L_p is continuous. On the other hand $T \subset \beta T$ has the discrete topology so $\{t\}$ is an open subset of βT. It follows that $t \in L_p(T)$ and there exists $s \in T$ with $ps = t$. But this implies that $p = ts^{-1} \in T$ and $q = s \in T$.

We end this section with an elementary proposition which speaks to the naturality of the construction of βT.

Proposition 1.4 Let:

(i) T be a semigroup,
(ii) $\emptyset \neq H \subset T$, and
(iii) $j : \beta H \to \beta T$ be the continuous extension to βH of the inclusion $H \to \beta T$.

Then:

(a) j is injective,
(b) im $j = \overline{H}$, and

(c) if H is a subsemigroup of T, then $j(pq) = j(p)j(q)$ for all $p, q \in \beta H$. (Thus we will identify βH with $\overline{H} \subset \beta T$.)

PROOF: (a) 1. Let $h_0 \in H$.

2. Let $\varphi\colon T \to \beta H$ be defined by $\varphi(t) = \begin{cases} t & \text{if } t \in H \\ h_0 & \text{if } t \notin H \end{cases}$.

3. Let $\hat{\varphi}\colon \beta T \to \beta H$ be the continuous extension of φ to βT.

4. Let $\psi = \hat{\varphi} \circ j\colon \beta H \to \beta H$.

5. $\psi(h) = h$ for all $h \in H$. (by 2, 3, 4)

6. $\psi(p) = p$ for all $p \in \beta H$. (by 3, (iii), **1.1**(iii))

7. j is injective. (by 4, 6)

(b) and (c) We leave these to the reader.

APPENDIX TO SECTION 1: ULTRAFILTERS AND THE CONSTRUCTION OF βT

Our goal here is the construction of the compact Hausdorff space βT, which is characterized up to homeomorphism by **1.1**. Those readers already familiar with ultrafilters will recall that a topological space X is compact if and only if every ultrafilter on X converges to a point in X (see **1.A.16** and **Ex. 1.5**). This motivates the approach we will take; in analogy with the construction of the real numbers as Cauchy sequences of rational numbers, βT will be identified as the collection of ultrafilters on T. We have attempted to make this presentation self-contained, so that filters and ultrafilters are defined, and the elementary properties necessary for the construction explicitly introduced. We make use of one of these properties, namely **1.A.8**, in the appendix to section 5. All the other sections of the book, while occasionally using the terminology of this appendix, rely only on the results of section 1 itself. In the interest of brevity, proofs of some of the results in this appendix are left as exercises for the reader. We begin with some background material on filters and ultrafilters.

Definition 1.A.1 Let T be a nonempty set and $\mathcal{F}$ a collection of nonempty subsets of T. We make the following definitions:

(a) $\mathcal{F}$ is a *filter base on* T if

$$F_1, \ldots, F_n \in \mathcal{F} \implies \text{there exists } F \in \mathcal{F} \text{ with } F \subset F_1 \cap \cdots \cap F_n.$$

(b) $\mathcal{F}^c = \{A \mid A \subset T \text{ and there exists } F \in \mathcal{F} \text{ with } F \subset A\}$.

(c) $\mathcal{F}$ is a *filter on* T if $\mathcal{F}$ is a filter base on T and $\mathcal{F}^c = \mathcal{F}$. Thus if $\mathcal{F}$ is a filter, then it has the *finite intersection property* (F.I.P.), meaning

$$F_1, \ldots, F_n \in \mathcal{F} \implies F_1 \cap \cdots \cap F_n \in \mathcal{F}.$$

(d) $\mathcal{U}$ is an *ultrafilter on* T if $\mathcal{U}$ is a filter on T such that

$$\mathcal{F} \text{ a filter on } T \text{ with } \mathcal{U} \subset \mathcal{F} \Longrightarrow \mathcal{U} = \mathcal{F}$$

(so that $\mathcal{U}$ is a maximal filter on T).

The neighborhoods of a point x in a topological space provide an important motivating example; we leave it as an exercise for the reader to verify this.

Example 1.A.2 Let X be a topological space and $x \in X$. Then the collection

$$\mathcal{N}_x = \{A \mid \text{there exists } U \text{ open in } X \text{ with } x \in U \subset A\}$$

is a filter on X. We refer to $\mathcal{N}_x$ as the *neighborhood filter at* x.

Another elementary example which plays a fundamental role in the construction of βT is the following:

Example 1.A.3 Let $t \in T$. Then the collection

$$h(t) = \{A \mid t \in A \subset T\}$$

is an ultrafilter on T. Moreover $h(t)$ is the only ultrafilter on T which contains the singleton set $\{t\}$. We refer to $h(t)$ as the *principal ultrafilter generated by* t.

PROOF: We leave the proof as an exercise for the reader.

According to **1.A.3**, every $t \in T$ generates an ultrafilter on T; we now observe that any filter is contained in some ultrafilter. Suppose that $\{\mathcal{F}_i \mid i \in I\}$ is a collection of filters on T, where I is a totally ordered set. Assume further that if $i < j \in I$, then $\mathcal{F}_i \subset \mathcal{F}_j$. (These assumptions amount to saying that this collection is an increasing chain of filters on T.) Then it is straightforward to check that the union $\bigcup_{i \in I} \mathcal{F}_i$ is a filter on T. This shows that every increasing chain of filters has a maximal element; hence as an immediate consequence of Zorn's lemma (see also **3.3** for a statement) every filter is contained in some maximal filter (i.e an ultrafilter). We state this result as a lemma for future reference:

Lemma 1.A.4 Let $\mathcal{F}$ be a filter (or filter base) on T. Then there exists an ultrafilter $\mathcal{U}$ on T such that $\mathcal{F} \subset \mathcal{U}$.

The next few results examine the structure of ultrafilters on T. In particular they allow us to characterize those filters which are ultrafilters. In fact a filter $\mathcal{F}$ is an ultrafilter if and only if for every $\emptyset \neq A \subset T$, either A or its complement lie in $\mathcal{F}$. (This is the content of **1.A.6** and **1.A.7**.)

Proposition 1.A.5 Let:

(i) $\mathcal{U}$ be an ultrafilter on T, and
(ii) $A \subset T$.

Then $A \in \mathcal{U}$ if and only if $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$.

PROOF: 1. Since $\mathcal{U}$ is a filter it is clear that $A \in \mathcal{U} \Rightarrow A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$.
2. Assume that $A \cap U \neq \emptyset$ for all $U \in \mathcal{U}$.
3. Let $\mathcal{G} = \{G \subset T \mid A \cap U \subset G$ for some $U \in \mathcal{U}\}$.
4. Let $G_1, \ldots, G_n \in \mathcal{G}$.
5. There exist $U_i \in \mathcal{U}$ such that $A \cap U_i \subset G_i$ for $1 \leq i \leq n$. (by 3, 4)
6. $U = U_1 \cap \cdots \cap U_n \in \mathcal{U}$. (by 5, (i))
7. $A \cap U \subset G_1 \cap \cdots \cap G_n$. (by 5, 6)
8. $G_1 \cap \cdots \cap G_n \in \mathcal{G}$. (3, 7)
9. $\mathcal{G}$ is a filter on T. (by 4, 8)
10. $\mathcal{U} \subset \mathcal{G}$. (by 3)
11. $\mathcal{U} = \mathcal{G}$. (by 10, (i))
12. $A \in \mathcal{U}$. (by 3, 11)

Corollary 1.A.6 Let:

(i) $\mathcal{U}$ be an ultrafilter on T, and
(ii) $A \subset T$.

Then either $A \in \mathcal{U}$ or $T \setminus A \in \mathcal{U}$.

PROOF: 1. Assume that $A \notin \mathcal{U}$.
2. There exists $U \in \mathcal{U}$ such that $A \cap U = \emptyset$. (by 1, **1.A.5**)
3. $U \subset T \setminus A$. (by 2)
4. $T \setminus A \in \mathcal{U}$. (by 3, (i))

Proposition 1.A.7 Let:

(i) $\mathcal{F}$ be a filter on T, and
(ii) $A \in \mathcal{F}$ or $T \setminus A \in \mathcal{F}$ for all $A \subset T$.

Then $\mathcal{F}$ is an ultrafiter on T.

PROOF: 1. Let $\mathcal{G}$ be a filter on T with $\mathcal{F} \subset \mathcal{G}$.
2. Let $G \in \mathcal{G}$.
3. $T \setminus G \notin \mathcal{G}$. (by 1, 2)
4. $T \setminus G \notin \mathcal{F}$. (by 1, 3)
5. $G \in \mathcal{F}$. (by 4, (ii))

6. $\mathcal{G} \subset \mathcal{F}$. (by 2, 5)
7. $\mathcal{F}$ is an ultrafilter on T. (by (i), 1, 5)

The following natural generalization of **1.A.6** will be useful here and is used in proposition **5.A.3** of the appendix to section 5.

Corollary 1.A.8 Let:

(i) $\mathcal{U}$ be an ultrafilter on T,
(ii) $A_1, \ldots, A_n$ be subsets of T, and
(iii) $A_1 \cup \cdots \cup A_n \in \mathcal{U}$.

Then there exists j with $A_j \in \mathcal{U}$.

PROOF: 1. Assume that $A_i \notin \mathcal{U}$ for all $i \neq j$.
2. $T \setminus A_i \in \mathcal{U}$ for all $i \neq j$. (by 1, (i), **1.A.6**)
3. $\bigcap_{i \neq j} (T \setminus A_i) \in \mathcal{U}$. (by 2, (i))
4. $A_j \cap \bigcap_{i \neq j} (T \setminus A_i) = (A_1 \cup \cdots \cup A_n) \cap \bigcap_{i \neq j} (T \setminus A_i) \in \mathcal{U}$. (by 3, (i), (iii))
5. $A_j \in \mathcal{U}$. (by 4, (i))
6. There exists j with $A_j \in \mathcal{U}$. (by 1, 5)

Having discussed a few of the elementary properties of ultrafilters, we are ready to define the Stone-Cech compactification βT of T. As a set βT simply consists of all the ultrafilters on T; the next step is to define a topology on βT. Describing this topology requires some notation.

Definition 1.A.9 Let T be a nonempty set. We define βT by

$$\beta T = \{\mathcal{U} \mid \mathcal{U} \text{ is an ultrafilter on } \mathcal{T}\}.$$

Definition and Notation 1.A.10 Let $\emptyset \neq A \subset T$. We define the *hull of* A by $h(A) = \{u \in \beta T \mid A \in u\}$.

Note that for $t \in T$ we have used the notation $h(t)$ for the **single element**

$$h(t) = \{A \mid t \in A \subset T\} \in \beta T,$$

whereas the hull $h(\{t\})$ as defined above is a **subset** of βT. This notation is justified by the fact that $h(\{t\}) = \{h(t)\}$ since $h(t)$ is the only ultrafilter which contains $\{t\}$. We will identify T with the subset $\{h(t) \mid t \in T\} \subset \beta T$ and thus write t for the element $h(t) \in \beta T$.

Note that if an ultrafilter $u \in h(A) \cap h(B)$, then $A \in u$ and $B \in u$. This implies that $A \cap B \in u$ and hence $u \in h(A \cap B)$. It follows that the collection

$$\{h(A) \mid A \subset T\}$$

is a base for a topology on βT. This gives us the following proposition.