# A FIRST COURSE IN COMPUTATIONAL ALGEBRAIC GEOMETRY

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# A FIRST COURSE IN COMPUTATIONAL ALGEBRAIC GEOMETRY

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# Preface

Most of mathematics is concerned at some level with setting up and solving various types of equations. Algebraic geometry is the mathematical discipline which handles solution sets of systems of polynomial equations. These are called algebraic sets.

By making use of a correspondence which relates algebraic sets to ideals in polynomial rings, problems concerning the geometry of algebraic sets can be translated into algebra. As a consequence, algebraic geometers have developed a multitude of often highly abstract techniques for the qualitative and quantitative study of algebraic sets, without, in the first instance, considering the equations. Modern computer algebra algorithms, on the other hand, allow us to manipulate the equations and, thus, to study explicit examples. In this way, algebraic geometry becomes accessible to experiments. The experimental method, which has proven to be highly successful in number theory, is now also added to the toolbox of the algebraic geometer.

In these notes, we discuss some of the basic operations in geometry and describe their counterparts in algebra. We explain how the operations can be carried out using computation, and give a number of explicit examples, worked out with the computer algebra system SINGULAR. In this way, our book may serve as a first introduction to SINGULAR, guiding the reader to performing his own experiments.

In detail, we proceed along the following lines:

The Prologue contains remarks on computer algebra systems in general with just a few examples of what can be computed in different application areas.

In Chapter 1, we focus on the geometry–algebra dictionary, illustrating its entries by including a number of SINGULAR examples.

Chapter 2 contains a discussion of the algorithms involved and gives a more thorough introduction to SINGULAR.

For the fun of it, in Chapter 3, we show how to find the solution of a well-posed Sudoku game by solving a corresponding system of polynomial equations.

Finally, in Chapter 4, we discuss a particular classification problem in group theory, and explain how a combination of theory and explicit computations leads to a solution of the problem. Here, algorithmic methods from group theory, number theory, and algebraic geometry are involved.

Due to the expository character of these notes, proofs are only included occasionally. For all other proofs, references are given.

For a set of Exercises, see

`http://www.mathematik.uni-kl.de/~pfister/Exercises.pdf`.

The notes grew out of a course we taught at the African Institute for the Mathematical Sciences (AIMS) in Cape Town, South Africa. Teaching at AIMS was a wonderful experience and we would like to thank all the students for their enthusiasm and the fun we had together. We very much appreciated the facilities at AIMS and we are grateful to its staff for constant support.

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