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Paul Fannon Vesna Kadelburg Ben Woolley and Stephen Ward
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1 Quadratic functions

In this chapter you will learn:

- about the shape and main features of graphs of quadratic functions
- about the uses of different forms of a quadratic function
- how to solve quadratic equations and simultaneous equations
- how to identify the number of solutions of a quadratic equation
- how to use quadratic functions to solve practical problems.

Introductory problem

A small dairy farmer wants to sell a new type of luxury cheese. After a fixed set-up cost of \$250, he can produce the cheese at a cost of \$9 per kilogram. He is able to produce up to 400 kg, but he plans to take advance orders and produce only what he can sell. His market research suggests that the amount he would be able to sell depends on the price in the following way: the amount decreases proportionally with the price; if he charged \$20 per kg he would not sell any, and if the cheese was free he would 'sell' the maximum 400 kg that he could produce. What price per kilogram should the farmer set in order to maximise his profit?

Problems like this, where we have to maximise or minimise a certain quantity, are known as optimisation problems. They are common in economics and business (e.g. minimising costs and maximising profits), biology (e.g. finding the maximum possible size of a population) and physics (e.g. electrons moving to the lowest energy state). The quadratic function is the simplest function with a maximum or minimum point, so it is often used to model such situations. Quadratic functions are also found in many natural phenomena, such as the motion of a projectile or the dependence of power on voltage in an electric circuit.

1A The quadratic form $y = ax^2 + bx + c$

A **quadratic function** has the general form $y = ax^2 + bx + c$ (where $a \neq 0$). In this chapter we will investigate graphs of quadratic functions and, in particular, how features of the graphs relate to the **coefficients** a, b and c.

A function is a rule that tells you what to do with any value you put in. We will study functions in general in chapter 4, but before then you will learn about some particular types of functions.

$x \in \mathbb{R}$ means that x can be any real number.

See Prior Learning section G on the CD-ROM for the meaning of such statements.

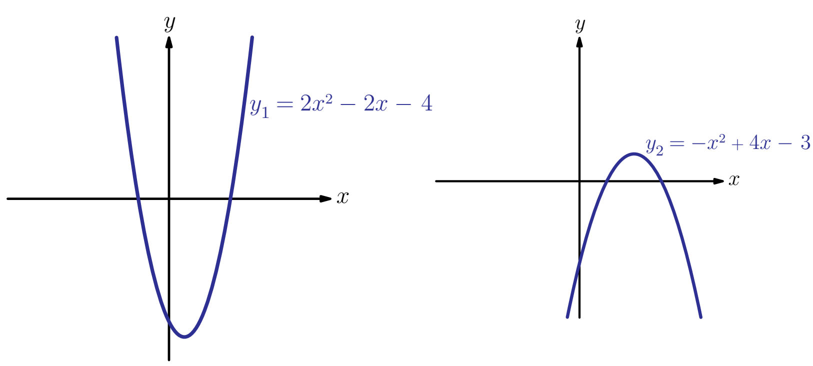
Let us look at two examples of quadratic functions:

$y_1 = 2x^2 - 2x - 4$ and $y_2 = -x^2 + 4x - 3$ $(x \in \mathbb{R})$

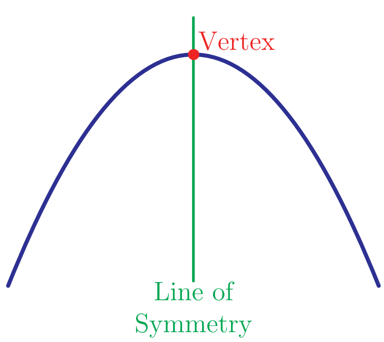
EXAM HINT

See Calculator Skills sheets 2 and 4 on the CD-ROM for how to sketch and analyse graphs on a graphic display calculator.

You can use your calculator to plot the two graphs:



The word 'quadratic' indicates that the term with the highest power in the equation is x^2. It comes from the Latin *quadratus*, meaning 'square'.



These two graphs have a similar shape, called a **parabola**. A parabola has a single turning point (called its vertex) and a vertical line of symmetry passing through the vertex. The most obvious difference between the two graphs above is that the first one has a minimum point whereas the second has a maximum point. This is due to the different signs of the x^2 term in y_1 and in y_2.

You can use your calculator to find the position of the vertex of a parabola. For the graphs above you should find that the coordinates of the vertices are (0.5, –4.5) and (2, 1); the lines of symmetry therefore have equations $x = 0.5$ and $x = 2$.

KEY POINT 1.1

For a quadratic function $f(x) = ax^2 + bx + c$:

If $a > 0$, $f(x)$ is a *positive* quadratic. The graph has a *minimum* point and goes *up* on both sides.

If $a < 0$, $f(x)$ is a *negative* quadratic. The graph has a *maximum* point and goes *down* on both sides.

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The constant coefficient (denoted by c here) gives the position of the y-intercept of the graph, that is, where the curve crosses the y-axis.

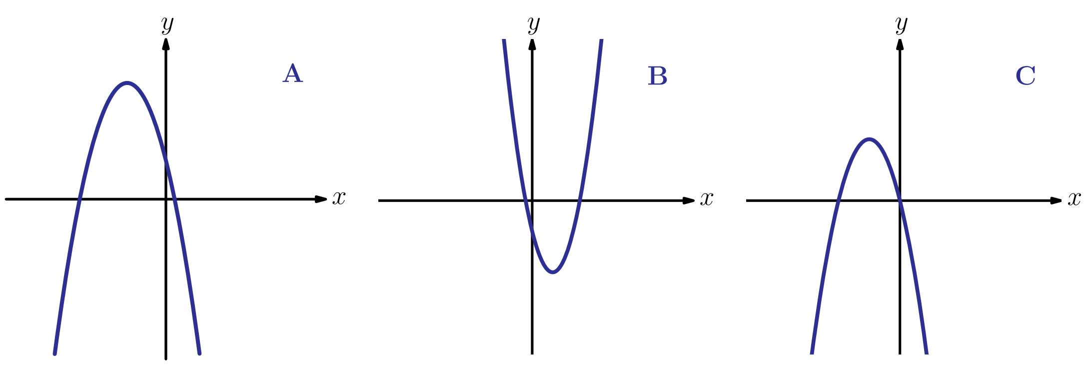
Worked example 1.1

Match each equation to the corresponding graph, explaining your reasons.

(a) $y = 3x^2 - 4x - 1$

(b) $y = -2x^2 - 4x$

(c) $y = -x^2 - 4x + 2$



Graph B is the only positive quadratic.

Graph B shows a positive quadratic, so graph B corresponds to equation (a).

We can distinguish between the other two graphs by their y-intercepts.

Graph A has a positive y-intercept, so graph A corresponds to equation (c). Graph C corresponds to equation (b).

Although we are mainly concerned with investigating how the features of a graph are determined by the coefficients in the equation, it is often useful to be able to do the reverse. In other words, given a graph, can we find the coefficients? The following example illustrates how to tackle this type of problem.

Finding the equation of a given graph is important in *mathematical modelling*, where often a graph is generated from experimental data and we seek an equation to describe it.

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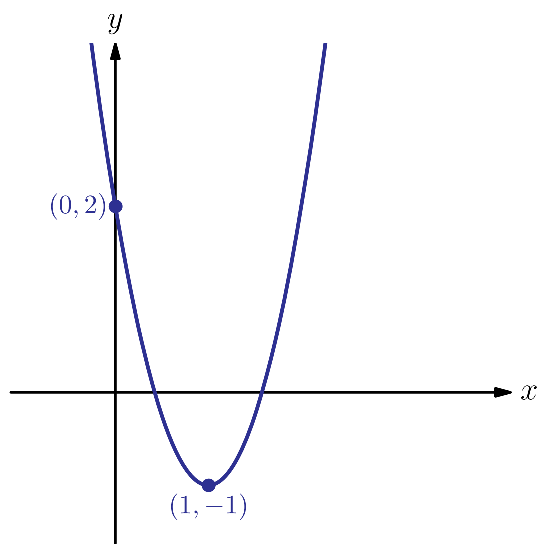
Worked example 1.2

The graph shown below has the equation $y = ax^2 - 6x + c$.



Find the values of c and a.

c is the y-intercept.

The y-intercept of the graph is $(0, 2)$, so $c = 2$

The coordinates of the vertex need to satisfy the equation of the graph.

The vertex is at $x = 1$ and $y = -1$, so

$-1 = a(1)^2 - 6(1) + 2$

$-1 = a - 4$

$a = 3$

The shape of the graph and the position of the y-intercept are the only two features we can read directly from the quadratic equation. We may also be interested in other properties, such as

- the position of the line of symmetry
- the coordinates of the vertex
- the x-intercepts.

In the next two sections we will see how rewriting the equation of the graph in different forms allows us to identify these features. In some of the questions below you will need to find them using your calculator.

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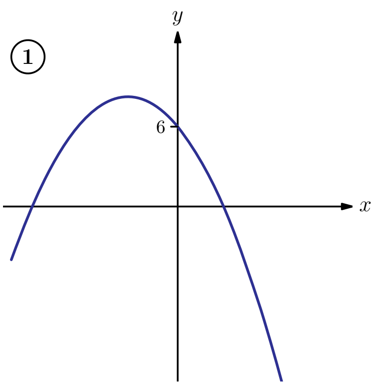
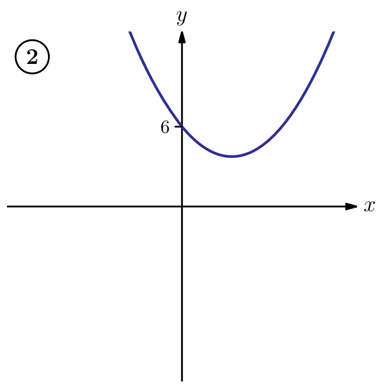
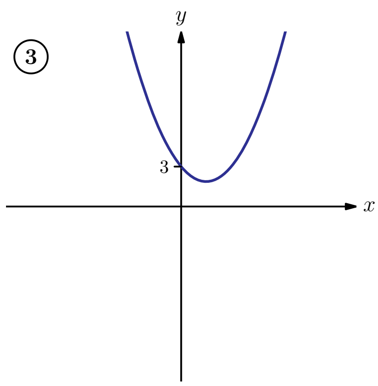
Exercise 1A

1. Match the equations to their corresponding graphs.

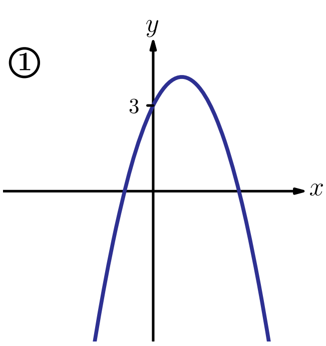
(i) A: $y=-x^2-3x+6$ B: $y=2x^2-3x+3$ C: $y=x^2-3x+6$

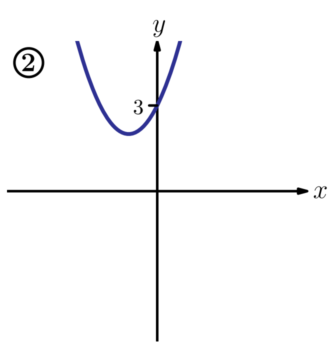


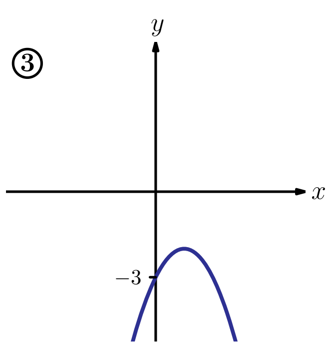




(ii) A: $y=-x^2+2x-3$ B: $y=-x^2+2x+3$ C: $y=x^2+2x+3$







2. Write the following quadratic expressions in the form ax^2+bx+c.

(a) (i) $2(x-1)(x+5)$ (ii) $5(x-1)(x-3)$

(b) (i) $-4(x+2)(4-x)$ (ii) $-(1-x)(2-x)$

(c) (i) $3(x-1)^2+3$ (ii) $4(x+2)^2-5$

(d) (i) $-4(x-1)^2-1$ (ii) $-2(x+2)^2-3$

See Prior Learning section K on the CD-ROM if you need to review the technique of expanding brackets.

3. Find the y-intercept of the graph of each equation.

(a) (i) $y=2(x-1)(x+3)$ (ii) $y=3(x+1)(x-1)$

(b) (i) $y=-3x(x-2)$ (ii) $y=-5x(x-1)$

(c) (i) $y=-(x-1)(x+2)$ (ii) $y=-3(x-1)(x+2)$

(d) (i) $y=2(x-3)^2+1$ (ii) $y=5(x-1)^2-3$

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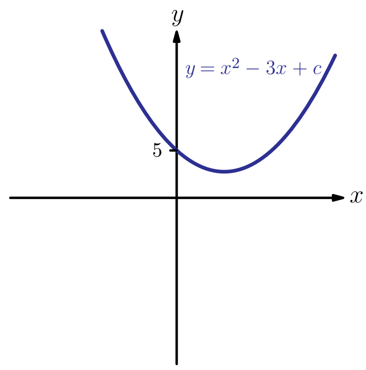
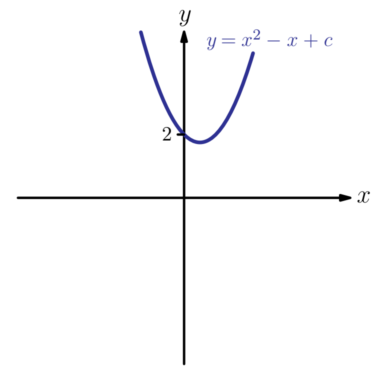
4. The diagrams show quadratic graphs and their equations. Find the value of c in each case.

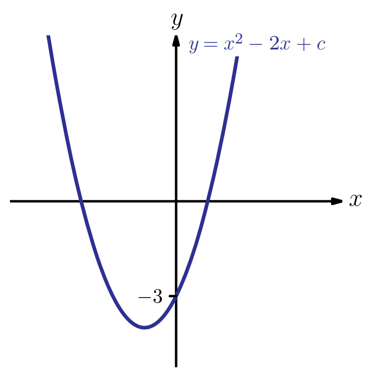
(a) (i)



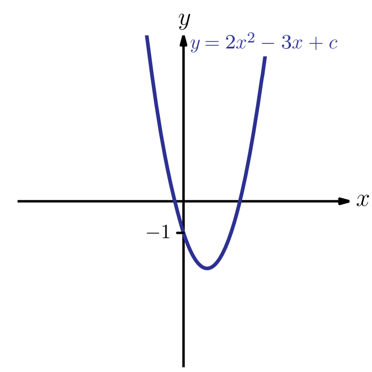
(ii)



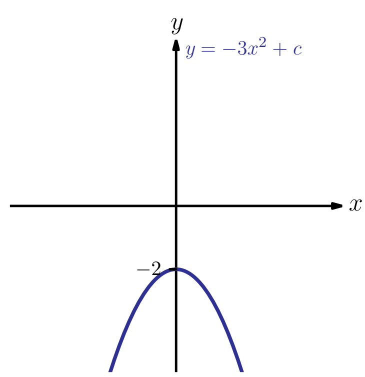
(b) (i)



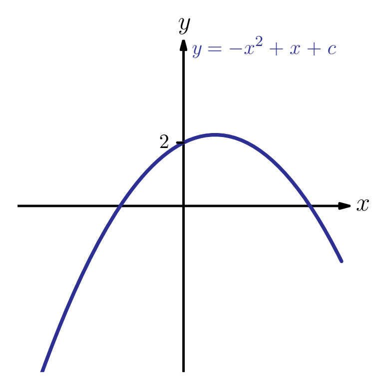
(ii)



(c) (i)

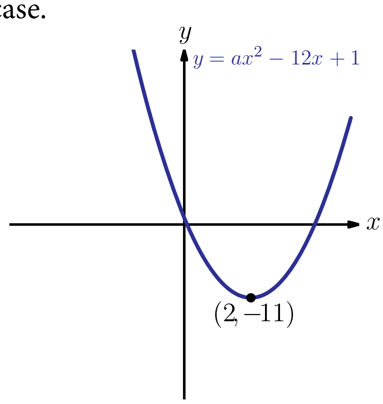


(ii)

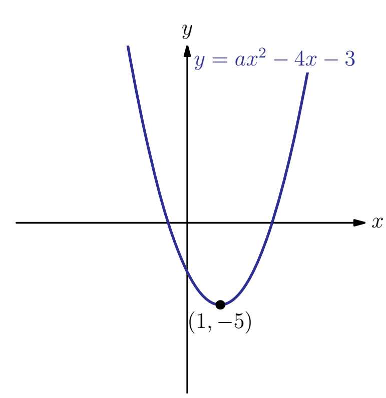


5. The diagrams show quadratic graphs and their equations. Find the value of a in each case.

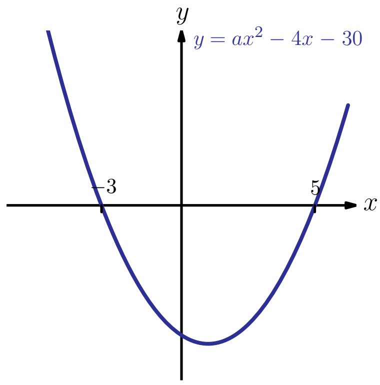
(a) (i)



(ii)



(b) (i)



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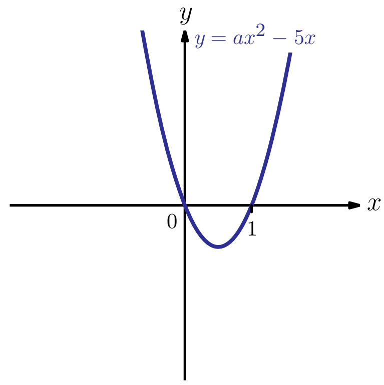
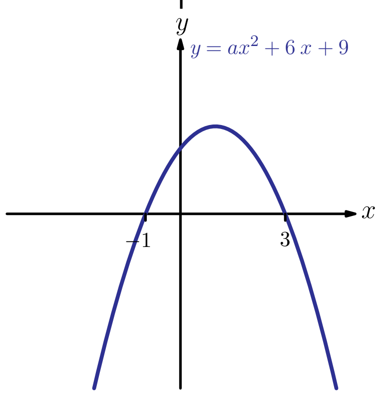
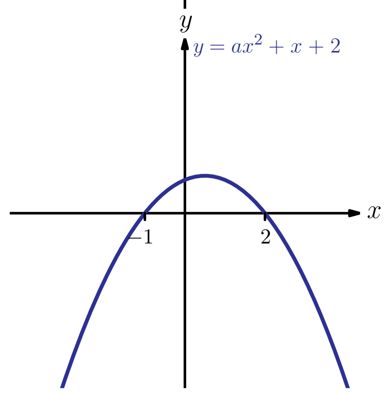


(c) (i)



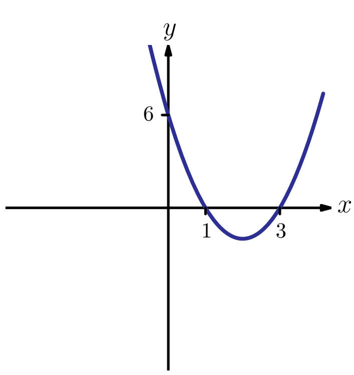
(ii)



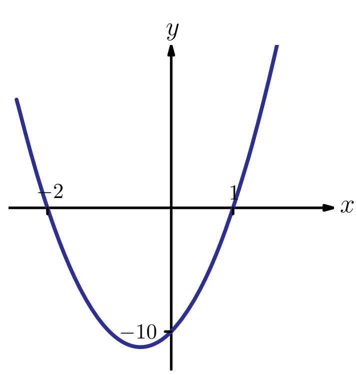
6. The diagrams show graphs of quadratic functions of the form $y = ax^2 + bx + c$. Write down the value of c and then find the values of a and b.

In this question you may need to solve simultaneous equations. See Prior Learning section Q on the CD-ROM if you need a reminder.

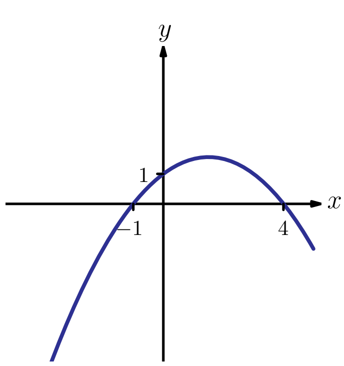
(a) (i)



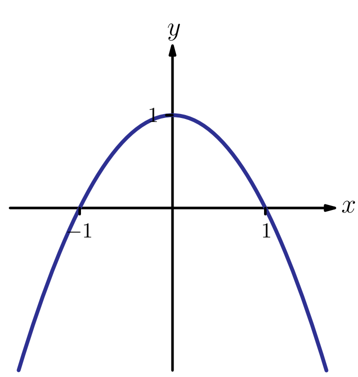
(ii)



(b) (i)



(ii)



7. For each of the following quadratic functions, find the coordinates of the vertex of the graph.

(a) (i) $y = 3x^2 - 4x + 1$ (ii) $y = 2x^2 + x - 4$

(b) (i) $y = -5x^2 + 2x + 10$ (ii) $y = -x^2 + 4x - 5$

8. Find the x-values for which $y = 0$.

(a) (i) $y = 3x^2 - 4x - 3$ (ii) $y = 4x^2 + x - 3$

(b) (i) $y = 4x + 2 - x^2$ (ii) $y = x + 5 - 2x^2$

(c) (i) $y = -2x^2 + 12x - 18$ (ii) $y = 2x^2 - 6x + 4.5$

9. Find the equation of the line of symmetry of the parabolas.

(a) (i) $y = x^2 - 4x + 6$ (ii) $y = 2x^2 + x + 5$

(b) (i) $y = 4 + 3x - 2x^2$ (ii) $y = 2 - x + 3x^2$

10. Find the values of x for which

(a) (i) $3x^2 + 4x - 7 = 15$ (ii) $x^2 + x - 1 = 3$

(b) (i) $4x + 2 = 3x^2$ (ii) $3 - 5x = x^2 + 2$

1B The completed square form $y = a(x - h)^2 + k$

It is often useful to write a quadratic function in a different form.

Worked example 1.3 below shows how to find the values of h and k.

Every quadratic function can be written in the form $y = a(x-h)^2 + k$. For example, you can check by multiplying out the brackets that $2x^2 - 2x - 4 = 2\left(x - \frac{1}{2}\right)^2 - \frac{9}{2}$. This second form of a quadratic equation, called the **completed square form**, allows us to find the position of the line of symmetry of the graph and the coordinates of the vertex. It can also be used to solve equations because x only appears once, in the squared term.

We know that squares are always positive, so $(x-h)^2 \geq 0$. It follows that for $y = a(x-h)^2 + k$:

- if $a > 0$, then $a(x-h)^2 \geq 0$ and so $y \geq k$; moreover, $y = k$ only when $x = h$
- if $a < 0$, then $a(x-h)^2 \leq 0$ and so $y \leq k$; moreover, $y = k$ only when $x = h$.

Hence the completed square form gives the extreme (maximum or minimum) value of the quadratic function, namely k, as

well as the value of x at which that extreme value occurs, h.
The point at which the extreme value occurs is called a **turning point** or **vertex**.

KEY POINT 1.2

A quadratic function $y = a(x-h)^2 + k$ has turning point $(h,\ k)$ and line of symmetry $x = h$.

For $a > 0$, $y \geq k$ for all x.

For $a < 0$, $y \leq k$ for all x.

The next example shows how the functions y_1 and y_2 from the previous section can be rearranged into completed square form.

Worked example 1.3

(a) Write $2x^2 - 2x - 4$ in the form $a(x-h)^2 + k$

(b) Hence write down the coordinates of the vertex and the equation of the line of symmetry of the graph $y_1 = 2x^2 - 2x - 4$.

Expand the brackets.
Compare coefficients with the given expression.

(a) $a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$
Comparing coefficients of x^2: $a = 2$
Comparing coefficients of x: $-2ah = -2$

But $a = 2$:

$\therefore -4h = -2$

$\Leftrightarrow h = \dfrac{1}{2}$

Comparing constants:
$ah^2 + k = -4$

But $a = 2, h = \dfrac{1}{2}$:

$\therefore \dfrac{1}{2} + k = -4$

$\Leftrightarrow k = -\dfrac{9}{2}$

Extract information from the turning point form (h, k).

(b) Line of symmetry is $x = \dfrac{1}{2}$ and the vertex is $\left(\dfrac{1}{2}, -\dfrac{9}{2}\right)$

We can also use the completed square form to solve equations. This is illustrated in the next example, which also shows you how to deal with negative coefficients.

Worked example 1.4

(a) Write $-x^2 + 4x - 3$ in the form $a(x - h)^2 + k$

(b) Hence solve the equation $y_2 = -8$

Expand the required form and compare coefficients

(a) $a(x-h)^2 + k = ax^2 - 2ahx + ah^2 + k$

Comparing the coefficients of x^2: $a = -1$

Comparing the coefficients of x:

$4 = -2ah$

$4 = 2h$

$\Leftrightarrow h = 2$

Comparing constants

$-3 = ah^2 + k$

$-3 = -4 + k$

$\Leftrightarrow \quad k = 1$

Therefore $-x^2 + 4x - 3 = -(x-2)^2 + 1$

Isolate the term containing x

(b) $y_2 = -(x-2)^2 + 1$

$-(x-2)^2 + 1 = -8$

$\Leftrightarrow -(x-2)^2 = -9$

$\Leftrightarrow (x-2)^2 = 9$

$\Leftrightarrow x - 2 = \pm 3$

$\therefore x = 5$ or $x = -1$

EXAM HINT

The line of symmetry (and the x-coordinate of the vertex) can also be found using the formula $x = -\frac{b}{2a}$, which is given in the Formula booklet. We will explain in section 1D where this formula comes from.

We can now label the lines of symmetry and the coordinates of the turning points on the graphs of y_1 and y_2.