Solutions for Chapter 1

1.1 The insurer will calculate the premium for a term or whole life insurance policy assuming that the policyholder is in relatively good health; otherwise, if the insurer assumed that all purchasers were unhealthy, the cost of insurance would be prohibitive to those customers who are healthy. The assumption then is that claims will be relatively rare in the first few years of insurance, especially since most policies are sold to lives in their 30s and 40s.

This means that the price is too low for a life who is very unwell, for whom the risk of a claim shortly after purchase might be 10 or 100 times greater than for a healthy life. The insurer therefore needs evidence that the purchaser is in good health, to avoid the risk that insurance is bought too cheaply by lives who have a much higher probability of claim.

The objective of underwriting is to produce a relatively homogeneous insured population when policies are issued. The risk that the policyholder purchases the insurance because they are aware that their individual risk is greater than that of the insured population used to calculate the premium, is an example of adverse selection risk. Underwriting is a way of reducing the impact of adverse selection for life insurance.

Adverse selection for an annuity purchaser works in the other direction – a life might buy an annuity if they considered their mortality was lighter than the general population. But, since adverse selection is likely to affect all lives purchasing annuities, more or less, it does not generate heterogeneity, and the impact can be managed by assuming lower overall mortality rates for annuitants.

In addition, the difference in the net cost to the insurer arising from adverse selection will be smaller compared with the term insurance example.

1.2 The insurer will be more rigorous with underwriting for term insurance than for whole life insurance because the potential financial consequence of adverse selection is greater. Note that the insurer expects few claims to arise from the term insurance portfolio. Premiums are small, relative to the death benefit, because the probability of payment of the death benefit is assumed to be small. For whole life insurance, premiums are substantially larger as payment of the death benefit is a certain event (ignoring surrenders). The only uncertainty is the timing of the benefit payment.

The main risk to the insurer is that a life with a very high mortality risk, much higher than the assumed insured population, purchases life insurance. It is likely in this case that the life will pay very few premiums, and the policy will involve a large death benefit payout with very little premium income. Since term insurance has much lower premiums for a given sum insured than whole life insurance, it is likely that such a policyholder would choose term insurance. Hence, the risk of adverse selection is greater for term insurance than for whole life insurance, and underwriting is used to reduce the adverse selection risk.

1.3 The principle of charging in advance for life insurance is to eliminate the potential for policyholders to benefit from short-term life insurance cover without paying for it. Suppose premiums were payable at the end of the policy year. A life could sign up for the insurance, and lapse the contract at the end of the year. The life would have benefited from free insurance cover for that year.

In addition, life insurance involves significant acquisition expenses. The first premium is used to meet some or all of these expenses.

Background note: The fact that the insurance for a policyholder did not result in a claim does not make it free to the insurer. The insurer's view is of a portfolio of contracts. Suppose 100 people buy term life insurance for one year, with a sum insured of \$1000, at a premium of \$11 each. The insurer expects a mortality rate of 1%, which means that, on average, one life out of the 100 dies. If all the policyholders pay their first year's premiums in advance, and one life dies, then the insurer receives \$1100 (plus some interest) and pays out \$1000. On the other hand, if premiums were due at the year end, it is possible that many of the 99 expected to survive might decide not to pay. It would be difficult and expensive for the insurer to pursue payment. The policyholders have benefited collectively from the insurance and the insurer has not been appropriately compensated.

1.4 (a) Without term insurance, the homeowner's dependents may struggle to meet mortgage payments in the even of the homeowner's death. The lending company wishes to reduce as far as possible the risk of having to foreclose on the loan. Foreclosure is expensive and creates hardship for the homeowner's family at the worst possible time. Term insurance is used to pay off the mortgage balance in the event of the homeowner's death, thus avoiding the foreclosure risk for both the lender and homeowner's family.

(b) If the homeowner is paying regular instalments of capital and interest to pay off the mortgage, then the term insurance sum insured will decrease as the loan outstanding decreases. The reduction in loan outstanding is slow in the early years of, say, a 25-year mortgage, but speeds up later. The reduction in the term insurance sum insured is therefore not linear. Different loan provisions, including interest-only loan periods, cliff-edge repayment schedules (where the interest is very low for some period and then increases substantially), fixed or variable interest rates, fixed or variable repayment instalments will all affect the sum insured.

(c) In Section 1.3.5 it is noted that around 2% of applicants for insurance are considered to be too high risk. If these lives are, in consequence, unable to purchase property, then that is a social cost for these lives that may not be acceptable.

1.5 In with-profit whole life insurance, the insurer invests the premiums, and excess investment returns over the minimum required to fund the original benefits are shared between the policyholders and the insurer.

With a cash bonus, the policyholder's share of profits can be paid out in cash, similar to a dividend on shares. In this case, the investments need to be realized (i.e. assets sold for cash). The payout is immediate.

With a reversionary bonus, the policyholder's share of profits is used to increase the sum insured. The assets can remain in the capital markets until the sum insured is due.

Cash Bonus System – Insurer Perspective
Advantages

- Bonuses are transparent and easy to explain to policyholders.

- It does not involve maintenance of records of payouts and does not impact schedules for surrender values.
- The prospect of cash bonuses may persuade policyholders to continue with their policies rather than surrender.

Disadvantages

- It creates a liquidity risk – that assets need to be sold to meet bonus expectations, possibly at unfavourable times.
- Investment proceeds are volatile; volatility in cash bonuses may be difficult to explain to policyholders. There may be a temptation to over-distribute in an attempt to smooth, that could cause long-term losses.
- There may be problems determining equitable payouts, resulting in possible policyholder grievances.

Cash Bonus System – Policyholder Perspective
Advantages

- Cash is immediate and it is easy to understand the distribution.

Disadvantages

- May not be tax efficient.
- The risks to the insurer may lead to under-distribution to avoid risk.
- Possible volatility of bonuses.

Reversionary Bonus System – Insurer Perspective
Advantages

- Assets remain invested as long as a policy is in force, reducing liquidity risk.
- Bonuses appear larger as they are generally delayed many years.
- Bonuses may not be paid in full if a policy is surrendered subsequently, allowing higher rates of bonus to be declared for remaining policyholders.
- Over-distribution can be mitigated with lower bonuses between the declaration year and the claim event.

Disadvantages

- More complex to value, to keep records.

- Policyholders may not understand the approach, and there may be resentment (e.g. on surrender).
- Difficult to determine an equitable distribution.
- Easy to over-declare, as profits are based on asset values which may subsequently decrease.
- It is difficult to reduce bonus rates, even when justified. This may lead to loss of new and existing business.

Reversionary Bonus System – Policyholder Perspective
Advantages

- It may be tax efficient to receive profit share with sum insured.
- The system allows more investment freedom for the insurer, with higher upside potential for the policyholder.

Disadvantages

- Difficult to understand, especially 'super-compound' systems.
- Possible loss of profit share on surrender.
- Opaque system of distribution. It is difficult to compare how different companies perform.

1.6 Insurers prefer policies to remain in force, as their profits from long-term business arise largely from the interest spread, which is the difference between the interest earned on the accumulated premiums, and the interest needed to support the benefits. After age 80 few policyholders will be receiving salary, so there is greater risk that the premiums will not be affordable. Policyholders may then surrender their policies, cutting off the profit stream to the insurer. By designing the contract such that no premiums are due after age 80, the insurer increases significantly the proportion of policies that remain in force at that time, which we call the **persistency**.

1.7 For a comprehensive answer, we need to understand Andrew's age, health and family responsibilities and support. The answers for an average 65-year old retiree in good health would be different from those for a 50-year old retiree in poor health. Also, we should consider the impact of governmental benefits (old age pension, social security, health costs), and any potential support from family in the event that he faces financial ruin.

In the absence of more detailed information, we assume that Andrew is a person in average health at an average retirement age of, say, 65. We also assume that the $500 000 represents the capital on which he wishes to live reasonably comfortably for the remainder of his life. We also ignore tax issues, though these are likely to be very significant in this kind of decision in practice.

Consider the risks Andrew faces at retirement.

(1) Outliving his assets – this is the risk that at some point the funds are all spent and Andrew must live on whatever government benefit or family support that might be available.

(2) Inflation risk – that is, that his standard of living is gradually eroded by increases in the cost of living that are not matched by increases in his income.

(3) Catastrophe costs – this is the risk that a large liability arises and Andrew does not have the assets (or cannot access the assets) to meet the costs. Examples might include the cost of health care for Andrew or a dependent (where health care is not freely available); catastrophic uninsured liability; cost of long-term care in older age.

Andrew may also have some 'wants' – for example

(1) Bequest – Andrew may want to leave some assets to dependents if possible.

(2) Flexible spending – Andrew may want the freedom of full access to all his capital at all times.

We now consider the options listed in the question in light of the risks and potential 'wants' listed.

(a) With a level life annuity, Andrew is assured of income for his whole life, and eliminates the risk of outliving his assets. However, he retains the inflation risk, and he may not have sufficient assets to meet catastrophe costs. If he uses all his capital for an annuity, there will be no bequest funds available on his death, and no flexibility in spending during his lifetime.

(b) As in (a), Andrew will not outlive his assets, and this option also covers inflation risk to some extent. There may be some residual inflation risk, as the cost of living increases that Andrew is exposed to may differ from the inflation adjustments applied to his annuity. In order to purchase the cost of living cover, Andrew will receive a significantly lower starting annual payment than under option (a). All other issues are similar to those under option (a).

(c) A 20-year annuity-certain will offer a similar or slightly higher benefit to a life annuity for a 65-year old man in average health. Andrew's life expectancy might be around 18 years, so on average the annuity will be sufficient to give Andrew a life income and allow a small bequest. An annuity-certain can be reasonably easily converted to cash in the event of a catastrophe or a change in circumstances. However, there is a significant risk that Andrew will live more than 20 years, and it will be difficult to manage the dramatic change in income at such an advanced age.

(d) Investing the capital and living off the interest would involve much risk. The interest income will be highly variable, and will be insufficient to live on in some years. If Andrew invests the capital in safe, stable long-term bonds, he might make only 2–3% after expenses (or less, this figure has been highly variable over the last 20 years) which would be insufficient if it is his only income. There is also reinvestment risk, as he could live longer than the longest income he could lock-in in the market, and there will be counterparty risk (that is, the risk that the borrower will default on the interest and capital) if his investment is not in solid risk-free assets.

If Andrew needs a higher income, he will have to take more risk. For example, he might invest in corporate bonds with counterparty risk, or he might put some of his capital in stocks, which have upside potential but downside risk. Using riskier investments would increase the volatility of his income and threaten his capital. If he invests heavily in shares, he may see negative returns in some years. This strategy just might not be sustainable.

Income would also not be inflation hedged, in general.

On the other hand, the capital would be accessible in the event of a catastrophe or for flexible spending (although that would raise the risk of outliving assets). This system would allow for a significant bequest, assuming that Andrew managed to live on the investment, but at the expense of income level and stability for Andrew. Also, Andrew would have the added complication of managing a portfolio of assets, or paying someone to manage them for him. On the other hand, purchasing an annuity involves substantial hidden expenses that would not be incurred under this option.

(e) \$40 000 is 8% of the capital. If this rate is higher than the interest rate achievable on capital, then Andrew will be drawing down the capital and risks outliving his assets. The income is not inflation hedged, but the system does allow spending flexibility. Other issues are as for option (d).

Solutions for Chapter 2

2.1 (a) $F_0(60) = 1 - \left(1 - \frac{60}{105}\right)^{1/5} = 0.1559.$

(b) $S_0(70)/S_0(30) = 0.8586.$

(c) $(S_0(90) - S_0(100))/S_0(20) = 0.1394.$

(d) We may use either

$$\mu_x = -\frac{1}{S_0(x)}\frac{d}{dx}S_0(x)$$

or

$$\mu_x = -\frac{d}{dx}\log S_0(x) = -\frac{d}{dx}\frac{1}{5}\log\left(1 - \frac{x}{105}\right) = \frac{1}{525-5x},$$

so $\mu_{50} = 0.0036.$

(e) We must solve

$$\frac{S_0(50+t)}{S_0(50)} = \frac{1}{2}$$

which is the same as

$$\left(1 - \frac{t}{55}\right)^{1/5} = \frac{1}{2}.$$

This gives $t = 53.28.$

(f) We have

$$\mathring{e}_{50} = \int_0^{55} {}_tp_{50}\,dt = \int_0^{55}\left(1 - \frac{t}{55}\right)^{1/5} dt = 45.83.$$

(g) We have

$$e_{50} = \sum_{t=1}^{54} {}_tp_{50} = \sum_{t=1}^{54} \left(1 - \frac{t}{55}\right)^{1/5} = 45.18.$$

2.2 (a) $G(x)$ can be written as

$$G(x) = \frac{(90-x)(x+200)}{18\,000}$$

and since $G(\omega) = 0$ at the limiting age (and $x > 0$), $\omega = 90$.

(b) First, we have $G(0) = 1$. Next, setting $x = 90$ we see that the function equals 0 at the limiting age. Third, the derivative of $G(x)$ is

$$\frac{-110-2x}{18\,000}$$

which is negative for $x > 0$. Hence all three conditions for a survival function are satisfied.

(c) $S_0(20)/S_0(0) = 0.8556$.

(d) The survival function is

$$\begin{aligned} S_{20}(t) &= \frac{S_0(20+t)}{S_0(20)} \\ &= \frac{18\,000 - 110(20+t) - (20+t)^2}{18\,000 - 110(20) - 20^2} \\ &= \frac{15\,400 - 150\,t - t^2}{15\,400}. \end{aligned}$$

(e) $(S_0(30) - S_0(40))/S_0(20) = 0.1169$.

(f) $\mu_x = -S_0'(x)/S_0(x)$. Using part (b) we obtain

$$\begin{aligned} \mu_x &= \left(\frac{110+2x}{18\,000}\right)\left(\frac{18\,000}{18\,000-110x-x^2}\right) \\ &= \frac{110+2x}{18\,000-110x-x^2} \end{aligned}$$

so that $\mu_{50} = 0.021$.

2.3 The required probability, ${}_{19|17}q_0$ in actuarial notation, is equal to

$$S_0(19) - S_0(36) = \frac{1}{10}\left(\sqrt{81} - \sqrt{64}\right) = 0.1.$$