OF

FUNCTIONAL EQUATIONS.

If a function α is of such a form, that, when it is twice performed on a quantity, the result is the quantity itself, or if $\alpha^2(x) = x$, then it is called a periodic function of the second order, if $\alpha^n(x) = x$, then it is termed a periodic function of the n^{th} order, thus when $\alpha(x) = a - x$ the second function, or

$$\alpha(\alpha x) = \alpha(a - x) = a - (a - x) = a - a + x = x.$$

If $\alpha(x) = \dfrac{1}{1-x}$,

then $\alpha^2 x = \alpha(\alpha x) = \dfrac{1}{1 - \dfrac{1}{1-x}} = \dfrac{1-x}{1-x-1} = \dfrac{x-1}{x}$,

and

$$\alpha^3 x = \alpha^2 \alpha x = \frac{\alpha x - 1}{\alpha x} = \frac{\dfrac{1}{1-x} - 1}{\dfrac{1}{1-x}} = 1 - \overline{1 - x} = x,$$

the first of these examples is a periodic function of the second, the last is a periodic function of the third order.

Prob. 1. To find periodic functions of the second order.

Since such functions must satisfy the equation $\psi^2 x = x$, we have

$$\psi x = \psi^{-1} x,$$

or ψ must be such a function, that it shall be the same as its inverse; if therefore $y = \psi x$, we have also $x = \psi^{-1} y = \psi y$, or if x and y are connected by some equation, it must be symmetrical relative to x and y; y or ψx must then be determined from the equation

$$* F\{\bar{x}, \overline{\psi x}\} = 0,$$

for instance, if $x + \psi x - a = 0$, $\psi x = a - x$,

or if $x \psi x = a^2$, $\psi x = \frac{a^2}{x}$.

Another method of determining such functions is as follows: since ψx is of such a form that $\psi^2 x = x$ any symmetrical function of x and ψx remain constant when x is changed into ψx thus

$$F\{\bar{x}, \overline{\psi x}\} \text{ becomes } F\{\overline{\psi x}, \overline{\psi^2 x}\} = F\{\overline{\psi x}, \bar{x}\},$$

if therefore, we can find any particular solution of the equation $\psi^2 x = x$, containing an arbitrary constant we may substitute such a function for it, but $\psi x = a - x$ is a particular solution therefore

$$\psi x = F(\bar{x}, \overline{\psi x}) - x,$$

or

$$x + \psi x = F(\bar{x}, \overline{\psi x}),$$

and by changing the arbitrary function into another of the same form, we find

$$F\{\bar{x}, \overline{\psi x}\} = 0,$$

as before.

These two methods of determining periodic functions of the second order, are not so convenient as a third process which can be extended to all orders.

* Bars placed above quantities under the functional sign, indicate that the function is symmetrical relative to those quantities.

Assume $\psi x = \phi^{-1} f \phi x$, then

$$\psi^2 x = \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^2 \phi x,$$

this must be equal to x or

$$\phi^{-1} f^2 \phi x = x,$$

this equation will be fulfilled if $f^2 v = v$, or if f is a particular solution, and if also ϕ^{-1} is such an inverse function that $\phi^{-1} \phi v = v$. If therefore ϕ is arbitrary, and f is a particular solution of $f^2 x = x$, then the solution of $\psi^2 x = x$ is

$$\psi x = \phi^{-1} f \phi x.$$

Ex. Let $f x = \frac{a}{x}$, then $\psi x = \phi^{-1} \left(\frac{a}{\phi x}\right)$,

if $f(x) = \frac{a - b x}{b + c x}$, $\quad \psi x = \phi^{-1} \left(\frac{a - b \phi x}{b + c \phi x}\right)$;

from these may easily be derived the following periodic functions of the second order,

$$\psi x = a - x \qquad \psi x = \frac{x}{x - 1}$$

$$\psi x = \frac{x - 2}{x - 1} \qquad \psi x = \frac{a^2}{x}$$

$$\psi x = \frac{1 - x}{1 + x} \qquad \psi x = \sqrt{1 - x^2}$$

$$\psi x = \frac{x + 1}{x - 1} \qquad \psi x = \frac{x}{\sqrt{x^2 - 1}},$$

$$\psi x = \tan^{-1} \left(\frac{\sin (a - x)}{\cos a . \cos x}\right) \qquad \psi x = \log (a - \varepsilon^x)$$

$$\psi x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi x = x - \log (\varepsilon^x - 1)$$

$$\psi x = \frac{x}{(x^n - a^n)^{\frac{1}{n}}} \qquad \psi x = \tan^{-1} (a - \tan x)$$

PROB. 2. Required periodic functions of the third order, or such as fulfil the equation $\psi^3 x = x$.

Assume $\psi x = \phi^{-1} f \phi x$, then the equation becomes

$$\psi^3 x = \phi^{-1} f \phi \phi^{-1} f \phi \phi^{-1} f \phi x = \phi^{-1} f^3 \phi x = x,$$

which will be verified if $f(v)$ is a particular solution of $f^3 v = v$, and if ϕ^{-1} is such an inverse value that $\phi^{-1} \phi v = v$, hence the solution of the equation is

$$\psi x = \phi^{-1} f \phi x,$$

one solution is $\frac{1}{1-x}$ and hence $\psi x = \phi^{-1}\left(\frac{1}{1-\phi x}\right)$ more particular cases are

$$\psi x = \frac{a^2}{a - x} \qquad \psi x = \frac{1 + x}{1 - 3x}$$

$$\psi x = \frac{a^2}{ac - c^2 x} \qquad \psi x = \frac{\sqrt{a x^2 - a^2}}{x}$$

$$\psi x = \frac{a x - a^2}{x} \qquad \psi x = \frac{1}{1 - x}$$

$$\psi x = \left(\frac{a^2}{a - x^n}\right)^{\frac{1}{n}} \qquad \psi x = -\log(1 - \varepsilon^x)$$

$$\psi x = \frac{(a x^n - a^2)^{\frac{1}{n}}}{x} \qquad \psi x = \log(a \varepsilon^x - a^2) - x$$

$$\psi x = \frac{a + b x}{c - \frac{a^2 + bc + c^2 x}{a}} \qquad \psi x = \log(\varepsilon^x - \varepsilon^c) - x + c.$$

PROB. 3. To find periodic functions of the n^{th} order, or to solve the equation $\psi^n x = x$.

Assume as before $\psi x = \phi^{-1} f \phi x$ then it becomes

$$\phi^{-1} f \phi \phi^{-1} f \phi \ldots \phi^{-1} f \phi x = \phi^{-1} f^n \phi x = x,$$

which is verified if f is a particular solution of $f^n x = x$, and if ϕ^{-1} is such an inverse function that $\phi^{-1}\phi x = x$.

It now remains to find particular solutions of $\psi^n x = x$ which may be accomplished in the following manner: let fx represent $\frac{a+bx}{c+dx}$ then the n^{th} function will be of the same form, or

$$f^n(x) = \frac{A_n + B_n x}{C_n + D_n x},$$

where A_n, B_n, C_n, D_n, are functions of a, b, c, d, and n, these may be so determined that $D_n = 0$, $A_n = 0$ and $B_n = C_n$ all which conditions are satisfied, if

$$d = -\frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}$$

hence

$$\phi x = \phi^{-1}\left\{\frac{a + b\phi x}{c - \frac{b^2 - 2bc\cos\frac{2k\pi}{n} + c^2}{\left(2 + 2\cos\frac{2k\pi}{n}\right)a}\phi x}\right\}$$

a more detailed account of this method of solution may be found in a paper by Mr. Horner in the Annals of Philosophy, Nov. 1817.

Instances of $\psi^4 x = x$ are

$$\psi x = \frac{1}{2}\,\frac{1}{1-x} \qquad \psi x = \frac{1+x}{1-x}$$

$$\psi x = \frac{2}{2-x} \qquad \psi x = \frac{2a^2}{2ac - c^2 x}$$

$$\psi x = 2\,\frac{x-1}{x} \qquad \psi x = \frac{a+bx}{c - \frac{b^2+c^2}{2a}x}$$

$$\psi x = \frac{\sqrt{2}}{\sqrt{2 - x^2}} \qquad \psi x = -\sqrt{\frac{1 - x^2}{1 + x^2}}$$

$$\psi x = \frac{(2 x^n - 2)^{\frac{1}{n}}}{x}$$

$$\psi x = \log 2 - x + \log (\varepsilon^x - 1).$$

All those cases which satisfy the equation $\psi^2 x = x$, also fulfil that of $\psi^4 x = x$, as well as all those which fulfil any of these equations $\psi^2 x = - x$, $\psi^2 x = \frac{1}{x}$, or more generally $\psi^2 x = \alpha x$, where αx is a particular solution of the equation $\psi^2 x = x$.

The following particular cases satisfy the equation $\psi^6 x = x$.

$$\psi x = \frac{1}{3(1 - x)} \qquad \psi x = \frac{3x - 1}{3x}$$

$$\psi x = \frac{3}{3 - x} \qquad \psi x = \frac{3a^2}{3ac - c^2 x}.$$

$$\psi x = 3\frac{x - 1}{x} \qquad \psi x = \frac{3 + 3x}{3 - x}$$

$$\psi x = \frac{a + bx}{c - \frac{b^2 - bc + c^2}{3a} x}$$

$$\psi x = \frac{1}{x}\left(x^n - \frac{1}{3}\right)^{\frac{1}{n}}$$

$$\psi x = \log 3 - x + \log (\varepsilon^x - 1).$$

The principle on which the solution of the functional equation $F\{x, \psi x, \psi \alpha x\} = 0$ depends, where $\alpha^2 x = x$, is that by substituting αx for x we have another equation $F\{\alpha x, \psi \alpha x, \psi x\} = 0$, between which and the given equation we may eliminate $\psi \alpha x$ and the result will be the value of ψx a few examples will illustrate this method.

(1). Given $\psi(x) + a\,\psi(-x) = x^n$

by putting $-x$ for x this becomes

$$\psi(-x) + a\,\psi(x) = (-x)^n,$$

and eliminating $\psi(-x)$, we have

$$\psi x - a^2\,\psi x = x^n - a(-x)^n,$$

hence

$$\psi x = \frac{1 - (-1)^n a}{1 - a^2} x^n.$$

(2). Given $\psi x - a\,\psi \frac{1}{x} = \varepsilon^x$

put $\frac{1}{x}$ for x, $\psi \frac{1}{x} - a\,\psi x = \varepsilon^{\frac{1}{x}}$

and

$$\psi x - a\,\varepsilon^{\frac{1}{x}} - a^2\,\psi x = \varepsilon^x,$$

$$\psi x = \frac{\varepsilon^x + a\,\varepsilon^{\frac{1}{x}}}{1 - a^2}$$

(3). Given $(\psi x)^2 . \psi \frac{1-x}{1+x} = c^2 x$

put $\frac{1-x}{1+x}$ for x, it becomes

$$\left(\psi \frac{1-x}{1+x}\right)^2 . \psi x = c^2 \frac{1-x}{1+x},$$

eliminating $\psi \frac{1-x}{1+x}$ by means of the former, we find

$$\psi x = \left(\frac{1+x}{1-x} c^2 x^2\right)^{\frac{1}{3}}.$$

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(4). Given $\psi x + \frac{1}{1 - x^2} \psi \sqrt{(1 - x^2)} = 1 + x^2$

putting $\sqrt{(1 - x^2)}$ for x, we have

$$\psi \sqrt{(1 - x^2)} + \frac{1}{x^2} \psi x = 2 - x^2,$$

and substituting this value of $\psi \sqrt{1 - x^2}$ in the former equation

$$\psi x + \frac{2 - x^2}{1 - x^2} - \frac{1}{x^2 - x^4} \psi x = 1 + x^2,$$

hence

$$\left(\frac{x^2 - x^4 - 1}{x^2 - x^4}\right) \psi x = 1 + x^2 - \frac{2 - x^2}{1 - x^2} = \frac{-1 + x^2 - x^4}{1 - x^2}$$

and $\psi x = x^2$.

(5). Given $\frac{\psi x}{1 + \psi x} + x \frac{\psi(-x)}{1 + \psi(-x)} = 1$

put $\psi_1 x = \frac{\psi x}{1 + \psi x}$ thus the equation becomes

$\psi_1 x + x \psi_1(-x) = 1$, and changing x into $-x$ we have $\psi_1(-x) - x \psi_1(x) = 1$, by which eliminating $\psi(-x)$ from the former, we find

$$\psi_1 x = \frac{1 - x}{1 + x^2},$$

hence

$$\psi x = \frac{\psi_1 x}{1 - \psi_1 x} = \frac{1 - x}{x + x^2}.$$

(6). Given $\psi x + \frac{1 + x}{x} \psi \frac{1}{x} = c,$

putting $\frac{1}{x}$ for x this becomes

$$\psi \frac{1}{x} + (1 + x)\, \psi x = x$$

and by eliminating $\psi \frac{1}{x}$, we have

$$\psi x = \frac{1}{1 + x + x^2} c$$

(7). Given $\psi x + x \psi (1 - x) = 1$,

putting $1 - x$ for x, we have

$$\psi (1 - x) + (1 - x)\, \psi (x) = 1,$$

whence, by elimination,

$$\psi x = \frac{1 - x}{1 - x(1 - x)} = \frac{1 - x}{1 - x + x^2}.$$

(8). Given $\dfrac{\psi x}{\psi x - x} + x \dfrac{\psi (1 - x)}{\psi (1 - x) + x - 1} = 1$,

put $\psi_1 x = \dfrac{\psi x}{\psi x - x}$, then will $\psi_1 (1 - x) = \dfrac{\psi (1 - x)}{\psi (1 - x) + x \quad 1}$,

and the equation becomes

$$\psi_1 x + x \psi_1 (1 - x) = 1,$$

the same as in the last example; let $f x$ represent the solution there found, then

$$\psi_1 x = f x = \frac{\psi x}{\psi x - x},$$

whence

$$\psi x = \frac{x f x}{f x - 1},$$

if we take for $f x$ its value $\dfrac{1 - x}{1 - x + x^2}$, we have

$$\psi x = \frac{x - 1}{x}.$$

In case the equation is symmetrical with regard to ψx and $\psi \alpha x$, the process of elimination apparently becomes illusory. By a peculiar artifice this difficulty may be overcome, and it happens rather singularly that in all these cases, the solution which is so obtained contains an arbitrary function, and in general the solution is the most extensive which the question admits of.

(9). Given $\psi x = \psi \frac{1}{x}$.

If we put $\frac{1}{x}$ for x, this is changed into $\psi \frac{1}{x} = \psi x$, the same as the given equation; it is therefore impossible to eliminate.

Let us now suppose $\psi x = a \psi \frac{1}{x} + b$,

which becomes the given equation when $a = 1$ and $b = 0$. By putting $\frac{1}{x}$ for x this is changed into

$$\psi \frac{1}{x} = a \psi x + b,$$

and eliminating $\psi \frac{1}{x}$, we have

$$\psi x = \frac{a b + b}{1 - a^2} = \frac{b}{1 - a},$$

if $b = 0$ and $a = 1$, this becomes a vanishing fraction whose value is any constant quantity c, and we have $\psi x = c$, which fulfils the equation. This is a very limited solution, but the following plan will lead us to much more general ones.

Take the equation

$$\psi x = a \psi \frac{1}{x} + v \phi x,$$

which coincides with the given one when $v = 0$ and $a = 1$; also ϕx is any arbitrary function of x; putting $\frac{1}{x}$ for x, we have