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PART I

Lagrangian mechanics

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Fundamental concepts

This book is about Lagrangians and Hamiltonians. To state it more formally, this book is about the variational approach to analytical mechanics. You may not have been exposed to the calculus of variations, or may have forgotten what you once knew about it, so I am not assuming that you know what I mean by, "the variational approach to analytical mechanics." But I think that by the time you have worked through the first two chapters, you will have a good grasp of the concept.

We begin with a review of introductory concepts and an overview of background material. Some of the concepts presented in this chapter will be familiar from your introductory and intermediate mechanics courses. However, you will also encounter several new concepts that will be useful in developing an understanding of advanced analytical mechanics.

1.1 Kinematics

A particle is a material body having mass but no spatial extent. Geometrically, it is a point. The position of a particle is usually specified by the vector $\mathbf{r}$ from the origin of a coordinate system to the particle. We can assume the coordinate system is *inertial* and for the sake of familiarity you may suppose the coordinate system is Cartesian. See Figure 1.1.

The velocity of a particle is defined as the time rate of change of its position and the acceleration of a particle is defined as the time rate of change of its velocity. That is,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}, \tag{1.1}$$

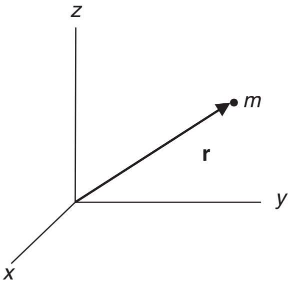

Figure 1.1 The position of a particle is specified by the vector **r**.

and

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}. \tag{1.2}$$

The equation for **a** as a function of $\mathbf{r}, \dot{\mathbf{r}}$, and t is called the equation of motion. The equation of motion is a second-order differential equation whose solution gives the position as a function of time, $\mathbf{r} = \mathbf{r}(t)$. The equation of motion can be solved numerically for any reasonable expression for the acceleration and can be solved analytically for a few expressions for the acceleration. You may be surprised to hear that the only general technique for solving the equation of motion is the procedure embodied in the Hamilton–Jacobi equation that we will consider in Chapter 6. All of the solutions you have been exposed to previously are special cases involving very simple accelerations.

Typical problems, familiar to you from your introductory physics course, involve falling bodies or the motion of a projectile. You will recall that the projectile problem is two-dimensional because the motion takes place in a plane. It is usually described in Cartesian coordinates.

Another important example of two-dimensional motion is that of a particle moving in a circular or elliptic path, such as a planet orbiting the Sun. Since the motion is planar, the position of the body can be specified by two coordinates. These are frequently the plane polar coordinates (r, θ) whose relation to Cartesian coordinates is given by the *transformation equations*

$$x = r\cos\theta,$$
$$y = r\sin\theta.$$

This is an example of a *point transformation* in which a point in the xy plane is mapped to a point in the $r\theta$ plane.

You will recall that in polar coordinates the acceleration vector can be resolved into a *radial* component and an *azimuthal* component

$$\mathbf{a} = \ddot{\mathbf{r}} = a_r \hat{\mathbf{r}} + a_\theta \hat{\boldsymbol{\theta}}. \tag{1.3}$$

Exercise 1.1 A particle is given an impulse which imparts to it a velocity v_0. It then undergoes an acceleration given by $a = -bv$, where b is a constant and v is the velocity. Obtain expressions for $v = v(t)$ and $x = x(t)$. (This is a one-dimensional problem.) Answer: $x(t) = x_0 + (v_0/b)\left(1 - e^{-bt}\right)$.

Exercise 1.2 Assume the acceleration of a body of mass m is given by $a = -(k/m)x$. (a) Write the equation of motion. (b) Solve the equation of motion. (c) Determine the values of the arbitrary constants (or constants of integration) if the object is released from rest at $x = A$. (The motion is called "simple harmonic.") Answer (c): $x = A \sin\left(\sqrt{k/m}t + \pi/2\right) = A \cos \sqrt{k/m}t$.

Exercise 1.3 In plane polar coordinates, the position is given by $\mathbf{r} = r\hat{\mathbf{r}}$. Obtain expressions for the velocity and acceleration in terms of $r, \theta, \hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$. (Hint: Express $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ in terms of $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$.) Answer: $\mathbf{a} = (\ddot{\mathbf{r}} - r\dot{\theta}^2)\hat{\mathbf{r}} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\boldsymbol{\theta}}$.

1.2 Generalized coordinates

In the preceding section we stated that the position of the particle was given by the vector $\mathbf{r}$. We assumed an inertial Cartesian coordinate system in which the components of $\mathbf{r}$ were (x, y, z). Of course, there are many other ways we could have specified the position of the particle. Some ways that immediately come to mind are to give the components of the vector $\mathbf{r}$ in cylindrical (ρ, ϕ, z) or in spherical coordinates (r, θ, ϕ). Obviously, a particular problem can be formulated in terms of many different sets of coordinates.

In three-dimensional space we need three coordinates to specify the position of a single particle. For a system consisting of two particles, we need six coordinates. A system of N particles requires $3N$ coordinates. In Cartesian coordinates, the positions of two particles might be described by the set of numbers $(x_1, y_1, z_1, x_2, y_2, z_2)$. For N particles the positions of all the particles are

$$(x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_N, y_N, z_N).$$

As you know, some problems are more easily solved in one coordinate system and some are more easily solved in another. To avoid being specific about the coordinate system we are using, we shall denote the coordinates by q_i and the corresponding velocities by $\dot{q}_i$. We call q_i the "generalized coordinates." Of course, for any particular problem you will chose an appropriate set of coordinates; in one problem you might use spherical coordinates in which case as i ranges from 1 to 3, the coordinates q_i take on the values r, θ, ϕ, whereas in another problem you might use cylindrical coordinates and $q_i = \rho, \phi, z$. To convert from Cartesian coordinates to generalized coordinates, you need to know the transformation equations, that is, you need to know the relations

$$\begin{aligned} q_1 &= q_1(x_1, y_1, z_1, x_2, y_2, \ldots, z_N, t) \\ q_2 &= q_2(x_1, y_1, z_1, x_2, y_2, \ldots, z_N, t) \\ &\vdots \\ q_{3N} &= q_{3N}(x_1, y_1, z_1, x_2, y_2, \ldots, z_N, t). \end{aligned} \tag{1.4}$$

The inverse relations are also called transformation equations. For a system of N particles we have

$$\begin{aligned} x_1 &= x_1(q_1, q_2, \ldots, q_{3N}, t) \\ y_1 &= y_2(q_1, q_2, \ldots, q_{3N}, t) \\ &\vdots \\ z_N &= z_N(q_1, q_2, \ldots, q_{3N}, t). \end{aligned} \tag{1.5}$$

It is often convenient to denote all the Cartesian coordinates by the letter x. Thus, for a single particle, (x, y, z) is written (x_1, x_2, x_3) and, for N particles, $(x_1, y_1, \ldots, z_N)$ is expressed as $(x_1, \ldots, x_n)$, where $n = 3N$.

We usually assume that given the transformation equations from the qs to the xs, we can carry out the inverse transformation from the xs to the qs, but this is not always possible. The inverse transformation is possible if the Jacobian determinant of Equations (1.4) is not zero. That is,

$$\frac{\partial(q_1, q_2, \ldots, q_n)}{\partial\,(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \partial q_1/\partial x_1 & \partial q_1/\partial x_2 & \cdots & \partial q_1/\partial x_n \\ & & \vdots & \\ \partial q_n/\partial x_1 & \partial q_n/\partial x_2 & \cdots & \partial q_n/\partial x_n \end{vmatrix} \neq 0.$$

The generalized coordinates are usually assumed to be *linearly independent,* and thus form a minimal set of coordinates to describe a problem. For example, the position of a particle on the surface of a sphere of radius a can be described in terms of two angles (such as the longitude and latitude). These two angles form a minimal set of linearly independent coordinates. However, we could also describe the position of the particle in terms of the three Cartesian coordinates, x, y, z. Clearly, this is not a minimal set. The reason is that the Cartesian coordinates are not all independent, being related by $x^2 + y^2 + z^2 = a^2$. Note that given x and y, the coordinate z is determined. Such a relationship is called a "constraint." We find that each equation of constraint reduces by one the number of independent coordinates.

Although the Cartesian coordinates have the property of being components of a vector, this is not necessarily true of the generalized coordinates. Thus, in our example of a particle on a sphere, two angles were an appropriate set of generalized coordinates, but they are not components of a vector. In fact, generalized coordinates need not even be *coordinates* in the usual sense of the word. We shall see later that in some cases the generalized coordinates can be components of the momentum or even quantities that have no physical interpretation.

When you are actually solving a problem, you will use Cartesian coordinates or cylindrical coordinates, or whatever coordinate system is most convenient for the particular problem. But for theoretical work one nearly always expresses the problem in terms of the generalized coordinates $(q_1, \ldots, q_n)$. As you will see, the concept of generalized coordinates is much more than a notational device.

Exercise 1.4 Obtain the transformation equations for Cartesian coordinates to spherical coordinates. Evaluate the Jacobian determinant for this transformation. Show how the volume elements in the two coordinate systems are related to the Jacobian determinant. Answer: $d\tau = r^2 \sin\theta dr d\theta d\phi$.

1.3 Generalized velocity

As mentioned, the position of a particle at a particular point in space can be specified either in terms of Cartesian coordinates (x_i) or in terms of generalized coordinates (q_i). These are related to one another through a transformation equation:

$$x_i = x_i(q_1, q_2, q_3, t).$$

Note that x_i, which is one of the three Cartesian coordinates of a specific particle, depends (in general) on *all* the generalized coordinates.

We now determine the components of velocity in terms of generalized coordinates.

The velocity of a particle in the x_i direction is v_i and by definition

$$v_i \equiv \frac{dx_i}{dt}.$$

But x_i is a function of the qs, so using the chain rule of differentiation we have

$$v_i = \sum_{k=1}^{3} \frac{\partial x_i}{\partial q_k}\frac{dq_k}{dt} + \frac{\partial x_i}{\partial t} = \sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial x_i}{\partial t}. \tag{1.6}$$

We now obtain a very useful relation involving the partial derivative of v_i with respect to $\dot{q}_j$. Since a mixed second-order partial derivative does not depend on the order in which the derivatives are taken, we can write

$$\frac{\partial}{\partial \dot{q}_j}\frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t}\frac{\partial x_i}{\partial \dot{q}_j}.$$

But $\frac{\partial x_i}{\partial \dot{q}_j} = 0$ because x does not depend on the generalized velocity $\dot{q}$. Let us take the partial derivative of v_i with respect to $\dot{q}_j$ using Equation (1.6). This yields

$$\begin{aligned}
\frac{\partial v_i}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j}\sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + \frac{\partial}{\partial \dot{q}_j}\frac{\partial x_i}{\partial t} \\
&= \frac{\partial}{\partial \dot{q}_j}\sum_k \frac{\partial x_i}{\partial q_k}\dot{q}_k + 0 \\
&= \sum_k \left(\frac{\partial}{\partial \dot{q}_j}\frac{\partial x_i}{\partial q_k}\right)\dot{q}_k + \sum_k \frac{\partial x_i}{\partial q_k}\frac{\partial \dot{q}_k}{\partial \dot{q}_j} = 0 + \sum_k \frac{\partial x_i}{\partial q_k}\delta_{ij} \\
&= \frac{\partial x_i}{\partial q_j}.
\end{aligned}$$

Thus, we conclude that

$$\frac{\partial v_i}{\partial \dot{q}_j} = \frac{\partial x_i}{\partial q_j}. \tag{1.7}$$

This simple relationship comes in very handy in many derivations concerning generalized coordinates. Fortunately, it is easy to remember because it states that *the Cartesian velocity is related to the generalized velocity in the same way as the Cartesian coordinate is related to the generalized coordinate.*

1.4 Constraints

Every physical system has a particular number of *degrees of freedom*. The number of degrees of freedom is the number of independent coordinates needed to completely specify the position of every part of the system. To describe the position of a free particle one must specify the values of three coordinates (say, x, y and z). Thus a free particle has three degrees of freedom. For a system of two free particles you need to specify the positions of both particles. Each particle has three degrees of freedom, so the system as a whole has six degrees of freedom. In general a mechanical system consisting of N free particles will have $3N$ degrees of freedom.

When a system is acted upon by forces of constraint, it is often possible to reduce the number of coordinates required to describe the motion. Thus, for example, the motion of a particle on a table can be described in terms of x and y, the constraint being that the z coordinate (defined to be perpendicular to the table) is given by $z =$ constant. The motion of a particle on the surface of a sphere can be described in terms of two angles, and the constraint itself can be expressed by $r =$ constant. *Each constraint reduces by one the number of degrees of freedom.*

In many problems the system is constrained in some way. Examples are a marble rolling on the surface of a table or a bead sliding along a wire. In these problems the particle is not completely free. There are forces acting on it which restrict its motion. If a hockey puck is sliding on smooth ice, the gravitational force is acting downward and the normal force is acting upward. If the puck leaves the surface of the ice, the normal force ceases to act and the gravitational force quickly brings it back down to the surface. At the surface, the normal force prevents the puck from continuing to move downward in the vertical direction. The force exerted by the surface on the particle is called a *force of constraint*.

If a system of N particles is acted upon by k constraints, then the number of generalized coordinates needed to describe the motion is $3N - k$. (The number of Cartesian coordinates is always $3N$, but the number of (independent) generalized coordinates is $3N - k$.)

A constraint is a relationship between the coordinates. For example, if a particle is constrained to the surface of the paraboloid formed by rotating a parabola about the z axis, then the coordinates of the particle are related by $z - x^2/a - y^2/b = 0$. Similarly, a particle constrained to the surface of a sphere has coordinates that are related by $x^2 + y^2 + z^2 - a^2 = 0$.

If the equation of constraint (the relationship between the coordinates) can be expressed in the form

$$f(q_1, q_2, \ldots, q_n, t) = 0, \tag{1.8}$$

then the constraint is called *holonomic*.[1]

The key elements in the definition of a holonomic constraint are: (1) the equals sign and (2) it is a relation involving the *coordinates*. For example, a possible constraint is that a particle is always outside of a sphere of radius a. This constraint could be expressed as $r \geq a$. This is *not* a holonomic constraint. Sometimes a constraint involves not just coordinates but also velocities or differentials of the coordinates. Such constraints are also not holonomic.

As an example of a non-holonomic constraint consider a marble rolling on a perfectly rough table. The marble requires five coordinates to completely describe its position and orientation, two linear coordinates to give its position on the table top and three angle coordinates to describe its orientation. If the table top is perfectly smooth, the marble can *slip* and there is no relationship between the linear and angular coordinates. If, however, the table top is rough there are relations (constraints) between the angle coordinates and the linear coordinates. These constraints will have the general form $dr = a d\theta$, which is a relation between differentials. If such differential expressions can be integrated, then the constraint becomes a relation between coordinates, and it is holonomic. But, in general, rolling on the surface of a plane does not lead to integrable relations. That is, in general, the *rolling* constraint is not holonomic because the rolling constraint is a relationship between *differentials*. The equation of constraint does not involve *only* coordinates. (But rolling in a straight line is integrable and hence holonomic.)

A holonomic constraint is an equation of the form of (1.8) relating the generalized coordinates, and it can be used to express one coordinate in terms of the others, thus reducing the number of coordinates required to describe the motion.

Exercise 1.5 A bead slides on a wire that is moving through space in a complicated way. Is the constraint holonomic? Is it scleronomous?

[1] A constraint that does not contain the time explicitly is called *scleronomous*. Thus $x^2 + y^2 + z^2 - a^2 = 0$ is both holonomic and scleronomous.