CHAPTER XII

COMPOUND MATRICES

[This Chapter, which is largely introductory to those which follow, contains definitions of compound and compartite matrices and a number of results to which reference may subsequently be made. In § 102 the primaries of any minor determinant of a matrix are defined, and §§ 104—107 deal with the possible ranks of a matrix which contains a given minor matrix.]

§ 98. Compound matrices.

We will use the notations

$$\phi = \begin{bmatrix} a, & b, & \dots\, c \\ a', & b', & \dots\, c' \\ \cdots & \cdots & \cdots \\ a'', & b'', & \dots\, c'' \end{bmatrix}^{\alpha,\beta,\dots\gamma}_{u,v,\dots w}, \quad \phi' = \overline{\underline{\begin{matrix} a, & a', & \dots\, a'' \\ b, & b', & \dots\, b'' \\ \cdots & \cdots & \cdots \\ c, & c', & \dots\, c'' \end{matrix}}_{\alpha,\beta,\dots\gamma}}^{u,v,\dots w}$$

for the matrices derived respectively from the schemes

$$\begin{bmatrix} [a]^{\alpha}_{u} & [b]^{\beta}_{u} & \dots\, [c]^{\gamma}_{u} \\ [a']^{\alpha}_{v} & [b']^{\beta}_{v} & \dots\, [c']^{\gamma}_{v} \\ \cdots & \cdots & \cdots \\ [a'']^{\alpha}_{w} & [b'']^{\beta}_{w} & \dots\, [c'']^{\gamma}_{w} \end{bmatrix}, \quad \left\lceil \begin{array}{ccc} \overline{\underline{a}_{\alpha}}^{u} & \overline{\underline{a'}_{\alpha}}^{v} & \dots\, \overline{\underline{a''}_{\alpha}}^{w} \\ \overline{\underline{b}_{\beta}}^{u} & \overline{\underline{b'}_{\beta}}^{v} & \dots\, \overline{\underline{b''}_{\beta}}^{w} \\ \cdots & \cdots & \cdots \\ \overline{\underline{c}_{\gamma}}^{u} & \overline{\underline{c'}_{\gamma}}^{v} & \dots\, \overline{\underline{c''}_{\gamma}}^{w} \end{array} \right\rfloor$$

by writing out the inner matrices in full and then discarding all the inner brackets; and for the determinoid of ϕ, which has the same value as the determinoid of ϕ', we will use the notation

$$\det \phi = \begin{pmatrix} a, & b, & \dots\, c \\ a', & b', & \dots\, c' \\ \cdots & \cdots & \cdots \\ a'', & b'', & \dots\, c'' \end{pmatrix}^{\alpha,\beta,\dots\gamma}_{u,v,\dots w}.$$

Thus ϕ and ϕ' are the two mutually conjugate matrices

$$\phi = \left[\begin{array}{cccccccccc}
a_{11} & a_{12} & \dots & a_{1\alpha} & b_{11} & b_{12} & \dots & b_{1\beta} & \dots\dots & c_{11} \;\; c_{12} \;\dots\; c_{1\gamma} \\
a_{21} & a_{22} & \dots & a_{2\alpha} & b_{21} & b_{22} & \dots & b_{2\beta} & \dots\dots & c_{21} \;\; c_{22} \;\dots\; c_{2\gamma} \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{u1} & a_{u2} & \dots & a_{u\alpha} & b_{u1} & b_{u2} & \dots & b_{u\beta} & \dots\dots & c_{u1} \;\; c_{u2} \;\dots\; c_{u\gamma} \\
a'_{11} & a'_{12} & \dots & a'_{1\alpha} & b'_{11} & b'_{12} & \dots & b'_{1\beta} & \dots\dots & c'_{11} \;\; c'_{12} \;\dots\; c'_{1\gamma} \\
a'_{21} & a'_{22} & \dots & a'_{2\alpha} & b'_{21} & b'_{22} & \dots & b'_{2\beta} & \dots\dots & c'_{21} \;\; c'_{22} \;\dots\; c'_{2\gamma} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a'_{v1} & a'_{v2} & \dots & a'_{v\alpha} & b'_{v1} & b'_{v2} & \dots & b'_{v\beta} & \dots\dots & c'_{v1} \;\; c'_{v2} \;\dots\; c'_{v\gamma} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a''_{11} & a''_{12} & \dots & a''_{1\alpha} & b''_{11} & b''_{12} & \dots & b''_{1\beta} & \dots\dots & c''_{11} \;\; c''_{12} \;\dots\; c''_{1\gamma} \\
a''_{21} & a''_{22} & \dots & a''_{2\alpha} & b''_{21} & b''_{22} & \dots & b''_{2\beta} & \dots\dots & c''_{21} \;\; c''_{22} \;\dots\; c''_{2\gamma} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a''_{w1} & a''_{w2} & \dots & a''_{w\alpha} & b''_{w1} & b''_{w2} & \dots & b''_{w\beta} & \dots\dots & c''_{w1} \;\; c''_{w2} \;\dots\; c''_{w\gamma}
\end{array}\right],$$

$$\phi' = \left[\begin{array}{cccccccccc}
a_{11} & a_{21} & \dots & a_{u1} & a'_{11} & a'_{21} & \dots & a'_{v1} & \dots\dots & a''_{11} \;\; a''_{21} \;\dots\; a''_{w1} \\
a_{12} & a_{22} & \dots & a_{u2} & a'_{12} & a'_{22} & \dots & a'_{v2} & \dots\dots & a''_{12} \;\; a''_{22} \;\dots\; a''_{w2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
a_{1\alpha} & a_{2\alpha} & \dots & a_{u\alpha} & a'_{1\alpha} & a'_{2\alpha} & \dots & a'_{v\alpha} & \dots\dots & a''_{1\alpha} \;\; a''_{2\alpha} \;\dots\; a''_{w\alpha} \\
b_{11} & b_{21} & \dots & b_{u1} & b'_{11} & b'_{21} & \dots & b'_{v1} & \dots\dots & b''_{11} \;\; b''_{21} \;\dots\; b''_{w1} \\
b_{12} & b_{22} & \dots & b_{u2} & b'_{12} & b'_{22} & \dots & b'_{v2} & \dots\dots & b''_{12} \;\; b''_{22} \;\dots\; b''_{w2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
b_{1\beta} & b_{2\beta} & \dots & b_{u\beta} & b'_{1\beta} & b'_{2\beta} & \dots & b'_{v\beta} & \dots\dots & b''_{1\beta} \;\; b''_{2\beta} \;\dots\; b''_{w\beta} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
c_{11} & c_{21} & \dots & c_{u1} & c'_{11} & c'_{21} & \dots & c'_{v1} & \dots\dots & c''_{11} \;\; c''_{21} \;\dots\; c''_{w1} \\
c_{12} & c_{22} & \dots & c_{u2} & c'_{12} & c'_{22} & \dots & c'_{v2} & \dots\dots & c''_{12} \;\; c''_{22} \;\dots\; c''_{w2} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
c_{1\gamma} & c_{2\gamma} & \dots & c_{u\gamma} & c'_{1\gamma} & c'_{2\gamma} & \dots & c'_{v\gamma} & \dots\dots & c''_{1\gamma} \;\; c''_{2\gamma} \;\dots\; c''_{w\gamma}
\end{array}\right]$$

More generally for the two mutually conjugate matrices derived in the same way from the schemes

$$\left[\begin{array}{cccc}
[a_{l\lambda}]^{\alpha}_{u} & [b_{m\mu}]^{\beta}_{u} & \dots & [c_{n\nu}]^{\gamma}_{u} \\
[a'_{r\rho}]^{\alpha}_{v} & [b'_{s\sigma}]^{\beta}_{v} & \dots & [c'_{t\tau}]^{\gamma}_{v} \\
\dots & \dots & \dots & \dots \\
[a''_{x\xi}]^{\alpha}_{w} & [b''_{y\eta}]^{\beta}_{w} & \dots & [c''_{z\zeta}]^{\gamma}_{w}
\end{array}\right] \quad \text{and} \quad \left[\begin{array}{cccc}
\lceil\lfloor a_{l\lambda}\rfloor\rceil^{u}_{\alpha} & \lceil\lfloor a'_{r\rho}\rfloor\rceil^{v}_{\alpha} & \dots & \lceil\lfloor a''_{x\xi}\rfloor\rceil^{w}_{\alpha} \\
\lceil\lfloor b_{m\mu}\rfloor\rceil^{u}_{\beta} & \lceil\lfloor b'_{s\sigma}\rfloor\rceil^{v}_{\beta} & \dots & \lceil\lfloor b''_{y\eta}\rfloor\rceil^{w}_{\beta} \\
\dots & \dots & \dots & \dots \\
\lceil\lfloor c_{n\nu}\rfloor\rceil^{u}_{\gamma} & \lceil\lfloor c'_{t\tau}\rfloor\rceil^{v}_{\gamma} & \dots & \lceil\lfloor c''_{z\zeta}\rfloor\rceil^{w}_{\gamma}
\end{array}\right]$$

we will use the notations

$$\begin{bmatrix} a_{l\lambda}, & b_{m\mu}, & \dots & c_{n\nu} \\ a'_{r\rho}, & b'_{s\sigma}, & \dots & c'_{t\tau} \\ \dots & \dots & \dots & \dots \\ a''_{x\xi}, & b''_{y\eta}, & \dots & c''_{z\zeta} \end{bmatrix}^{\alpha, \beta, \dots \gamma}_{u, v, \dots w} \quad \text{and} \quad \begin{array}{c} \hline a_{l\lambda}, \; a'_{r\rho}, \; \dots \; a''_{x\xi} \\ b_{m\mu}, \; b'_{s\sigma}, \; \dots \; b''_{y\eta} \\ \dots\dots\dots \\ c_{n\nu}, \; c'_{t\tau}, \; \dots \; c''_{z\zeta} \\ \hline \end{array}^{u, v, \dots w}_{\alpha, \beta, \dots \gamma} ;$$

and the determinoid of the first of these matrices, which has the same value as the determinoid of the second matrix, will be denoted by

$$\begin{pmatrix} a_{l\lambda}, & b_{m\mu}, & \dots & c_{n\nu} \\ a'_{r\rho}, & b'_{s\sigma}, & \dots & c'_{t\tau} \\ \dots & \dots & \dots & \dots \\ a''_{x\xi}, & b''_{y\eta}, & \dots & c''_{z\zeta} \end{pmatrix}^{\alpha, \beta, \dots \gamma}_{u, v, \dots w} .$$

These notations are generalisations of those used for augmented matrices in the first volume.

Ex. i.
$$\begin{bmatrix} a, & x \\ b, & y \\ c, & z \end{bmatrix}^{2, 3}_{3, 1, 2} \quad \text{and} \quad \begin{array}{c} \hline a, \; b, \; c \\ x, \; y, \; z \\ \hline \end{array}^{3, 1, 2}_{2, 3}$$

are by definition the respective matrices

$$\begin{bmatrix} a_{11} & a_{12} & x_{11} & x_{12} & x_{13} \\ a_{21} & a_{22} & x_{21} & x_{22} & x_{23} \\ a_{31} & a_{32} & x_{31} & x_{32} & x_{33} \\ b_{11} & b_{12} & y_{11} & y_{12} & y_{13} \\ c_{11} & c_{12} & z_{11} & z_{12} & z_{13} \\ c_{21} & c_{22} & z_{21} & z_{22} & z_{23} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{11} & a_{21} & a_{31} & b_{11} & c_{11} & c_{21} \\ a_{12} & a_{22} & a_{32} & b_{12} & c_{12} & c_{22} \\ x_{11} & x_{21} & x_{31} & y_{11} & z_{11} & z_{21} \\ x_{12} & x_{22} & x_{32} & y_{12} & z_{12} & z_{22} \\ x_{13} & x_{23} & x_{33} & y_{13} & z_{13} & z_{23} \end{bmatrix} .$$

Ex. ii.
$$\begin{bmatrix} a_{pq}, & x_{\kappa\lambda} \\ b_{uv}, & y_{\mu\nu} \\ c_{rs}, & z_{\sigma\tau} \end{bmatrix}^{2, 3}_{3, 1, 2} \quad \text{and} \quad \begin{array}{c} \hline a_{pq}, \; b_{uv}, \; c_{rs} \\ x_{\kappa\lambda}, \; y_{\mu\nu}, \; z_{\sigma\tau} \\ \hline \end{array}^{3\; 1, 2}_{2, 3}$$

are by definition the respective matrices

$$\begin{bmatrix} a_{p_1q_1} & a_{p_1q_2} & x_{\kappa_1\lambda_1} & x_{\kappa_1\lambda_2} & x_{\kappa_1\lambda_3} \\ a_{p_2q_1} & a_{p_2q_2} & x_{\kappa_2\lambda_1} & x_{\kappa_2\lambda_2} & x_{\kappa_2\lambda_3} \\ a_{p_3q_1} & a_{p_3q_2} & x_{\kappa_3\lambda_1} & x_{\kappa_3\lambda_2} & x_{\kappa_3\lambda_3} \\ b_{u_1v_1} & b_{u_1v_2} & y_{\mu_1\nu_1} & y_{\mu_1\nu_2} & y_{\mu_1\nu_3} \\ c_{r_1s_1} & c_{r_1s_2} & z_{\sigma_1\tau_1} & z_{\sigma_1\tau_2} & z_{\sigma_1\tau_3} \\ c_{r_2s_1} & c_{r_2s_2} & z_{\sigma_2\tau_1} & z_{\sigma_2\tau_2} & z_{\sigma_2\tau_3} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} a_{p_1q_1} & a_{p_2q_1} & a_{p_3q_1} & b_{u_1v_1} & c_{r_1s_1} & c_{r_2s_1} \\ a_{p_1q_2} & a_{p_2q_2} & a_{p_3q_2} & b_{u_1v_2} & c_{r_1s_2} & c_{r_2s_2} \\ x_{\kappa_1\lambda_1} & x_{\kappa_2\lambda_1} & x_{\kappa_3\lambda_1} & y_{\mu_1\nu_1} & z_{\sigma_1\tau_1} & z_{\sigma_2\tau_1} \\ x_{\kappa_1\lambda_2} & x_{\kappa_2\lambda_2} & x_{\kappa_3\lambda_2} & y_{\mu_1\nu_2} & z_{\sigma_1\tau_2} & z_{\sigma_2\tau_2} \\ x_{\kappa_1\lambda_3} & x_{\kappa_2\lambda_3} & x_{\kappa_3\lambda_3} & y_{\mu_1\nu_3} & z_{\sigma_1\tau_3} & z_{\sigma_2\tau_3} \end{bmatrix} .$$

NOTE 1. *Scalar matrices.*

For a scalar matrix of order m in which the elements of the leading diagonal all have the value k, all other elements having the value 0, we have used the notation $k\,[1]_m^m$. Whenever k is a scalar quantity, and not a symbol which has no meaning until suffixes are attached, we will understand that

$$[k]_m^m = k\,[1]_m^m = \begin{bmatrix} k & 0 & \dots & 0 \\ 0 & k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k \end{bmatrix}.$$

It is convenient to use this alternative notation for a scalar matrix when the scalar matrix is a component of a compound matrix. In cases where any doubt is possible it will be expressly stated when this meaning is to be attached to $[k]_m^m$.

Further whenever k is a scalar quantity and $[a]_m^n$ is a matrix, it will be understood that

$$[ka]_m^n = k\,[a]_m^n.$$

NOTE 2. *Quasi-scalar matrices.*

In the case of quasi-scalar matrices we shall sometimes use such notations as the following:

$${}^1[k]_m = {}^1\overline{\underline{k}}_m = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_m \end{bmatrix}, \qquad {}^1[\sqrt{k}]_m = {}^1\overline{\underline{\sqrt{k}}}_m = \begin{bmatrix} \sqrt{k_1} & 0 & \dots & 0 \\ 0 & \sqrt{k_2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{k_m} \end{bmatrix},$$

$${}^1[k^{-1}]_m = {}^1\overline{\underline{k^{-1}}}_m = \begin{bmatrix} k_1^{-1} & 0 & \dots & 0 \\ 0 & k_2^{-1} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_m^{-1} \end{bmatrix}, \qquad {}^1[k^p]_m = {}^1\overline{\underline{k^p}}_m = \begin{bmatrix} k_1^p & 0 & \dots & 0 \\ 0 & k_2^p & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_m^p \end{bmatrix}.$$

NOTE 3. *Zero matrices.*

The symbol $[0]_m^n$ will be used to denote a zero matrix whose horizontal and vertical orders are m and n, i.e. to denote the matrix $[a]_m^n$ all of whose elements are 0's. This notation will usually only be required when $[0]_m^n$ is a component of a compound matrix.

NOTE 4. *One-rowed matrices.*

For the one-rowed matrix $[x_1 x_2 \dots x_p\ y_1 y_2 \dots y_q \dots z_1 z_2 \dots z_r]$ and its conjugate we shall sometimes use the notations

$$[x,\ y,\ \dots\ z]_{p,\ q,\ \dots\ r} \quad \text{and} \quad \begin{matrix} x \\ y \\ \vdots \\ z \end{matrix}\ \Big\rfloor_{p,\ q,\ \dots\ r}$$

respectively. The commas separating the suffixes $p,\ q,\ \dots\ r$ will distinguish these from the single suffix notations for a matrix described in § 2.

Since $[1]_m^n$ is undefined except when $m=n$, we may define $[1]_1^m$ and $\overline{\underline{1}}_m^1$ by the equations

$$[1]_m^1=[1,\ 1,\ \dots\ 1]_1^{1,1,\dots 1}=[1\ 1\ \dots\ 1],\quad \overline{\underline{1}}_m^1=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}_{1,1,\dots 1}^1=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix},$$

where in each case there are m 1's.

NOTE 5. *Matrices one of whose orders is zero.*

Neither of the orders m and n of a matrix $[a]_m^n$ can be less than 1; but for the sake of generality we shall often proceed as if the value 0 were admissible for m and n. Then if either of the orders of a matrix is 0, the matrix is non-existent; it can be regarded as having rank 0; it can be replaced as a factor by the scalar number 0; and its conjugate reciprocal is also a matrix one of whose orders is 0.

Ex. iii. $[9]_3^3=9[1]_3^3=\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix};\quad [-\sqrt{2}]_3^3=-\sqrt{2}\,[1]_3^3=\begin{bmatrix}-\sqrt{2},&0,&0\\0,&-\sqrt{2},&0\\0,&0,&-\sqrt{2}\end{bmatrix}.$

Ex. iv. $$\begin{bmatrix}a,&-5\\0,&2b\end{bmatrix}_{3,2}^{4,3}=\begin{bmatrix}a_{11},&a_{12},&a_{13},&a_{14},&-5,&0,&0\\a_{21},&a_{22},&a_{23},&a_{24},&0,&-5,&0\\a_{31},&a_{32},&a_{33},&a_{34},&0,&0,&-5\\0,&0,&0,&0,&2b_{11},&2b_{12},&2b_{13}\\0,&0,&0,&0,&2b_{21},&2b_{22},&2b_{23}\end{bmatrix}.$$

Ex. v. ${}^1[k]_m\,{}^1[k]_m={}^1[k^2]_m;\quad {}^1[\sqrt{k}]_m\,{}^1[\sqrt{k}]_m={}^1[k]_m;\quad {}^1[k]_m\,{}^1[k^{-1}]_m=[1]_m^m.$

Ex. vi. $$\overline{\underline{\begin{matrix}x\\1\end{matrix}}}_{m,1}=\begin{bmatrix}x_1\\x_2\\\vdots\\x_m\\1\end{bmatrix};\quad [x,\ 1]_m^1=[x_1\ x_2\ \dots\ x_m\ 1].$$

Ex. vii. $$\overline{\underline{\begin{matrix}x\\1\end{matrix}}}_{m,1}^{r}=\begin{bmatrix}x_{11}&x_{21}&\dots&x_{r1}\\x_{12}&x_{22}&\dots&x_{r2}\\\dots&\dots&\dots&\dots\\x_{1m}&x_{2m}&\dots&x_{rm}\\1&1&\dots&1\end{bmatrix};\quad [x,\ 1]_r^{m,1}=\begin{bmatrix}x_{11}&x_{12}&\dots&x_{1m}&1\\x_{21}&x_{22}&\dots&x_{2m}&1\\\dots&\dots&\dots&\dots&\dots\\x_{r1}&x_{r2}&\dots&x_{rm}&1\end{bmatrix}.$$

§ 99. Multiplication of compound matrices.

If we call $u, v, \dots w$ and $\alpha, \beta, \dots \gamma$ the partial orders of the compound matrices ϕ and ϕ' in § 98, then the product of two compound matrices whose partial passivities are the same and arranged in the same order can be evaluated in a manner analogous to that in which we evaluate the product of two simple matrices which have the same passivities. For we have

$$\begin{bmatrix} a, & b, & \dots & c \\ a', & b', & \dots & c' \\ \dots & \dots & \dots & \dots \\ a'', & b'', & \dots & c'' \end{bmatrix}_{m,n,\dots p}^{r,s,\dots t} \begin{bmatrix} \alpha, & \alpha', & \dots & \alpha'' \\ \beta, & \beta', & \dots & \beta'' \\ \dots & \dots & \dots & \dots \\ \gamma, & \gamma', & \dots & \gamma'' \end{bmatrix}_{r,s,\dots t}^{u,v,\dots w} = \begin{bmatrix} x, & y, & \dots & z \\ x', & y', & \dots & z' \\ \dots & \dots & \dots & \dots \\ x'', & y'', & \dots & z'' \end{bmatrix}_{m,n,\dots p}^{u,v,\dots w},$$

where

$$[x]_m^u = [a]_m^r [\alpha]_r^u + [b]_m^s [\beta]_s^u + \dots + [c]_m^t [\gamma]_t^u,$$

$$[y]_m^v = [a]_m^r [\alpha']_r^v + [b]_m^s [\beta']_s^v + \dots + [c]_m^t [\gamma']_t^v,$$

$$[y'']_p^v = [a'']_p^r [\alpha']_r^v + [b'']_p^s [\beta']_s^v + \dots + [c'']_p^t [\gamma']_t^v,$$

and so on.

This method of evaluating the product is particularly advantageous when some of the component matrices are zero matrices or scalar matrices.

Ex. i. Since $[a]_m^n [1]_n^n = [1]_m^m [a]_m^n = [a]_m^n$, we see at once that

$$\begin{bmatrix} a, & b, & c \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{bmatrix}_{2,3,2}^{4,3,2} \begin{bmatrix} \alpha, & 0, & 0 \\ \beta, & 1, & 0 \\ \gamma, & 0, & 1 \end{bmatrix}_{4,3,2}^{2,3,2} = \begin{bmatrix} x, & b, & c \\ \beta, & 1, & 0 \\ \gamma, & 0, & 1 \end{bmatrix}_{2,3,2}^{2,3,2},$$

where $$[x]_2^2 = [a]_2^4 [\alpha]_4^2 + [b]_2^3 [\beta]_3^2 + [c]_2^2 [\gamma]_2^2.$$

Ex. ii. If $i = \sqrt{-1}$, then

$$\begin{bmatrix} 1, & 0, & 0, & 0 \\ 0, & 1, & i, & 0 \end{bmatrix}_{r-\rho,\rho}^{r-\rho,\rho,\rho,s} \begin{bmatrix} 0, & 0 \\ 0, & 1 \\ 0, & i \\ 1, & 0 \end{bmatrix}_{r-\rho,\rho,\rho,s}^{s,\rho} = 0.$$

Ex. iii. If k is a scalar quantity, we have

$$[k, x]_m^{m,n} \begin{bmatrix} -x \\ k \end{bmatrix}_{m,n}^{n} = -[k]_m^m [x]_m^n + [x]_m^n [k]_n^n = -k[x]_m^n + k[x]_m^n = 0.$$

§ 100. Compartite matrices.

A matrix whose elements all vanish except those lying in a number of mutually complementary minors will be called a compartite matrix, and those mutually complementary minors will be called the *parts* of the matrix.

Every such matrix can be converted by derangements of its horizontal and vertical rows into a matrix of the form

$$\phi = [e]_m^n = \begin{bmatrix} a, & 0, & \dots & 0 \\ 0, & b, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & c \end{bmatrix}_{p,\,q,\,\dots\,r}^{u,\,v,\,\dots\,w}.$$

A matrix ϕ having this special form will be called a *compartite matrix in standard form.* The parts of ϕ are the matrices

$$A = [a]_p^u, \quad B = [b]_q^v, \quad \dots \quad C = [c]_r^w$$

or any derangements of them. We shall speak of $A, B, \dots C$ themselves, taken in this order, as the *successive parts* of ϕ.

Ex. i. *There exists a connection between the horizontal (or vertical) rows of a compartite matrix when and only when there exists a connection between the horizontal (or vertical) rows of one of its parts.*

Since there will be no loss of generality in supposing the compartite matrix to be in standard form, it will be sufficient to prove this theorem for the matrix ϕ shown above.

An equation of the form

$$[\lambda, \mu, \dots \nu]_{p,\,q,\,\dots\,r} [e]_m^n = 0$$

is satisfied when and only when all the equations

$$[\lambda]_p \, [a]_p^u = 0, \quad [\mu]_q \, [b]_q^v = 0, \; \dots \; [\nu]_r \, [c]_r^w = 0$$

are satisfied; and the one-rowed matrix $[\lambda, \mu, \dots \nu]_{p,\,q,\,\dots\,r}$ contains a non-vanishing element when and only when one of its minors $[\lambda]_p, [\mu]_q, \dots [\nu]_r$ contains a non-vanishing element. Consequently there exists a connection between the horizontal rows of ϕ when and only when there exists a connection between the horizontal rows of one of its parts.

The truth of the theorem for vertical rows can be proved in a similar way.

Ex. ii. If the parts of ϕ are all square matrices, so that $u=p$, $v=q$, ... $w=r$, then the determinant of ϕ is equal to the product of the determinants of its successive parts. Also the determinant of any matrix of which ϕ is a derangement can only differ in sign from this product.

Ex. iii. Every minor of ϕ is a compartite matrix whose parts are minors of the parts of ϕ.

Theorem. *The rank of a compartite matrix is the sum of the ranks of its parts.*

It will be sufficient to prove this theorem for the matrix ϕ shown above.

Let the parts $A, B, \dots C$ of ϕ have ranks $\alpha, \beta, \dots \gamma$ respectively, and let $R = \alpha + \beta + \dots + \gamma$. Further let $(a_{\kappa\lambda})_\alpha^\alpha$, $(b_{\mu\nu})_\beta^\beta$, ... $(c_{\sigma\tau})_\gamma^\gamma$ be non-vanishing minor determinants of orders $\alpha, \beta, \dots \gamma$ of $A, B, \dots C$. Then the compartite matrix

$$\phi' = \begin{bmatrix} a_{\kappa\lambda}, & 0, & \dots & 0 \\ 0, & b_{\mu\nu}, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & c_{\sigma\tau} \end{bmatrix}_{\alpha, \beta, \dots \gamma}^{\alpha, \beta, \dots \gamma}$$

is a square minor of ϕ of order R whose determinant, which is equal to the product $(a_{\kappa\lambda})_\alpha^\alpha \, (b_{\mu\nu})_\beta^\beta \, \dots (c_{\sigma\tau})_\gamma^\gamma$, does not vanish. Therefore the rank of ϕ cannot be less than R.

Again if ϕ'' is any square minor of ϕ of order greater than R, then ϕ'' must contain either more than α horizontal rows of A, or more than β horizontal rows of $B, \dots$ or more than γ horizontal rows of C. Consequently there must be a connection between the horizontal rows of one of the parts of the compartite matrix ϕ'', and therefore a connection between the horizontal rows of ϕ'' itself, i.e. ϕ'' must be degenerate. Thus ϕ cannot contain a non-vanishing minor determinant of order greater than R.

It follows that ϕ must have rank R, where R is the sum of the ranks of the parts of ϕ.

NOTE. *Compartite matrices whose parts are quasi-scalar matrices.*

For a compartite matrix in standard form whose successive parts are the quasi-scalar matrices ${}^1[a]_p$, ${}^1[b]_q$, $\dots$ ${}^1[c]_r$ we shall sometimes use one of the notations

$${}^1[a,\ b,\ \dots c]_{p,\, q,\, \dots r}, \qquad {}^1\begin{bmatrix} a, & 0, & \dots & 0 \\ 0, & b, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & c \end{bmatrix}_{p,\, q,\, \dots r}.$$

§ 101. The conjugate reciprocals and inverses of certain matrices.

The following examples can be regarded as exercises in the multiplication of compound matrices.

Ex. i. *The conjugate reciprocals of the respective matrices*

$$\begin{bmatrix} a, & b \\ 0, & 1 \end{bmatrix}_{r,s}^{r,s}, \quad \begin{bmatrix} b, & a \\ 1, & 0 \end{bmatrix}_{r,s}^{s,r}, \quad \begin{bmatrix} 0, & 1 \\ a, & b \end{bmatrix}_{s,r}^{r,s}, \quad \begin{bmatrix} 1, & 0 \\ b, & a \end{bmatrix}_{s,r}^{s,r}$$

are

$$\ulcorner\begin{matrix} \underline{A}, & \underline{B} \\ 0, & \Delta \end{matrix}\lrcorner_{r,s}^{r,s}, \quad \ulcorner\begin{matrix} 0, & \Delta \\ A, & B \end{matrix}\lrcorner_{s,r}^{r,s}, \quad \ulcorner\begin{matrix} B, & A \\ \Delta, & 0 \end{matrix}\lrcorner_{r,s}^{s,r}, \quad \ulcorner\begin{matrix} \Delta, & 0 \\ B, & A \end{matrix}\lrcorner_{s,r}^{s,r},$$

where $\ulcorner A \lrcorner_r^r$ *is the conjugate reciprocal of* $[a]_r^r$; $\Delta = (a)_r^r$; $[\Delta]_s^s = \Delta\,[1]_s^s$; *and*

$$\ulcorner B \lrcorner_r^s = -\ulcorner A \lrcorner_r^r \, [b]_r^s.$$

It will be observed that $-B_{ij}$ is the determinant formed when the jth vertical row of Δ is replaced by the ith vertical row of $[b]^s_r$.

If these results are true when the matrices are undegenerate, i.e. when $\Delta \neq 0$, they must be true in general. Hence we can at once verify them by multiplication; for we have

$$\begin{bmatrix} a, & b \\ 0, & 1 \end{bmatrix}^{r,s}_{r,s} \overline{\underline{\begin{matrix} A, & B \\ 0, & \Delta \end{matrix}}}^{r,s}_{r,s} = \begin{bmatrix} \Delta, & 0 \\ 0, & \Delta \end{bmatrix}^{r,s}_{r,s} = \Delta[1]^{r+s}_{r+s},$$

and similarly in the other cases.

To prove the first result directly, assume that $\Delta \neq 0$, and let the conjugate reciprocal of $\begin{bmatrix} a, & b \\ 0, & 1 \end{bmatrix}^{r,s}_{r,s}$ be $\begin{bmatrix} x, & y \\ z, & w \end{bmatrix}^{r,s}_{r,s}$, so that

$$\begin{bmatrix} a, & b \\ 0, & 1 \end{bmatrix}^{r,s}_{r,s} \begin{bmatrix} x, & y \\ z, & w \end{bmatrix}^{r,s}_{r,s} = \Delta \begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}^{r,s}_{r,s}$$

Then $$[a]^r_r\,[x]^r_r + [b]^s_r\,[z]^r_s = \Delta[1]^r_r, \quad [a]^r_r\,[y]^s_r + [b]^s_r\,[w]^s_s = 0,$$

$$[z]^r_s = 0, \quad [w]^s_s = \Delta[1]^s_s.$$

Using the last two equations the first two reduce to

$$[a]^r_r\,[x]^r_r = \Delta[1]^r_r, \quad [a]^r_r\,[y]^s_r = -\Delta[b]^s_r.$$

Prefixing $\overline{\underline{A}}^r_r$, the conjugate reciprocal of $[a]^r_r$, on both sides, we obtain

$$[x]^r_r = \overline{\underline{A}}^r_r, \quad [y]^s_r = -\overline{\underline{A}}^r_r\,[b]^s_r.$$

The remaining three results can be deduced, or proved in similar ways.

Ex. ii. The inverses of the same matrices, when they are undegenerate, are

$$\overline{\underline{\begin{matrix} A, & B \\ 0, & 1 \end{matrix}}}^{r,s}_{r,s}, \quad \overline{\underline{\begin{matrix} 0, & 1 \\ A, & B \end{matrix}}}^{r,s}_{s,r}, \quad \overline{\underline{\begin{matrix} B, & A \\ 1, & 0 \end{matrix}}}^{s,r}_{r,s}, \quad \overline{\underline{\begin{matrix} 1, & 0 \\ B, & A \end{matrix}}}^{s,r}_{s,r},$$

where $\overline{\underline{A}}^r_r$ is the inverse of $[a]^r_r$, and $\overline{\underline{B}}^s_r = -\overline{\underline{A}}^r_r\,[b]^s_r$.

Ex. iii. *The conjugate reciprocals of the respective matrices*

$$\begin{bmatrix} a, & 0 \\ b, & 1 \end{bmatrix}^{r,s}_{r,s}, \quad \begin{bmatrix} b, & 1 \\ a, & 0 \end{bmatrix}^{r,s}_{s,r}, \quad \begin{bmatrix} 0, & a \\ 1, & b \end{bmatrix}^{s,r}_{r,s}, \quad \begin{bmatrix} 1, & b \\ 0, & a \end{bmatrix}^{s,r}_{s,r}$$

are

$$\overline{\underline{\begin{matrix} A, & 0 \\ B, & \Delta \end{matrix}}}^{r,s}_{r,s}, \quad \overline{\underline{\begin{matrix} 0, & A \\ \Delta, & B \end{matrix}}}^{s,r}_{r,s}, \quad \overline{\underline{\begin{matrix} B, & \Delta \\ A, & 0 \end{matrix}}}^{r,s}_{s,r}, \quad \overline{\underline{\begin{matrix} \Delta, & B \\ 0, & A \end{matrix}}}^{s,r}_{s,r},$$

where $\overline{\underline{A}}^r_r$ *is the conjugate reciprocal of* $[a]^r_r$; $\Delta = (a)^r_r$; $[\Delta]^s_s = \Delta[1]^s_s$; *and*

$$\overline{\underline{B}}^r_s = -[b]^r_s\,\overline{\underline{A}}^r_r.$$

These results can be verified by multiplication, or proved as in Ex. i.

Ex. iv. The conjugate reciprocal of $\begin{bmatrix}1, & a\\ 0, & 1\end{bmatrix}_{r,s}^{r,s}$ is $\begin{bmatrix}1, & -a\\ 0, & 1\end{bmatrix}_{r,s}^{r,s}$.

Ex. v. The conjugate reciprocal of $\begin{bmatrix}a, & 0\\ 0, & 1\end{bmatrix}_{r,s}^{r,s}$ is $\underline{\overline{\begin{matrix}A, & 0\\ 0, & \Delta\end{matrix}}}{}_{r,s}^{r,s}$,

where $\underline{\overline{A}}{}_r^r$ is the conjugate reciprocal of $[a]_r^r$, and $\Delta=(a)_r^r$.

Ex. vi. The conjugate reciprocal of the matrix

$$M=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} & a_{1,m+1}\\ a_{21} & a_{22} & \dots & a_{2m} & a_{2,m+1}\\ \dots & \dots & \dots & \dots & \dots\\ a_{m1} & a_{m2} & \dots & a_{mm} & a_{m,m+1}\\ 0 & 0 & \dots & 0 & 1\end{bmatrix}\quad\text{is}\quad M'=\begin{bmatrix} A_{11} & A_{21} & \dots & A_{m1} & A_{m+1,1}\\ A_{12} & A_{22} & \dots & A_{m2} & A_{m+1,2}\\ \dots & \dots & \dots & \dots & \dots\\ A_{1m} & A_{2m} & \dots & A_{mm} & A_{m+1,m}\\ 0 & 0 & \dots & 0 & \Delta\end{bmatrix},$$

where $\Delta=(a)_m^m$; $\underline{\overline{A}}{}_m^m$ is the conjugate reciprocal of $[a]_m^m$;

and
$$\begin{bmatrix}A_{m+1,1}\\ A_{m+1,2}\\ \vdots\\ A_{m+1,m}\end{bmatrix}=-\underline{\overline{A}}{}_m^m\begin{bmatrix}a_{1,m+1}\\ a_{2,m+1}\\ \vdots\\ a_{m,m+1}\end{bmatrix}.$$

In particular $-A_{m+1,u}$ is the determinant formed when the uth vertical row of $(a)_m^m$ or Δ is replaced by the last vertical row of $[a]_m^{m+1}$.

The matrix M is included in the matrices considered in Ex. i.

Ex. vii. *The conjugate reciprocal of* $\begin{bmatrix}a, & 0, & b\\ c, & 1, & d\\ 0, & 0, & 1\end{bmatrix}_{r,s,t}^{r,s,t}$ *is* $\underline{\overline{\begin{matrix}A, & 0, & B\\ C, & \Delta, & D\\ 0, & 0, & \Delta\end{matrix}}}{}_{r,s,t}^{r,s,t}$,

where $\Delta=(a)_r^r$; $\underline{\overline{A}}{}_r^r$ *is the conjugate reciprocal of* $[a]_r^r$; *and*

$$\underline{\overline{B}}{}_r^t=-\underline{\overline{A}}{}_r^r\,[b]_r^t,\qquad \underline{\overline{C}}{}_s^r=-[c]_s^r\,\underline{\overline{A}}{}_r^r,\qquad \underline{\overline{D}}{}_s^t=[c]_s^r\,\underline{\overline{A}}{}_r^r\,[b]_r^t-\Delta\,[d]_s^t.$$

We can prove this by putting

$$\begin{bmatrix}a, & 0, & b\\ c, & 1, & d\\ 0, & 0, & 1\end{bmatrix}_{r,s,t}^{r,s,t}=\begin{bmatrix}\alpha, & \beta\\ 0, & 1\end{bmatrix}_{r+s,t}^{r+s,t},$$

where
$$[\alpha]_{r+s}^{r+s}=\begin{bmatrix}a, & 0\\ c, & 1\end{bmatrix}_{r,s}^{r,s},\qquad [\beta]_{r+s}^{t}=\begin{bmatrix}b\\ d\end{bmatrix}_{r,s}^{t},$$

and applying Exs. i and iii; or we can verify it by multiplication.