Chapter 1

INTRODUCTION

It is convenient to distinguish two classes of equipment for carrying out numerical calculations by mechanical or electrical means. One class consists of those devices which operate by translating numbers into physical quantities of which the numbers are the *measures*, on specified scales, operating with these quantities, and finally *measuring* some physical quantity to get the required result. For example, a product xy may be evaluated by adjusting a variable resistance to have the value x ohms, then adjusting the current through it to have the value y milliamps., and measuring the potential difference across the resistance in millivolts (fig. 1). Other examples of devices of this class are the slide-rule, various

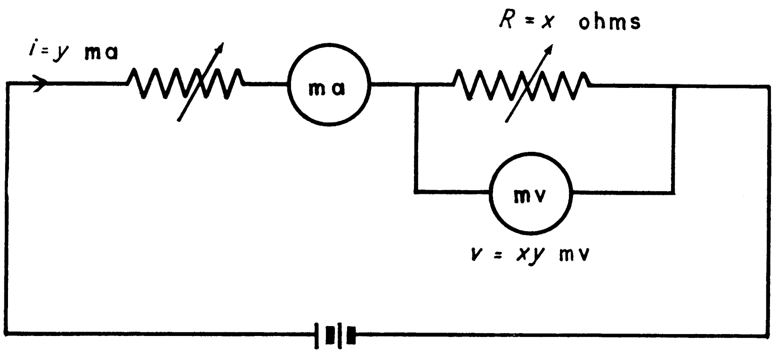


FIG. 1

forms of planimeter, integraph and harmonic analyser, the isograph, differential analyser, and cinema integraph.

The other class consists of those devices which take and operate with numbers directly in their digital form, usually, but not necessarily, by counting discrete objects such as the teeth of a gear wheel, or discrete events such as electrical pulses. Examples are the Brunsviga, Marchant, and other desk calculating machines, and standard I.B.M. equipment as used for computing purposes.

I have found it convenient to distinguish the two classes by the terms "instruments" and "machines" respectively; a corresponding distinction is made in the Encyclopaedia Britannica (14th Edition), in which the two classes of equipment are considered in different articles entitled "Mathematical Instruments" and "Calculating Machines" respectively. In America a similar distinction is made between "analogue machines" and "digital machines," but the former term is, I think, usually restricted to the larger and more elaborate instruments; I have never heard a slide-rule referred to as an "analogue machine," though in my classification it is certainly an "instrument."

Of course, a single large piece of equipment may contain components of an "instrumental" character and others of a "machine" character, so that the whole may be of a composite character. But at the present stage of the art and science of the design and use of computing equipment, the distinction is a convenient one.

The two kinds of equipment have their characteristic advantages and limitations. Any one "instrument" is restricted to a rather narrow range of kinds of calculations; for example, a slide-rule to multiplication, division, and the evaluation of powers and roots; a differential analyser to the integration of ordinary differential equations. And further the accuracy of an instrument is restricted by the mechanical and electrical accuracy of its components and by the attainable accuracy of physical measurement of the result. On the other hand, it is possible to design instruments to deal with continuous variables, and in particular to carry out integration as a continuous process.

A "machine" can only handle numbers expressed in digital form to a finite number of significant figures; it cannot deal with continuous variables or continuous processes as such, and, in particular, in using a machine integration has to be replaced by summation over a finite number of finite intervals. On the other hand, a machine can be designed to work to any finite degree of accuracy without difficulty; in order to get, on a desk calculating machine such as a Marchant, a result accurate to 10 significant decimal figures, it is not necessary for any component to be constructed, or any measurement made, to an accuracy of 1 part in 10^{10}.

With an instrument, it is only possible to put an approximate question and to get an approximate answer. For example, with a slide-rule it is not possible to multiply exactly 2 by exactly 2.5; all one can do is to multiply a number in the range 2.000 ± 0.002, say, by one in the range 2.500 ± 0.002, and get a result perhaps in the range 5.000 ± 0.005. On the other hand, with a machine it is normally *only* possible to put an exact question and get an exact result (provided, of course, that the machine is operated and operating correctly). For example, we cannot multiply π by $\sqrt{2}$ on a machine; we can multiply 3.14159 by 1.41421, perhaps obtaining this product either exactly to ten decimal places or rounded off to, say, five places as required. The latter result might be regarded as an approximate answer to the question "find the product of 3.14159 by 1.41421," but from the point of view of how it is obtained on a machine, it is better regarded as an exact answer to another exact question, "find the integer part and first five decimals of $(3.14159 \times 1.41421 + 0.000005)$."

There have been considerable developments in both instruments and machines in the last twenty years. In the field of instruments the outstanding development has been the differential analyser, an instrument for evaluating by mechanical means the solution of ordinary differential equations. In the field of machines the main developments have been in the direction of large digital machines designed to carry out, rapidly and automatically, extended sequences of individual arithmetical operations.

The developments in these two fields have been almost independent, and their characters are so different that they can best be considered separately. The differential analyser will be the subject of Chapters 2 and 3, some other instruments will be considered in Chapter 4, and various machines will form the subject of Chapters 5 to 8.

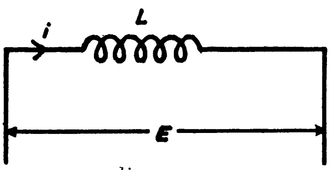
Chapter 2

THE DIFFERENTIAL ANALYSER

2.1. The Nature of the Problem of Instrumental Solution of Differential Equations

Before considering the differential analyser, it will be as well to examine the general nature of the situation with which it deals, and what are the characteristic features of problems giving rise to this situation.

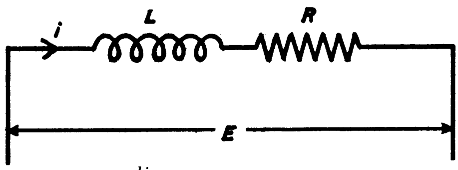
In many applications of mathematics to problems of pure and applied science, there occurs the idea of a *rate of change*, usually with respect to time or to space, of one or more of the relevant quantities, and then the idea of integration is involved in the process of finding the *total* change in a time or space interval from the *rate* of change, which will usually be varying through the interval. In contexts of this nature there are two different kinds of situation which may arise, of which simple examples are provided by the responses of two different circuits to an applied voltage, varying in a known way with time (fig. 2).



$$L\frac{di}{dt} = E(t)$$

$$i = (1/L)\int E(t)dt$$

FIG. 2a



$$L\frac{di}{dt} + Ri = E(t)$$

$$i = (1/L)\int [E(t) - Ri]\,dt$$

FIG. 2b

If the voltage is applied to a circuit with inductance but with negligible resistance (as in fig. 2a), the time rate of change of current at any moment depends only on the value of the voltage at that moment, which is given and is independent of the value the current itself happens to have. Then we are concerned simply with the evaluation of an integral whose integrand is a known function of the independent variable.

But it very often happens that the rate of change of a quantity at any moment (or point) depends on the magnitude of that quantity itself at that moment (or point). For example, if the voltage is applied to a circuit with inductance and resistance (as in fig. 2b), the rate of change of current depends on the instantaneous value of the current itself, as well as on the voltage. The formal expression of such a situation is what is called in mathematics a *differential equation*, and the determination of the current at any time involves the solution of this differential equation. This solution can be regarded as the result of evaluating an integral in which the integrand at any time depends in a definite way on the result of the integration up to that time. For example in the circuit illustrated in fig. 2b, the equation for the current can be written

$$i = (1/L) \int [E(t) - Ri]\, dt,$$

and in the integral here, the current i to be found occurs in one contribution to the integrand.

This aspect of a differential equation is not prominent in the conventional formal treatment of such equations, but it expresses rather closely the way in which it is often best to consider their mechanical solution. Indeed, from the point of view of mechanical integration, it is just this feature which distinguishes the evaluation of a solution of a differential equation from the evaluation of a simple integral of a known function of the independent variable. Thus the essential points in a mechanical instrument for integrating differential equations are an integrating mechanism and means of furnishing to that mechanism as integrand a quantity depending in a definite way on the value of the integral calculated by it.

2.2. Integrating Mechanisms

Any continuously variable gear can be used as an integrating mechanism. For suppose that in fig. 3 the rectangle represents any mechanism giving a continuously variable gear ratio $1:n$ between the rotations of driving and driven shafts. Then for a rotation dx turns of the driving shaft, at gear ratio $1:n$, the rotation of the driven shaft is $n\,dx$ turns. If the gear ratio n is changing as the driving shaft is rotating, the total rotation of the driven shaft is the sum of the elements of rotation $n\,dx$; that is, it is $\int n\, dx$ turns.

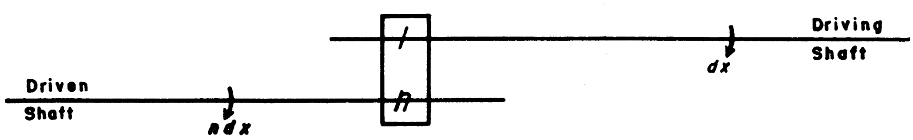


FIG. 3. Continuously variable gear as integrator.

To be suitable for incorporation in a differential analyser, such a mechanism should be able to be set accurately to any gear ratio n in its range and should include both positive and negative values of n, including $n = 0$, in this range. The form used in most differential analysers consists of a vertical wheel driven by a horizontal disc (fig. 4),

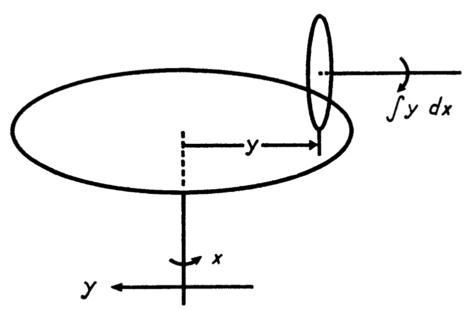
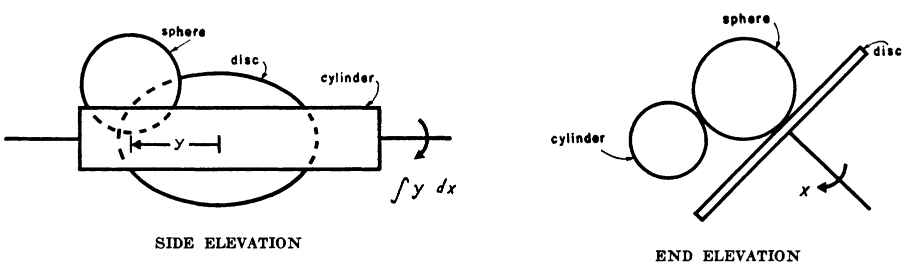


FIG. 4

the wheel and disc being so mounted that the distance between the centre of rotation of the disc and the point of contact of the wheel with it can be varied. The gear ratio between the rotations of disc and wheel is proportional to this distance, which is usually called the "displacement" of the integrator; this must therefore be made proportional to the integrand of the integral to be evaluated. The rotation of the disc, usually called the "rotation" of the integrator, represents the variable of integration and must be made proportional to it. It is interesting that a planimeter based on this integrating mechanism is considerably older than the now more familiar Amsler planimeter (see ref. 31).



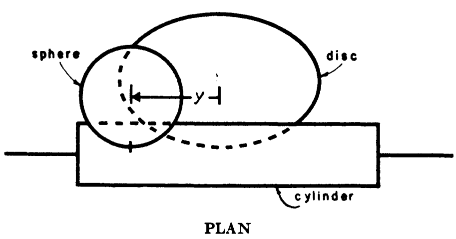


FIG. 5. James Thomson's integrator.

Another form of integrator (fig. 5), devised by James Thomson, brother of Lord Kelvin, consists of an inclined disc, a horizontal cylinder not in contact with it, and a sphere in contact with both (104). The sphere can be moved along the cylinder, and makes contact with the disc along its horizontal diameter. It gives a drive from disc to cylinder at a gear ratio which varies proportionately to the distance from the centre of the disc to the point at which the sphere is in contact with it.

2.3. The General Idea of the Differential Analyser

The differential analyser consists essentially of a number of integrating mechanisms which can be connected together so as to evaluate solutions of ordinary differential equations.

The idea of connecting together integrating mechanisms for this purpose is not by any means new. It was originally formulated, clearly and in considerable detail, by Kelvin in two papers published as long ago as 1876 (106, 107). But the realisation of this idea in the form of a practical working instrument was an achievement of Dr. Bush and an able group working with him at M.I.T. nearly twenty years ago (17) (see also refs.

27, 47, 49, 58); and it should be added that their conception of such a machine was independent of Kelvin's.

It is interesting to follow in Kelvin's papers the development of his thought on the subject. His brother, James Thomson, had recently designed the integrating mechanism shown in fig. 5, and Kelvin had seen how it could be adapted to evaluate continuously the integral of the product of two functions, and had outlined its application to a harmonic analyser (105).

His first paper on its application to the integration of differential equations is concerned with the equation

$$\frac{d}{dx}\left[\frac{1}{F(x)}\frac{dy}{dx}\right] + y = 0,$$

of which he says: "On account of the great importance of this equation in mathematical physics" (of which he gives examples) "I have long endeavoured to obtain a means of facilitating its practical solution. . . ." After some general remarks he continues, "A ready means of obtaining approximate results which shall show the general character of the solutions . . . has always seemed to me a desideratum. Therefore I have made many attempts to plan a mechanical integrator which should give solutions by successive approximations." He then shows how an instrument consisting of two of James Thomson's integrators, connected together so that the result of the integration performed by the first forms the integrand for the second, will provide such a result by an iterative process, as follows. If an integrand y_0 is fed to the first of the two integrators of such an instrument, the output from the second is

$$y_1 = \int F(x)\,[c - \int y_0\,dx]\,dx;$$

if this is recorded, and subsequently fed as integrand to the first integrator, the output from the second is

$$y_2 = \int F(x)\,[c - \int y_1\,dx]\,dx$$

and so on. The process is continued until there is no appreciable difference between input and output, and the common input and output is then the solution of the equation.

Then comes the inspiration, and here again it is worth quoting Kelvin's own words. "So far I had gone and was feeling satisfied, feeling I had done what I wished to do for many years. But then came a pleasing surprise. Compel agreement between the function fed into the double machine and that given out by it." He then shows how, in principle, this can be done by making a second interconnection between the two integrators so that the output of the second is used continuously as the integrand of the first, and that this interconnected system of integrators evaluates the solution of the equation directly. He continues: "Thus I was led to a conclusion which was wholly unexpected; and it seems to me very remarkable that the general differential equation of the second order with variable coefficients may be rigorously, continuously, and in a single process solved by a machine."

It is clear from the context that here he has in mind only linear second-order equations. But in a second paper he is concerned with the extension to linear equations of any order, and in an addendum includes in principle the further extension to non-linear equations, even including the idea of three-dimensional cams for feeding in functions of two independent variables.

Harmonic analysers using James Thomson's integrating mechanism as suggested by Kelvin have been constructed (108). But as far as I have been able to ascertain, no steps were taken at the time for realising his conception of an instrument for solving differential equations, and, as already mentioned, it was left to Dr. Bush and his team at M.I.T. to develop in the differential analyser a machine which could be built and which would be accurate and reliable in operation.

2.4. General Structure of the Earlier Forms of Differential Analyser

A differential analyser consists of a number of units, each of which carries out an operation which can be regarded as a translation into mechanical terms of a process (integration, addition, etc.) which may be required in the mechanical integration of a differential equation, and some means of interconnecting these units. Each unit is driven by the rotation of one or more shafts, and the result of its operation is the rotation of a shaft driven by it. Each shaft represents one of the quantities occurring in the equation to be solved, and the total rotation of each shaft measures the corresponding quantity, on an assigned scale. The interconnections between the units are made in such a way that the relations between the rotations of the various shafts form a translation into mechanical terms of the equation to be solved. Then the rotation of one shaft, representing the independent variable, drives the remainder of the shafts in accordance with the equation represented by these interconnections.

In the original differential analyser, and in others which have been built on the same general plan though with various differences of detailed design, the connections between the units are purely mechanical, by means of shafts and gearing. A later development will be considered in §2.5.

The instrument with which I personally have been most closely associated is that installed at the University of Manchester, England, in 1935 (87); this is of the earlier type with purely mechanical interconnections, for which it will serve as an example. A similar instrument, at the

Mathematical Laboratory of the University of Cambridge, is illustrated in fig. 6.

The main units, the integrators, are arranged in pairs in cases along the near side. Means must be provided for supplying the instrument with information about functional relationships between variables occurring in

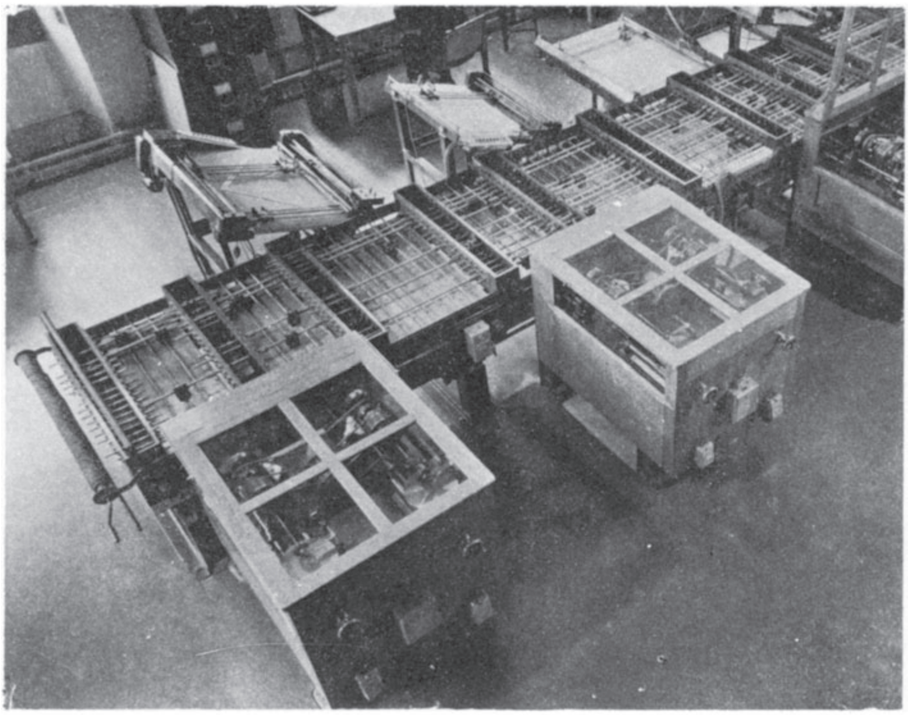


FIG. 6. Differential Analyser at Mathematical Laboratory, University of Cambridge.

the equation, as for example the relation between voltage and current in a non-linear resistive element in a circuit. In some cases this can be done by generating the required relation by the solution of an auxiliary equation. In others it is necessary to supply the information from an "input table"; four such tables are on the far side of the main frame in fig. 6. Means must also be provided for recording the result; this can be done either in graphical form on an "output table" or in numerical form on a set of counters.

The shafts directly driving or driven by the various units lie across the length of the main frame, and are connected through cross-drives and longitudinal shafts. Adding units can be incorporated in the various gear trains in these interconnections.

Some close-up views of individual components are given in figs. 7, 8, 10. Figure 7 shows two integrators with associated equipment. Each integrator is a precision form of continuously variable gear of the disc-and-wheel type already mentioned (fig. 4). In the interconnections between the various units required in the solution of a differential equation, the output from an integrator may have to drive several other units, and this may form much too heavy a load to be driven directly by the friction between the disc and wheel of an integrator. Hence on the output side

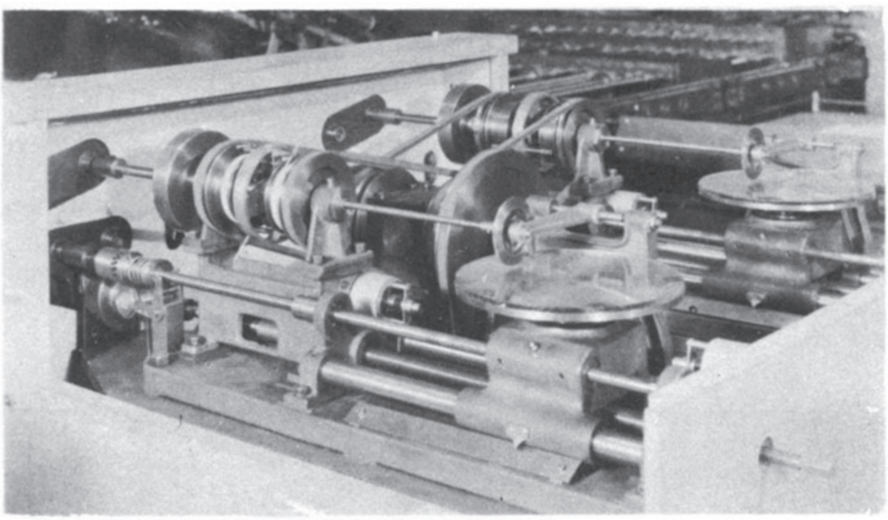


Fig. 7. Integrator and torque amplifier.

there is a torque amplifier, which drives the output shaft at the same speed as the integrating wheel, but with a much greater torque. In the instrument illustrated in fig. 6, as in Bush's original differential analyser, the torque amplifier is a purely mechanical servo operating on the capstan principle. In some other more recent instruments this has been replaced by an electrical servo, for example by an amplidyne system controlled by an optical signal measuring the angular error between the positions of the integrating wheel and of the output shaft.

Figure 8 shows an input table. A bridge spans the table and can be moved perpendicularly to its length by the input shaft to the table. The bridge supports a carriage which can be moved along its length by the rotation of a handle, and which carries an index. A graph, representing the information to be fed to the differential analyser, is placed on the table. As the bridge is traversed across the table by the rotation of the input shaft, an operator turns the handle so as to keep the index on the