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Preface

Hyperbolic geometry is a classical subject of pure mathematics. It was invented as the first example of a non-euclidean geometry, and hence played a significant role in the development of modern geometry. Soon, however, its manifold connections with other branches of mathematics and science in general became apparent. Subsequently, it played a major role in such diverse fields as number theory, representation theory, complex functions, ergodic theory, dynamical systems, string theory, quantum chaos, and cosmology.

One of the earliest appearances of hyperbolic geometry in applications arose in connection with ergodic theory and dynamical systems, when in 1898 Hadamard realised that the negative curvature of hyperbolic surfaces produced an unstable and erratic behaviour of the geodesic motion. As a specific example, in 1924 Artin considered the geodesic billiard motion in an infinite triangle on the hyperbolic plane. He constructed a symbolic coding of the geodesics in terms of continued fractions and for the first time proved a quasi ergodic behaviour of the geodesics. Subsequently, many developments in ergodic theory and dynamical systems began with studies of geodesic flows on hyperbolic manifolds.

Beginning in the 1940s, a further far reaching development was initiated by the work of Maass, Selberg and others. Maass considered the spectral theory of the Laplace Beltrami operator on a hyperbolic surface, and Selberg discovered a close connection between the eigenvalues of the Laplacian and the closed geodesics on the surface, as expressed by the celebrated Selberg trace formula. This relation can be viewed as an analogue of the Riemann-von Mangoldt explicit formula that connects the non-trivial zeros of the Riemann zeta function with the prime numbers. The explicit formula plays an important role in studies of the distribution of prime numbers. On the other hand, it is also used to investigate properties of the Riemann zeros, most notably in connection with the famous Riemann hypothesis.

Roughly at the same time, but independently of these developments, in physics the statistical correlations of energy eigenvalues of complicated quantum systems, like heavy nuclei, were found to be the same as the eigenvalue correlations of certain random matrices. This observation was later extended to eigenvalue correlations of quantum Hamiltonians with only a few degrees of freedom, when their classical counterparts possessed a chaotic dynamical behaviour. In 1984 this connection was brought in a condensed form and since then is known as the conjecture of Bohigas, Giannoni and Schmit. Based on these developments the field of quantum chaos emerged which is, in general, concerned with the implications of classical chaos for the dynamical properties of quantum systems. In this context new light was also shed on the similarities between Laplace eigenvalues and Riemann zeros. Montgomery had proven that certain correlations of the Riemann zeros were the same as in random matrix theory. Subsequently, Odlydzko's extensive calculations of Riemann zeros revealed that these correlations follow the random matrix behaviour to an amazing extent.

In the context of quantum chaos, Gutzwiller soon realised that the force free motion of a particle on a hyperbolic surface provides an ideal model system. On the one hand, the quantum system is defined in terms of the Laplace Beltrami operator on the surface which, in suitable units, serves as the quantum Hamiltonian. On the other hand, the geodesic flow on the surface models the classical motion of a particle. Due to the negative curvature this is strongly chaotic, i.e., ergodic, mixing, and uniformly hyperbolic. In this context, the Selberg trace formula can be seen as a connection between the classical and the quantum motion, and hence it can serve as an ideal tool to relate both descriptions of the physical (model) system. Although in the form of the Gutzwiller trace formula a similar connection exists for practically every physical system, in general such a relation is only of an asymptotic (semiclassical) nature. It is an extremely remarkable feature of hyperbolic manifolds that the Selberg trace formula is an identity rather than an asymptotic relation. This property is related to a number of further peculiarities such as, e.g., the connection of the trace formula with Ruelle transfer operators that usually only appear in ergodic theory. These are some of the major reasons why the motion on hyperbolic manifolds emerged as a preferred model in the field of quantum chaos.

This volume now gathers lectures on various aspects of hyperbolic geometry and its applications, mainly to quantum chaos. The contributions are based on lectures held by experts in the field during the International School *Quantum Chaos on Hyperbolic Manifolds*. This school was organised in the framework of the European Research Training Network *Mathematical Aspects of Quantum Chaos* and took place on Schloß Reisensburg in Günzburg (Germany), 4–11 October 2003. The lectures are intended for graduate students and young researchers, as well as for those experienced scientists who want to learn about the subject in some detail. The first contribution is an extended introduction to the geometry of hyperbolic surfaces. This is followed by a detailed exposition of the Selberg trace formula for compact surfaces in the second contribution. An application of the trace formula to correlations of Laplace eigenvalues is the subject of the third contribution. An alternative approach to Laplace eigenvalues that is based on ergodic theory is explained in the following contribution. The subject of the fifth and the sixth contribution are computations of Laplace eigenfunctions on certain non-compact surfaces. The final contribution then deals with three-dimensional hyperbolic manifolds and their applications in cosmology.

We are indebted to the authors of the contributions gathered here, who spent so much effort in preparing their lectures with great care and enthusiasm. We would also like to thank them for their patience during the editorial process of this volume. In addition, we are grateful to Holger Then for his help in preparing this volume. Further thanks go to the European Commission for their financial support through the Research Training Network *Mathematical Aspects of Quantum Chaos*, no. HPRN-CT-2000-00103 of the IHP Programme, which initially brought us all together.

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