CHAPTER XI.

Systems of Equations of the First Order and their Reduced Forms in the Vicinity of Singularities of the Derivatives.

145. In most of the preceding investigations, the discussion has been restricted to a single equation of the first order. It is clear that many of the processes, with appropriate modifications and obvious extensions, can be applied to a system of equations of the first order: and accordingly it is possible, in making such applications, to deal more briefly both with the explanations and the details of the processes.

When the number of dependent variables is two, an appropriate geometrical illustration is provided by the skew curves in ordinary space which satisfy two differential equations of the first order. The full development of this illustration requires the restriction that the variables, dependent and independent, shall have real values. As this restriction prevents the full consideration of the functional relation between the variables, the geometrical mode of regarding the variables will be adopted chiefly for purposes of illustration: largely for the same reason that, in the corresponding problems involving only one dependent variable, the geometrical association of plane curves with the equations was adopted for the subsidiary purpose of illustration.

The general theorem in Chap. II establishes the existence of the integrals of a system of equations of the first order, determined by the condition that the integrals shall assume assigned values when the independent variable acquires its initial value. It effectively gives an expression for the integral, in the form of power-series valid over a finite domain: there being a precedent hypothesis that the aggregate of values, assigned as initial values to all the variables, constitutes an ordinary combination for the functions, which are the values of the respective derivatives as given by the differential equations. But if the combination of initial values should not have the character of an ordinary combination for the functions in question, then further investigation is required in order to afford knowledge of the integrals for values of the variables in the vicinity of such a combination.

146. Consider, in the first instance, a set of equations involving $n-1$ dependent variables and one independent variable, say $x_1, x_2, \ldots, x_n$ in all; and let them be

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n},$$

where, for the present purpose, each of the functions $X_1, X_2, \ldots, X_n$ vanishes when $x_1 = 0$, $x_2 = 0$, $\ldots$, $x_n = 0$. Let t denote a new parametric variable, such that the common value of the n fractions is

$$= \frac{dt}{t}.$$

Taking only the simplest case, so that in each of the functions X the terms of lowest dimensions which occur, when the functions are expanded in power-series, are of the first order, let

$$X_s = \sum_{i=1}^{n} a_{si} x_i + \text{terms of higher orders.}$$

Change the variables from x to y by means of the homogeneous linear substitution

$$y_r = \sum_{j=1}^{n} c_{rj} x_j, \qquad (r = 1, 2, \ldots, n),$$

where the n^2 coefficients c are disposable constants. Thus

$$t\frac{dy_r}{dt} = \sum_{j=1}^{n} c_{rj} X_j$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} c_{rj} (a_{ji} x_i + \text{terms of higher order}),$$

for all values of r. If possible, let a set of constants c be chosen so that

$$\sum_{j=1}^{n} \sum_{i=1}^{n} c_{rj} a_{ji} x_i = \lambda y_r$$

$$= \sum_{j=1}^{n} \lambda c_{rj} x_j;$$

this choice can be made if the constants c satisfy the equations

$$\sum_{j=1}^{n} c_{rj} a_{js} = \lambda c_{rs},$$

for $s = 1, 2, \ldots, n$. These n equations are linear and homogeneous in the n constants $c_{r1}, c_{r2}, \ldots, c_{rn}$: in order that they may be consistent, we must have

$$A = \begin{vmatrix} a_{11} - \lambda, & a_{21}\ , & a_{31}\ , & \ldots \\ a_{12}\ , & a_{22} - \lambda, & a_{32}\ , & \ldots \\ a_{13}\ , & a_{23}\ , & a_{33} - \lambda, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0,$$

an equation in λ of degree n.

The importance of this equation does not lie mainly in the comparative simplicity of the forms of differential equations which are suggested as possible equivalents of the original system, but is rather constituted by an invariantive property which it possesses, viz. whatever be the intermediate linear transformations effected upon the variables x, the roots λ of the determinantal equation are entirely independent of those intermediate transformations. The formal proof of this property is practically identical with the formal proof of the invariantive character of the fundamental equation appertaining to a singularity of an ordinary linear equation of order n; it is as follows*.

Let a new set of variables be given by the relations

$$z_m = \sum_{s=1}^{n} g_{ms} x_s,$$

where the determinant

$$\Gamma = \begin{vmatrix} g_{11}, & g_{12}, & \ldots \\ g_{21}, & g_{22}, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix}$$

is different from zero; and let the new equations be

$$\frac{dz_1}{Z_1} = \frac{dz_2}{Z_2} = \ldots = \frac{dz_n}{Z_n} = \frac{dt}{t},$$

where

$$Z_s = \sum_{i=1}^{n} b_{si} z_i + \text{terms of higher orders.}$$

* This is similar to the proof given by Hamburger, *Crelle*, t. LXXVI (1873), p. 115, for the corresponding result connected with linear equations.

The corresponding equation, for the reduction to the simpler form with a multiplier λ, is

$$B = \begin{vmatrix} b_{11}-\lambda, & b_{21}, & b_{31}, & \dots \\ b_{12}, & b_{22}-\lambda, & b_{32}, & \dots \\ b_{13}, & b_{23}, & b_{33}-\lambda, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = 0.$$

It is clear that the highest term in powers of λ, viz. $(-\lambda)^n$, is the same in A as in B.

We have

$$\begin{aligned} \sum_{p=1}^{n} g_{mp} X_p &= t \sum_{p=1}^{n} g_{mp} \frac{dx_p}{dt} \\ &= t\frac{dz_m}{dt} \\ &= Z_m \\ &= \sum_{i=1}^{n} b_{mi} z_i + \text{terms of higher orders} \\ &= \sum_{i=1}^{n} b_{mi} \sum_{s=1}^{n} g_{is} x_s + \text{terms of higher orders}; \end{aligned}$$

whence, on equating to one another the coefficients of the first powers of the variables x on the two sides of the equation thus obtained, we find

$$\sum_{p=1}^{n} g_{mp} a_{pt} = \sum_{p=1}^{n} b_{mp} g_{pt} = f_{mt},$$

say, for all values of m and t. Now

$$\Gamma A = \begin{vmatrix} g_{11}, & g_{12}, & g_{13}, & \dots \\ g_{21}, & g_{22}, & g_{23}, & \dots \\ g_{31}, & g_{32}, & g_{33}, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \begin{vmatrix} a_{11}-\lambda, & a_{21}, & a_{31}, & \dots \\ a_{12}, & a_{22}-\lambda, & a_{32}, & \dots \\ a_{13}, & a_{23}, & a_{33}-\lambda, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

$$= \begin{vmatrix} f_{11}-g_{11}\lambda, & f_{12}-g_{12}\lambda, & f_{13}-g_{13}\lambda, & \dots \\ f_{21}-g_{21}\lambda, & f_{22}-g_{22}\lambda, & f_{23}-g_{23}\lambda, & \dots \\ f_{31}-g_{31}\lambda, & f_{32}-g_{32}\lambda, & f_{33}-g_{33}\lambda, & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix};$$

and

$$B\Gamma = \begin{vmatrix} b_{11}-\lambda, & b_{12}\ , & b_{13}\ , & \dots \\ b_{21}\ , & b_{22}-\lambda, & b_{23}\ , & \dots \\ b_{31}\ , & b_{32}\ , & b_{33}-\lambda, & \dots \\ \dots & \dots & \dots & \end{vmatrix} \begin{vmatrix} g_{11}, & g_{21}, & g_{31}, & \dots \\ g_{12}, & g_{22}, & g_{32}, & \dots \\ g_{13}, & g_{23}, & g_{33}, & \dots \\ \dots & \dots & \dots & \end{vmatrix}$$

$$= \begin{vmatrix} f_{11}-g_{11}\lambda, & f_{12}-g_{12}\lambda, & f_{13}-g_{13}\lambda, & \dots \\ f_{21}-g_{21}\lambda, & f_{22}-g_{22}\lambda, & f_{23}-g_{23}\lambda, & \dots \\ f_{31}-g_{31}\lambda, & f_{32}-g_{32}\lambda, & f_{33}-g_{33}\lambda, & \dots \\ \dots & \dots & \dots & \end{vmatrix} = \Gamma A,$$

so that $B=A$. The equation $A=0$ is therefore absolutely invariantive for all intermediate linear transformations of the variables.

The invariance of this fundamental equation makes it possible to construct a canonical equivalent of the original system of equations.

147. If ω be a simple root of this equation, some (or it may be any) $n-1$ of the preceding n homogeneous linear equations determine the ratios of the n coefficients $c_{r1}, \dots, c_{rn}$; so that, taking one of the coefficients arbitrarily, say c_{rr}, the others are known, and the consequent value of y is obtained.

First, suppose that the roots of the determinantal equation of degree n are distinct from one another; denote them by $\lambda_1, \lambda_2, \dots, \lambda_n$, so that each of them is simple. After the above explanation, each root determines a substitution (c_{rj}) and a consequent new variable y; let y_r be the variable determined by λ_r, for $r=1, \dots, n$. Conversely, the variables $x_1, \dots, x_n$ are linearly expressible in terms of the variables $y_1, \dots, y_n$; so that an aggregate of terms of any order in x becomes, after substitution, an aggregate of terms of the same order in y. Moreover, the expression for $t\dfrac{dy_r}{dt}$ is

$$\frac{dy_r}{dt} = \lambda_r y_r + \text{terms of order higher than the first in } x;$$

hence after substitution, the system of differential equations is

$$\frac{dy_1}{Y_1} = \frac{dy_2}{Y_2} = \dots = \frac{dy_n}{Y_n} = \frac{dt}{t},$$

where
$$Y_r = \lambda_r y_r + \eta_r,$$

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and η_r is an aggregate of terms of order higher than the first in the variables $y_1, \ldots, y_n$.

This is the canonical form when the n roots of the determinantal equation are distinct.

Next, suppose that a root of the determinantal equation is repeated, say $\lambda = \omega$, and that its multiplicity is κ. Then the equations

$$\sum_{j=1}^{n} c_{rj} a_{js} = \omega c_{rs}$$

are consistent with one another, and they provide a set of coefficients c, though the set may not be uniquely determined. Taking them in some definite form, let them be associated with a variable y_1; and transform the variables from $x_1, x_2, \ldots, x_n$ to $y_1, x_2, \ldots, x_n$. The differential equations then take the form

$$t\frac{dy_1}{dt} = \omega y_1 + \eta_1',$$

$$t\frac{dx_s}{dt} = \beta_s y_1 + \sum_{j=2}^{n} a_{sj}' x_j + \eta_s', \qquad (s = 2, \ldots, n),$$

where $\eta_1', \eta_2', \ldots, \eta_n'$ are the aggregates of terms of order higher than the first on the respective sides. The determinantal equation now is

$$\begin{vmatrix} \omega - \lambda, & 0 \;, & 0 \;, & \ldots \\ \beta_2 \;, & a_{22}' - \lambda, & a_{32}' \;, & \ldots \\ \beta_3 \;, & a_{23}' \;, & a_{33}' - \lambda, & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{vmatrix} = 0;$$

as the equation is invariantive, the root ω is of multiplicity κ, and therefore the equation

$$\begin{vmatrix} a_{22}' - \lambda, & a_{32}' \;, & \ldots \\ a_{23}' \;, & a_{33}' - \lambda, & \ldots \\ \ldots & \ldots & \ldots \end{vmatrix} = 0$$

has the root ω of multiplicity $\kappa - 1$. Hence equations

$$\sum_{j=2}^{n} c_{rj}' a_{js}' = \omega c_{rs}' \qquad \begin{matrix} (s = 2, \ldots, n), \\ (r = 2, \ldots, n), \end{matrix}$$

are consistent with one another; and they provide a set of coefficients c', though the set may not be uniquely determined. Taking them in some definite form, let them be associated with a variable y_2; and transform the variables from $x_2, x_3, \ldots, x_n$ to $y_2, x_3, \ldots, x_n$. The differential equations then take the form

$$t\frac{dy_2}{dt} = \beta_2 y_1 + \omega y_2 + \eta_2'',$$

$$t\frac{dx_s}{dt} = \beta_s y_1 + \gamma_s y_2 + \sum_{j=3}^{n} a_{sj}'' x_j + \eta_s'', \qquad (s = 3, \ldots, n),$$

where $\eta_2'', \eta_3'', \ldots, \eta_n''$ are the aggregates of terms of order higher than the first on the respective sides.

Proceeding in this way, we see that κ variables $y_1, \ldots, y_\kappa$ can be associated with a root $\lambda = \omega$, of multiplicity κ in the determinantal equation, in such a way as to replace κ variables $x_1, \ldots, x_\kappa$, and to transform the original system of differential equations to the form

$$t\frac{dy_1}{dt} = \omega y_1 + \eta_1,$$

$$t\frac{dy_2}{dt} = \alpha_{21} y_1 + \omega y_2 + \eta_2,$$

$$t\frac{dy_3}{dt} = \alpha_{31} y_1 + \alpha_{32} y_2 + \omega y_3 + \eta_3,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$t\frac{dy_\kappa}{dt} = \alpha_{\kappa 1} y_1 + \alpha_{\kappa 2} y_2 + \ldots + \alpha_{\kappa, \kappa-1} y_{\kappa-1} + \omega y_\kappa + \eta_\kappa,$$

so far as concerns the derivatives of the κ variables indicated.

Taking the simple roots in turn, the transformation is effected for a variable associated with each of them; and the expression for the corresponding derivative is monomial in terms of the first order on the right-hand side. Taking the multiple roots of the determinantal equation in turn, with each of them is associated a group of variables in number equal to the multiplicity of the root; when the transformation is effected, the expressions for the derivatives of the corresponding variables are of the above form.

148. The reduction indicated is effective if, in the expressions for the regular functions $X_1, \ldots, X_n$, which are valid in the vicinity of $0, 0, \ldots, 0$, a common zero of all of them, the terms of the first order occur in quite general form. We shall not discuss the case in which the terms of the first order occur only in specialised forms: nor shall we discuss, for an unrestricted number of variables, the case in which the terms of lowest order that occur

are of dimensions higher than the first. For this investigation, it will be sufficient to refer to Königsberger's treatise *.

The simplest instance of the preceding class of equations is furnished by a system of n equations in n dependent variables, which occur only linearly and homogeneously in the equations.

When the system is of the form

$$\frac{dy_{n-1}}{dx} = \sum_{r=0}^{n-1} P_{n-r} y_r,$$

$$\frac{dy_r}{dx} = y_{r+1}, \qquad (r = 0, 1, \ldots, n-2),$$

it can immediately be replaced by a single linear equation of order n. A complete discussion of such an equation and of the properties of its integrals is reserved for a separate section.

The next most important instance is that in which the system has the form

$$\frac{dy_r}{dx} = \sum_{i=1}^{n} A_{ir} y_i,$$

where the coefficients A_{ij} are functions of x alone. An ample discussion of the properties and characteristics of the integrals of such a system, leading to results associated with the ordinary linear equation of order n to which reference has just been made, will be found in various papers by Sauvage; a general exposition of the results is given by him in a separately published volume†. Among these, the most important are:—

(i) The establishment of the existence of integrals of the equation in the vicinity of an ordinary point of the coefficients A;

* *Lehrbuch der Theorie der Differentialgleichungen mit einer unabhängigen Variabeln*, (Leipzig, Teubner, 1889); see, in particular, Chapter V.

Upon this subject, reference may also be made to the following authorities:

Picard, *Cours d'Analyse*, t. III, chap. I.

Horn, *Crelle*, t. CXVI (1896), pp. 265—306, *ib.* t. CXVII (1897), pp. 104—128, 254—266.

Poincaré, *Inaugural Dissertation*, (1879).

Bendixson, *Stockh. Öfv.*, t. LI (1894), pp. 141—151.

Further references will be found in the works quoted.

† *Théorie générale des systèmes d'équations différentielles linéaires et homogènes*, (Paris, Gauthier-Villars, 1895).

A memoir by Grünfeld, *Wiener Akad. Denkschr.* t. LIV, ii Abth., (1888), pp. 93—104, in which he follows the earlier work of Sauvage, dispensing with the use of

(ii) A proof that equations, which have regular* integrals in the vicinity of $x=0$, are of the form

$$x\frac{dy_r}{dx}=\sum_{j=1}^{n} a_{ri}y_i, \qquad (r=1, \ldots, n),$$

where a_{ri} is regular in the vicinity of the point $x=0$. This is the canonical form for such equations in the vicinity of the point $x=0$; and it is the form which must be possessed by the equations in the vicinity of every non-regular point of the coefficients A;

(iii) The construction of the solution when the tests for regularity are satisfied;

(iv) A generalisation, to the class of equations considered, of the main properties established by Fuchs and others for the ordinary linear differential equation of the nth order.

The investigations will not be repeated here, as they follow somewhat closely the development of the theory of ordinary linear equations, which the present writer hopes to discuss in detail in another volume.

Form of Equations in the Vicinity of Non-ordinary Combinations of Values for the Derivatives.

149. In order to give some indication of the mode of obtaining integrals of simultaneous equations, involving more than one dependent variable, it is desirable to shew how the processes, applied in preceding chapters to the discussion of the integral of a single equation, can be extended so as to be effective for such a purpose. Accordingly, we shall discuss in some detail, though without attempting the same completeness as in the case of a single equation, the case when the system involves one independent variable z and two dependent variables u and v, and when it

Weierstrass's theory of bilinear forms, and also a paper by the same author, Schlömilch's *Zeitschrift*, t. XXXVI (1891), pp. 21—33, may be consulted.

The form of the equations which have "regular" integrals, and the characteristics of the regular integrals of such equations, are discussed by Königsberger in his treatise (quoted p. 8, note) pp. 441—469: also in two memoirs by Horn, *Math. Annalen*, t. XXXIX (1891), pp. 391—408, *ib.* t. XL (1892), pp. 527—550.

Reference may also be made to Picard, *Cours d'Analyse*, t. III, ch. I, II; and to Jordan, *Cours d'Analyse*, t. III, ch. II, Section 1.

* In the same sense as in § 30.

therefore contains two independent equations. The general discussion of Chapter I shews that, when the equations are given in the form

$$\left.\begin{aligned} F\left(\frac{du}{dz}, \frac{dv}{dz}, u, v, z\right) = 0 \\ G\left(\frac{du}{dz}, \frac{dv}{dz}, u, v, z\right) = 0 \end{aligned}\right\},$$

then, except in the vicinity of what may be called branch-values for $\frac{du}{dz}$ and $\frac{dv}{dz}$ given by these two algebraical equations, we may take

$$\frac{du}{dz} = f(u, v, z), \qquad \frac{dv}{dz} = g(u, v, z),$$

as representing the equations. The nature of the integrals in the vicinity of any assigned values depends upon the character of the functions f and g in such a vicinity.

Cauchy's existence-theorem gives the character of the integrals of the system

$$\frac{du}{dz} = f(u, v, z), \qquad \frac{dv}{dz} = g(u, v, z),$$

which are regular functions of $z - c$ in the vicinity of c, and acquire values α and β respectively when $z = c$: provided $u = \alpha$, $v = \beta$, $z = c$ are an ordinary combination of values for the functions f and g. It therefore is necessary to discuss the character of the integrals in the vicinity of values, which constitute a non-ordinary combination of those functions. For this purpose, we take $u - \alpha$, $v - \beta$, $z - c$ as new variables and, assuming the corresponding transformations effected, we can consider 0, 0 as initial values assigned to the dependent variables, when the independent variable vanishes.

Now $u = 0$, $v = 0$, $z = 0$, when not an ordinary combination of values for both f and g, can give rise to several alternatives. Denoting

(i) an ordinary combination for one function by O:

(ii) an accidental singularity of the first kind for one function by A_1:

(iii) an accidental singularity of the second kind for one function by A_2:

(iv) an essential singularity by E: