MATRICES

AND

DETERMINOIDS

University of Calcutta
Readership Lectures

MATRICES AND DETERMINOIDS

BY

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PREFACE

THE author's chief aim in writing this book was to give a systematic account of certain applications of matrices, particularly of rectangular matrices as distinguished from square matrices, and thereby to illustrate the very great advantages gained by using them in almost all branches of Mathematics. It originated in a habit of using matrices freely in the solution of problems in Algebra, Geometry and Applied Mathematics, and is based on the very extensive manuscript acquired in doing so. To give a satisfactory answer to the frequently propounded question 'What is a matrix ?', it seemed advisable to commence with some account of the theory. Accordingly the course of Readership Lectures in which this work was first made public was divided into two halves, the first half dealing with the theory, and the second half with the applications. The theoretical portion has been constantly increased, in the first place by abstractions from the applications, and in the second place by incorporating the work of other writers. As a consequence the applications have been driven further back, though they still remain the ultimate object of the book.

The first volume contained the foundations of a Calculus of Matrices in which the operations are addition, subtraction and multiplication, and the result of performing any number of these operations with any rectangular matrices whatever is always a completely determinate matrix. It also contained:

an account of the properties of the determinoid of a matrix, which becomes the determinant of the matrix in the particular case when the matrix is square;

an account of the solution of matrix equations of the first degree, including as a special case the solution of systems of linear algebraic equations;

a precise statement of the Law of Cancellation of matrix factors in a matrix equation.

If we have an equation $ab=0$ in which a and b are scalar numbers, then, by the Law of Cancellation in Algebra, if either one of the two factors a and

b is not 0, it can be cancelled. The formulation of the corresponding Law of Cancellation when a and b are matrices is very necessary for the applications, and was essential to any advance in the general theory of rectangular matrices. This Law of Cancellation is included as a particular case in the important generalisation of Chapter XV regarding the possible ranks of a matrix product.

The continuation of the work has been greatly hindered by untoward circumstances, above all by the difficulty of obtaining sufficient leisure for the final preparation of the manuscript for the press. On this account, and because of the growth of the manuscript, it has been decided with reluctance to publish as a second volume the first half of that portion of the complete work which deals in greater detail with the Theory of Matrices. Accordingly this second volume contains those parts of the theory which naturally precede any investigation of the special properties of functional matrices, i.e. matrices whose elements are rational integral functions of a finite number of variables. It deals almost exclusively with matrices whose elements are constants, which may be arbitrary parameters, and with those transformations of such matrices which are classed as equigradent. It does not however contain all the properties of such matrices. There remain many properties which it will be more convenient to consider after a preliminary study of functional matrices.

The language used in this volume is frequently geometrical, especially in the later chapters. Any set of matrices which are vertically equivalent to one another are regarded as defining a 'spacelet' which is completely represented by any one of them, usually by one which is undegenerate, the spacelet being a 'point' when the common rank of the matrices is 1. Any property of a matrix which remains unaltered when the matrix is replaced by any matrix vertically equivalent to it, is a property of a spacelet, and conversely all properties of spacelets are properties of matrices. Although these definitions are not geometrical, their geometrical interpretations, which will be fully discussed in a later volume, are quite evident. We therefore speak of 'spacelets' and 'points' from the outset as if they were geometrical concepts, and the chapters dealing with them will serve to lighten subsequent chapters on the geometrical applications of matrices. In the chapter on equigradent transformations it is shown that every such transformation of a matrix whose elements are constants corresponds to a linear transformation of the variables in a bilinear or quadratic algebraic form; and therefore everything in that chapter has an immediate algebraic application. Thus although the present volume is avowedly restricted to the *theory* of matrices, it actually contains a large number of geometrical applications, and it also implicitly contains a large number of algebraic applications to which attention will subsequently be directed.

The geometrical applications which occur can be divided into two classes. The first class contains properties which depend only on the notion of connection, and are invariant in every equigradent (or projective) transformation of the points of space. These include the *rank* of a spacelet, the *paratomy* of two spacelets, and the relations between spacelets represented by the terms *intersection* and *join* and by the terms *incident* and *connected*. The second class contains properties which depend on the notion of orthogonality, and are invariant in every semi-unit transformation of the points of space, i.e. in every equigradent transformation which leaves the absolute quadric unaltered. These include the *extravagance* of a spacelet, the *orthotomy* and *cross rank* of two spacelets, and the relations between spacelets represented by the terms *core* and *plenum* and by the terms *orthogonal* and *normal*.

It is often desirable to know what a general theorem concerning matrices becomes in the special case in which the matrices are real. When the theorem is one involving only rational operations on the elements of the matrices, all reference to the special case is rendered unnecessary by enunciating the general theorem for matrices whose elements lie in any domain of rationality Ω; for the special case is then obtained by simply taking Ω to be the domain of all real numbers. Since however the real domain has special properties not possessed by other domains, there are special properties of real matrices which cannot be obtained in this way.

In the figures which are given in some of the articles, spacelets and their intersections are represented by areas. Such representations, though necessarily very imperfect, can be used in simple cases as aids to the imagination. It should particularly be noted that a shaded area always represents a completely extravagant spacelet, i.e. one which, being orthogonal with itself, is a generating spacelet of the absolute quadric.

The references to Vol. II contained in Vol. I have been vitiated by the alterations and re-arrangements of the text which have been made since Vol. I was published; but a use of the Index will probably remove any inconvenience caused by this.

As this work has been built up on an independent plan and based chiefly on applications, it is for the most part not easy to ascribe definite sources to the various articles or the suggestions for them. The following is a list of the books which have had most influence on the work as a whole:

Bôcher's *Introduction to Higher Algebra*,

Heffter and Koehler's *Lehrbuch der Analytischen Geometrie*,

Muth's *Elementarteiler*,

Netto's *Vorlesungen über Algebra*,

Veronese's *Fondamenti di geometria a piu dimensioni*,

Whitehead's *Universal Algebra*.

My indebtedness to these and other writers will be more easily recognised in those portions of the work, occurring chiefly in Vol. III, which are interpolations in the original scheme. Amongst the few articles in the present volume which admit of more detailed references may be mentioned § 120 which was written after reading the appendix in Heffter and Koehler's *Analytische Geometrie*, and § 159 which was written after reading a paper by Schläfli in *Crelle's Journal* for 1866 to which a reference is given in Muth's *Elementarteiler*. The addition of Appendix B was suggested by reading a paper by Mr Haripada Datta in Vol. XXXIV of the *Proceedings of the Edinburgh Mathematical Society*.

A few remarks are added concerning the contents of the individual chapters.

Chapter XII (the first chapter of the present volume) contains the notations for compound and compartite matrices, definitions of the primaries of a minor determinant and of primary superdeterminants and primary subdeterminants, a description of the elimination of a variable from a system of inequalities, and the determination of the possible ranks of a matrix containing a given minor matrix. The notations for compound matrices and their determinoids are the complete generalisations of the notations used in Vol. I.

Chapter XIII deals with relations between the elements and minor determinants of a matrix. Starting with the determination of the connections between the short rows of an undegenerate matrix, we are led to all the most useful relations, and these are finally seen to be all particular cases of, or immediately deducible from, the fundamental identity of § 116. A brief review of the relations and the reasons for their utility is given in Appendix A; and other summaries will be found in the Index under the headings Relations, Standard identities, and Standard equations. Those of the relations which are identities in the elements are of course applicable to functional matrices.

Chapter XIV gives an account of some special properties of square matrices. The earlier articles deal with the properties of two co-joint complete matrices of the minor determinants of a square matrix, and the later articles with symmetric and skew-symmetric matrices. Appendix B should be read in conjunction with the articles on skew-symmetric matrices.

Chapter XV deals with the possible ranks of the product matrix and the factor matrices in any matrix product. The theorem of § 133 and the final results of §§ 135 and 137 constitute the complete generalisation of the Law of Cancellation for matrices. In proving the latter results an indication is given of methods of determining all solutions of any matrix equation of the form $X_1 X_2 \ldots X_n = C$. The concluding articles deal with the equivalences of matrices, on which the definition of a spacelet is based, and with the joins, intersections and connections of matrices and spacelets.

Chapter XVI deals with equigradent transformations of matrices, and with the reduction of a matrix of given rank r whose elements are constants to standard forms by equigradent transformations. The simplest standard form is a similar matrix which is conventionally equal to the unit matrix $[1]_r^r$; and every matrix of rank r with constant elements can be derived from the unit matrix $[1]_r^r$ by an equigradent transformation. The reductions of symmetric and skew-symmetric matrices by symmetric equigradent transformations receive special attention.

Chapter XVII deals with the solution of matrix equations of the second degree. One of the most important results obtained in this chapter is the general formula for all solutions of any assigned rank ρ of the symmetric equation $\overline{\underline{x}}{}^{s}_{m}[x]^{m}_{s} = \overline{\underline{a}}{}^{r}_{m}[a]^{m}_{r}$ in which $[a]^{m}_{r}$ is a given matrix of rank r. The general theory of extravagant matrices is largely based on this result, which leads at once to the reductions of the next chapter.

Chapter XVIII deals in the first place with the extravagances of any matrix whose elements are constants, and with certain special kinds of equigradent transformations, a review of which is given in Appendix C. The result of which most use is made is the reduction of a matrix whose elements are constants to a standard form by a unilaterally semi-unit equigradent transformation. This reduction re-appears in the reduction of a matrix to an equivalent undegenerate matrix whose long rows are mutually orthogonal, or to one which is the join of a core and a semi-unit matrix; in the corresponding representations of a spacelet as a join of mutually orthogonal unconnected points; and in the discussion of the properties of mutually normal undegenerate matrices. Further it enables us to complete the discussion of the unconnected mutually orthogonal solutions of any system of homogeneous linear algebraic equations, which was left unfinished in Chapter XI.

Chapter XVIII deals in the second place with the extravagances of spacelets and with semi-unit transformations of the points of space. The extravagance of a spacelet (or the degree of its orthogonality with itself) is that property of it which is next in importance to its rank. It is invariant in every semi-unit transformation of the points of space, and can be interpreted as being the rank of contact of the spacelet with the absolute quadric. A spacelet which has the greatest extravagance consistent with its rank is either completely extravagant or plenarily extravagant. A completely extravagant spacelet is orthogonal with itself, and is therefore a generating spacelet of the absolute quadric; a plenarily extravagant spacelet contains all points orthogonal with itself. With every spacelet is associated a completely extravagant spacelet, called its *core*, which is the locus of all points which lie in the given spacelet and are orthogonal with it, i.e. the locus of the points in which the given spacelet touches the absolute quadric; and a plenarily extravagant spacelet, called its *plenum*, which is the smallest spacelet containing the given spacelet and all points orthogonal with it. A sharp distinction is drawn between mutually orthogonal spacelets and mutually normal spacelets. A given spacelet has one and only one normal, whereas an indefinite number of spacelets are orthogonal with it.

Chapter XIX deals chiefly with the mutual orthotomy of two spacelets, or the degree of their mutual orthogonality. The most interesting results in it are those relating to the greatest possible orthotomy of two spacelets. It is shown that the mutual orthotomy of two arbitrary spacelets of given ranks is greatest when each of them is incident with the normal to the other, i.e. in that one of the two following mutually exclusive cases which is possible:

(1) when the two spacelets are mutually orthogonal; this being the case when their complete intersection is a completely extravagant spacelet ω_π, and the spacelets are the joins of ω_π with two mutually orthogonal non-intersecting spacelets lying in the plenum of ω_π, i.e. orthogonal with ω_π;

(2) when the normals to the two spacelets are mutually orthogonal; this being the case when the complete intersection of their normals is a completely extravagant spacelet ω_π, and the normals are the joins of ω_π with two mutually orthogonal non-intersecting spacelets lying in the plenum of ω_π, i.e. orthogonal with ω_π.

Further it is shown that the mutual orthotomy of two spacelets of given ranks which have a given complete intersection ω_p with core ω_π is greatest in that one of the two following mutually exclusive cases which is possible:

(1) when the two spacelets lie in the plenum of ω_p and are the joins of ω_p with two mutually orthogonal non-intersecting spacelets orthogonal with ω_p;

(2) when the normals to the two spacelets are mutually orthogonal; this being the case when the complete intersection ω_κ of their normals (whose rank κ is known) lies in ω_π, and the normals are the joins of ω_κ with two mutually orthogonal non-intersecting spacelets orthogonal with ω_p (whose join is necessarily complementary to ω_κ in the normal to ω_p).

The corresponding simpler results for real spacelets are also given. Another noteworthy result is the independence of the extravagances of two spacelets of given ranks which have a given complete intersection. All the theorems of Chapter XIX can be applied to common metrical space Ω_{n+1} of n dimensions when we define the paratomy and orthotomy of two spacelets of Ω_{n+1} to be those of their infinite sub-spaces, i.e. those of their intersections with the (homogeneous) infinite sub-space ω_n of Ω_{n+1}.

I owe many thanks to the authorities of the University of Calcutta who have generously undertaken the publishing of this volume, and have now with the sanction of the Governments of Bengal and India selected me as Hardinge Professor of Mathematics in the University. In consequence of the additional leisure thus secured to me from this time it is hoped that there will be no long interval before the appearance of the third volume, completing the theory of matrices and clearing the way for the applications. My special gratitude is due to Sir Asutosh Mukhopadhyay for his stimulating interest and encouragement.

Finally I desire to acknowledge my indebtedness to the officials and staff of the Cambridge University Press for the very great care bestowed on the printing.

C. E. CULLIS.

CALCUTTA,
February, 1918.

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$$\overline{\underline{x}}_{m}[x]_{m}=[c]_{m}^{m}, \quad \overline{\underline{x}}_{m}^{r}[x]_{r}^{m}=[c]_{m}^{m}, \quad \overline{\underline{x}}_{m}^{s}[x]_{s}^{m}=[c]_{m}^{m},$$

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CORRIGENDA

Page	4,	line	8 :	*For* "component", *read* "constituent".
„	6,	„	10 :	*For* "component", *read* "constituent".
„	89,	„	2 :	Interchange the two extreme factors on the right of the equation.
„	94,	„	7 :	*For* (A) , *read* (A′).
„	152,	„	9 :	*For* "reciprocal" , *read* "conjugate reciprocal".
„	152,	„	10 :	*For* "reciprocal" , *read* "conjugate reciprocal".
„	176,	„	1 :	*For* "matrix" , *read* "matrix equation".
„	299,	„	21 :	*For* "equigradent", *read* "equigradent in the real domain".

N.B. The space occupied by a matrix is counted as one line.