I. Hyperbolic Geometry

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Summary. The aim of these lectures is to provide a self-contained introduction into the geometry of the hyperbolic plane and to develop computational tools for the construction and the study of discrete groups of non-Euclidean motions.

The first lecture provides a minimal background in the classical geometries: spherical, hyperbolic and Euclidean. In Lecture 2 we present the Poincaré model of the hyperbolic plane. This is the most widely used model in the literature, but possibly not always the most suitable one for effective computation, and so we present in Lectures 4 and 5 an independent approach based on what we call the *matrix model*. Between the two approaches, in Lecture 3, we introduce the concepts of a Fuchsian group, its fundamental domains and the hyperbolic surfaces. In the final lecture we bring everything together and study, as a special subject, the construction of hyperbolic surfaces of genus 2 based on geodesic octagons.

The course has been designed such that the reader may work himself linearly through it. The prerequisites in differential geometry are kept to a minimum and are largely covered, for example by the first chapters of the books by Do Carmo [8] or Lee [15]. The numerous exercises in the text are all of a computational rather than a problem-to-solve nature and are, hopefully, not too hard to do.

Lecture 1:

The Classical Geometries in Dimension 2

It was long believed that the homogeneity and isotropy of our physical space in which we live, are characteristic features of Euclidean geometry, and it was not until the works of Bolyai, Gauss and Lobatchewski in the early nineteenth century that it became clear that other equally homogeneous geometries exist. These, in the meanwhile classical, geometries will be introduced here in a unified form, where for simplicity we restrict ourselves to dimension two.

1.1 The ε-Hyperboloïd Model

The starting point is the following bilinear form in $\mathbb{R}^3$, where $\varepsilon \in \mathbb{R}$ is a constant, $\varepsilon \neq 0$,

$$h_\varepsilon(x, y) = x_1 y_1 + x_2 y_2 + \frac{1}{\varepsilon} x_3 y_3. \tag{1}$$

Here $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ may be interpreted either as points of $\mathbb{R}^3$ or as vectors at some point $p \in \mathbb{R}^3$. Along with this bilinear form comes the surface

$$\mathbb{S}^2_\varepsilon = \{x \in \mathbb{R}^3 \mid \varepsilon(x_1^2 + x_2^2) + x_3^2 = 1\}. \tag{2}$$

When $\varepsilon > 0$, $\mathbb{S}^2_\varepsilon$ is an ellipsoid, when $\varepsilon < 0$, $\mathbb{S}^2_\varepsilon$ is a two-sheeted hyperboloïd with the two sheets

$$\mathbb{S}^{2+}_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{S}^2_\varepsilon \mid x_3 > 0\}, \quad \mathbb{S}^{2-}_\varepsilon = \{(x_1, x_2, x_3) \in \mathbb{S}^2_\varepsilon \mid x_3 < 0\}.$$

We will now verify that $\mathbb{S}^2_\varepsilon$ with h_ε is a Riemannian manifold, study its isometry group, and learn how the geodesics look like.

Isometries

An economic way to carry out this program is to begin with the transformations of $\mathbb{R}^3$ which preserve h_ε. We use the following notation. $\mathrm{M}(3, \mathbb{R})$ is the set of all 3×3 matrices with real coefficients. Any $M \in \mathrm{M}(3, \mathbb{R})$ has three columns m_1, m_2, m_3, and we write $M = (m_1, m_2, m_3) = (m_{ij})$. A special case is the identity matrix (e_1, e_2, e_3), where e_1, e_2, e_3 is the standard basis of $\mathbb{R}^3$. As the columns of M are members of $\mathbb{R}^3$ we may apply h_ε to them, and so the following definition makes sense,

$$\mathrm{O}_\varepsilon(3) = \{(m_1, m_2, m_3) \in \mathrm{M}(3, \mathbb{R}) \mid h_\varepsilon(m_i, m_j) = h_\varepsilon(e_i, e_j)\}. \tag{3}$$

Observing that $Me_i = m_i$, $i = 1, 2, 3$, and using that multiplication by a matrix is a linear operation we easily see that for $M \in \mathrm{M}(3, \mathbb{R})$ we have

$$M \in \mathrm{O}_\varepsilon(3) \iff h_\varepsilon(Mx, My) = h_\varepsilon(x, y), \ \forall x, y \in \mathbb{R}^3$$

(with x and y written as column vectors). From this we further see that if $L, M \in \mathrm{O}_\varepsilon(3)$, then also $LM \in \mathrm{O}_\varepsilon(3)$ and $M^{-1} \in \mathrm{O}_\varepsilon(3)$. Hence, $\mathrm{O}_\varepsilon(3)$ is a group. It is called the *orthogonal group of* h_ε.

Example 1.1. The following matrices belong to $\mathrm{O}_\varepsilon(3)$,

$$S_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad N_t = \begin{pmatrix} \mathbf{C}_\varepsilon(t) & 0 & \mathbf{S}_\varepsilon(t) \\ 0 & 1 & 0 \\ -\varepsilon\mathbf{S}_\varepsilon(t) & 0 & \mathbf{C}_\varepsilon(t) \end{pmatrix}, \quad \alpha, t \in \mathbb{R},$$

where $\mathbf{C}_\varepsilon$ and $\mathbf{S}_\varepsilon$ are the generalized trigonometric functions

$$\mathbf{C}_\varepsilon(t) = \begin{cases} \cos\sqrt{\varepsilon}\,t, & \text{if } \varepsilon > 0; \\ \cosh\sqrt{|\varepsilon|}\,t, & \text{if } \varepsilon < 0; \end{cases} \qquad \mathbf{S}_\varepsilon(t) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}}\sin\sqrt{\varepsilon}\,t, & \text{if } \varepsilon > 0; \\ \dfrac{1}{\sqrt{|\varepsilon|}}\sinh\sqrt{|\varepsilon|}\,t, & \text{if } \varepsilon < 0. \end{cases}$$

The Taylor series expansion of these functions for $t \to 0$ is

$$\mathbf{C}_\varepsilon(t) = 1 - \frac{\varepsilon t^2}{2!} + \cdots, \qquad \mathbf{S}_\varepsilon(t) = t - \frac{\varepsilon t^3}{3!} + \cdots. \tag{4}$$

Exercise 1.2.
(a) Any $M \in \mathrm{O}_\varepsilon(3)$ maps $\mathbb{S}^2_\varepsilon$ onto itself.
(b) For any $p, q \in \mathbb{S}^2_\varepsilon$ there exists $M \in \mathrm{O}_\varepsilon(3)$ such that $Mp = q$.
(c) $R_\alpha R_\beta = R_{\alpha+\beta}$, for all $\alpha, \beta \in \mathbb{R}$.
(d) $N_s N_t = N_{s+t}$, for all $s, t \in \mathbb{R}$.
(e) S_1, S_2, S_3 and the R_α's and N_t's generate $\mathrm{O}_\varepsilon(3)$.
(Hint: in (b) show first that there exists α and t such that $N_t R_\alpha p = \pm e_3$.)

We will now introduce a Riemannian metric on $\mathbb{S}^2_\varepsilon$ such that $\mathrm{O}_\varepsilon(3)$ operates by isometries on it. (For the concept of a Riemannian metric and worked out examples, see e.g. the third chapter in Lee's book [15].)

For $p \in \mathbb{S}^2_\varepsilon$ we denote by $T_p\mathbb{S}^2_\varepsilon$ the space of all vectors at p which are tangent to $\mathbb{S}^2_\varepsilon$. In the special case $p = e_3$ this space is easy to describe:

$$v = (v_1, v_2, v_3) \in T_{e_3}\mathbb{S}^2_\varepsilon \iff v_3 = 0.$$

For the other points in $\mathbb{S}^2_\varepsilon$ it will not be necessary to describe $T_p\mathbb{S}^2_\varepsilon$ explicitly; it suffices to know, by Exercise 1.2(b), that there exists $M \in \mathrm{O}_\varepsilon(3)$ sending the tangent vectors at e_3 to the tangent vectors at p, and vice-versa.

Definition 1.3. For any given $p \in \mathbb{S}^2_\varepsilon$ and for all $v, w \in T_p\mathbb{S}^2_\varepsilon$ we define $g_\varepsilon(v, w) = h_\varepsilon(v, w)$.

Theorem 1.4. *$\mathbb{S}^2_\varepsilon$ endowed with the product g_ε is a Riemannian 2-manifold and $\mathrm{O}_\varepsilon(3)$ acts by isometries. If $v, w \in T_p\mathbb{S}^2_\varepsilon$ and $v', w' \in T_{p'}\mathbb{S}^2_\varepsilon$ are orthonormal bases, then there exists a unique $\phi \in \mathrm{O}_\varepsilon(3)$ satisfying $\phi p = p'$, $\phi v = v'$, $\phi w = w'$.*

Proof. To prove that g_ε is a Riemannian metric on $\mathbb{S}^2_\varepsilon$ it only remains to show that g_ε is positive definite on any $T_p\mathbb{S}^2_\varepsilon$. For $p = e_3$ this is clearly the case because for $v, w \in T_{e_3}\mathbb{S}^2_\varepsilon$ the product $g_\varepsilon(v, w)$ coincides with the ordinary scalar product. For any other $p \in \mathbb{S}^2_\varepsilon$ it now suffices to recall (Exercise 1.2(b)) that there exists $M \in \mathrm{O}_\varepsilon(3)$ sending $T_p\mathbb{S}^2_\varepsilon$ to $T_{e_3}\mathbb{S}^2_\varepsilon$.

Since $\mathrm{O}_\varepsilon(3)$ preserves h_ε it preserves g_ε, and this means that $\mathrm{O}_\varepsilon(3)$ acts by isometries.

For the two orthonormal bases we take $M, M' \in \mathrm{O}_\varepsilon(3)$ such that $Mp = e_3$ and $M'p' = e_3$. Now Mv, Mw and $M'v', M'w'$ are orthonormal bases of $T_{e_3}\mathbb{S}^2_\varepsilon$ with respect to g_ε and thus also with respect to the ordinary scalar product. We find therefore an isometry S either of the form $S = \mathbb{R}_\alpha$ or $S = \mathbb{R}_\alpha S_1$ such that $SMv = M'v'$ and $SMw = M'w'$. Setting $\phi = (M')^{-1}SM$ we get the desired isometry. The uniqueness is left as an exercise (use Theorem 1.5 below). □

For any Riemannian manifold $\mathcal{M}$ we denote by $\mathrm{Isom}(\mathcal{M})$ its *isometry group* i.e. the group of all isometries $\phi : \mathcal{M} \to \mathcal{M}$.

We shall use the following standard fact from Riemannian geometry which we state here without giving a proof.

Theorem 1.5. *Let $\mathcal{M}$ be an n-dimensional connected Riemannian manifold, $p \in \mathcal{M}$ a point, and $v_1, \ldots, v_n$ a basis of $T_p\mathcal{M}$. If $\phi, \psi \in \mathrm{Isom}(\mathcal{M})$ send p to the same image point and $v_1, \ldots, v_n$ to the same image vectors, then $\phi = \psi$.*

(The theorem is actually seldom stated, but it is a simple consequence of the fact that the so-called exponential map is a local diffeomorphism (Proposition 2.9 in [8], or Lemma 5.10 in [15]).

The Hyperboloïd

If $\varepsilon < 0$, then $\mathbb{S}^2_\varepsilon$ has two isometric sheets $\mathbb{S}^{2+}_\varepsilon$ and $\mathbb{S}^{2-}_\varepsilon$. An isometry between the two is e.g. given by S_3 in Example 1.1. We shall therefore restrict ourselves from now on to one of the two sheets. To treat all cases in a uniform manner we use the following notation.

Definition 1.6.

$$\mathrm{H}^2_\varepsilon = \begin{cases} \mathbb{S}^2_\varepsilon, & \text{if } \varepsilon > 0; \\ \mathbb{S}^{2+}_\varepsilon, & \text{if } \varepsilon < 0. \end{cases}$$

We call H^2_ε the *ε-hyperboloïd*. Together with g_ε it is a connected two-dimensional Riemannian manifold.

Geodesics

In this part we use some elements from Riemannian geometry such as the Levi-Civita connection to characterize the geodesics of H^2_ε. The reader who is not familiar with this may skip the proofs and use the conclusion of Theorem 1.8 as an ad hoc definition of a geodesic.

In what follows, all geodesics are understood to be parametrized by unit speed on a maximal possible interval that contains 0. Frequently such a geodesic will be identified with the set of its points.

Lemma 1.7. *The following curve is a geodesic of* H^2_ε*,*

$$\eta(t) = N_t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S}_\varepsilon(t) \\ 0 \\ \mathbf{C}_\varepsilon(t) \end{bmatrix}, \quad t \in \mathbb{R}.$$

Proof. We denote by $\dot\eta(t) = \frac{d}{dt}\eta(t)$ the tangent vectors of η. The isometry $S = S_2$ fixes η point-wise, and therefore $\nabla_{(S\circ\eta)\dot{}(t)}(S\circ\eta)\dot{}(t) = \nabla_{\dot\eta(t)}\dot\eta(t)$.

On the other hand, η satisfies $g_\varepsilon(\dot\eta(t),\dot\eta(t)) = \text{constant} = 1$, and the rule $\frac{d}{dt}g_\varepsilon(\dot\eta(t),\dot\eta(t)) = 2g_\varepsilon(\nabla_{\dot\eta(t)}\dot\eta(t),\dot\eta(t))$ implies $g_\varepsilon(\nabla_{\dot\eta(t)}\dot\eta(t),\dot\eta(t)) = 0$, i.e. $\nabla_{\dot\eta(t)}\dot\eta(t)$ is orthogonal to η and therefore $\nabla_{(S\circ\eta)\dot{}(t)}(S\circ\eta)\dot{}(t) = S(\nabla_{\dot\eta(t)}\dot\eta(t)) = -\nabla_{\dot\eta(t)}\dot\eta(t)$.

Altogether $\nabla_{\dot\eta(t)}\dot\eta(t) = 0$, i.e. η is a geodesic. □

Theorem 1.8. *The geodesics in* H^2_ε *are the curves* $\mathrm{H}^2_\varepsilon \cap G$*, where* G *is a plane through* O *that intersects* H^2_ε*.*

Proof. Let E be the plane through O, e_1, e_3. Then $\mathrm{H}^2_\varepsilon \cap E = \{\eta(t) \mid t \in \mathbb{R}\}$, where η is the geodesic from Lemma 1.7. If G is any plane intersecting H^2_ε, then by Theorem 1.4, there exists $\phi \in \mathrm{O}_\varepsilon(3)$ such that $\phi(E) = G$. (Since ϕ is a linear map any plane through O is mapped to a plane through O !) It follows that $\mathrm{H}^2_\varepsilon \cap G = \phi(\mathrm{H}^2_\varepsilon \cap E) = \phi(\eta)$ is a geodesic.

Conversely, let γ be an arbitrary geodesic on H^2_ε. By Theorem 1.4, there exists $\psi \in \mathrm{O}_\varepsilon(3)$ sending $\eta(0)$ to $\gamma(0)$ and $\dot\eta(0)$ to $\dot\gamma(0)$. The uniqueness theorem for geodesics then implies that $\gamma = \psi(\eta)$. It follows that $\gamma = \psi(\mathrm{H}^2_\varepsilon \cap E) = \mathrm{H}^2_\varepsilon \cap G$ where G is the plane $G = \psi(E)$. □

Corollary 1.9. *Any geodesic on* H^2_ε *is the image of* η *under some* $\psi \in \mathrm{O}_\varepsilon(3)$*.*
□

Polar Coordinates

For the next consideration we set

$$d_\varepsilon = \begin{cases} \pi/\sqrt{\varepsilon}, & \text{if } \varepsilon > 0; \\ \infty, & \text{if } \varepsilon \le 0. \end{cases} \tag{5}$$

For any given $\sigma \in [0, 2\pi]$, the curve

$$\rho \mapsto \gamma_\sigma(\rho) = R_\sigma\eta(\rho), \quad 0 \le \rho < d_\varepsilon,$$

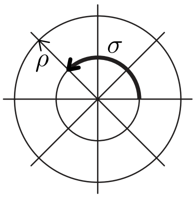
(where η is again as in Lemma 1.7) is a unit speed geodesic. For any given $\rho > 0$ the curve

$$\sigma \mapsto c_\rho(\sigma) = R_\sigma\eta(\rho) = \begin{bmatrix} \cos(\sigma)\mathbf{S}_\varepsilon(\rho) \\ \sin(\sigma)\mathbf{S}_\varepsilon(\rho) \\ \mathbf{C}_\varepsilon(\rho) \end{bmatrix}$$

has tangent vectors

$$\dot c_\rho(\sigma) = \begin{bmatrix} -\sin(\sigma)\mathbf{S}_\varepsilon(\rho) \\ \cos(\sigma)\mathbf{S}_\varepsilon(\rho) \\ 0 \end{bmatrix},$$

and $g_\varepsilon(\dot c_\rho(\sigma), \dot c_\rho(\sigma))^{1/2} = \mathbf{S}_\varepsilon(\rho)$. One also easily checks that $g_\varepsilon(\dot c_\rho(\sigma), \dot\gamma_\sigma(\rho)) = 0$. This shows that if we use ρ and σ as new coordinates for points in H^2_ε, then the expression for the Riemannian metric g_ε is as follows, where we use the differential notation ds_ε for the length of an infinitesimal segment with components $d\rho$ and $d\sigma$:



$$ds^2_\varepsilon = d\rho^2 + \mathbf{S}^2_\varepsilon(\rho)\, d\sigma^2. \tag{6}$$

The coordinates ρ and σ are called *geodesic polar coordinates*. The figure illustrates them as ordinary polar coordinates in the Euclidean plane. The radial straight line of argument σ corresponds to γ_σ, the circle of radius ρ corresponds to c_ρ. The length of c_ρ is $2\pi\,\mathbf{S}_\varepsilon(\rho)$.

The polar coordinates here have been centered at point e_3; applying $\mathrm{O}_\varepsilon(3)$ we may center them at any other point of H^2_ε.

Exercise 1.10. Let $f(\rho, \sigma) = \gamma_\sigma(\rho)$.

(a) In the case $\varepsilon > 0$ the mapping $f :]0, d_\varepsilon[\times [0, 2\pi[\to \mathrm{H}^2_\varepsilon \smallsetminus \{\pm e_3\}$ is one-to-one and onto.
(b) In the case $\varepsilon < 0$ the mapping $f :]0, \infty[\times [0, 2\pi[\to \mathrm{H}^2_\varepsilon \smallsetminus \{e_3\}$ is one-to-one and onto.

Exercise 1.11. Use (6) to show that for any r in the interval $]0, d_\varepsilon[$ the restriction of the geodesic γ_σ to the interval $[0, r]$ is the shortest curve on $\mathbb{S}^2_\varepsilon$ from $\gamma_\sigma(0)$ to $\gamma_\sigma(r)$.

Use Corollary 1.9 to conclude that this is true for all geodesics in $\mathbb{S}^2_\varepsilon$.

Distances

Definition 1.12. For $p, q \in \mathrm{H}^2_\varepsilon$ the *distance* $d(p, q)$ is defined as the infimum over the lengths of all arcs from p to q on H^2_ε.

In the next theorem d_ε is again $\pi/\sqrt{\varepsilon}$ if $\varepsilon > 0$ and ∞ otherwise (see (5)).

Theorem 1.13. *For any $p, q \in \mathrm{H}^2_\varepsilon$, $p \neq q$, the following holds.*

(i) $d(p, q) \leq d_\varepsilon$.

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(ii) *There exists a geodesic through* p, q*, and on this geodesic an arc of length* $d(p, q)$ *from* p *to* q*.*

(iii) *If* $d(p, q) < d_\varepsilon$*, then this geodesic and the arc are unique.*

Proof. There exists $\phi \in \mathrm{O}_\varepsilon(3)$ such that $\phi(p) = e_3$. By Exercise 1.10, $\phi(q) = \gamma_\sigma(r)$ for some $\sigma \in [0, 2\pi[$ and $r \leq d_\varepsilon$. By Exercise 1.11, $r = d(p, q)$. This yields (i) and (ii). If $r < d_\varepsilon$, then $\phi(q) \neq -e_3$ and so the plane through $O, e_3, \phi(q)$ is unique. By Theorem 1.8 this implies that the geodesic through e_3 and $\phi(q)$ is unique. If $\varepsilon < 0$, then γ_σ contains only one arc from $e_3 = \phi(p)$ to $\phi(q)$. If $\varepsilon > 0$, then there are two such arcs, one of length r, the other of length $2d_\varepsilon - r > r$. This yields (iii). □

Displacement Lengths and Angles of Rotation

From the rule $N_s N_t = N_{s+t}$ (Exercise 1.2(d)) we see that N_s shifts any point $\eta(t) = N_t e_3$ to $N_{s+t} e_3 = \eta(t+s)$. This shows that η is an invariant geodesic of N_s, and for any point $\eta(t)$ on this geodesic the distance to its image is equal to $|s|$.

More generally, if $M = \phi N_s \phi^{-1}$, where $\phi \in \mathrm{O}_\varepsilon(3)$ maps η to some geodesic γ_M of H^2_ε, then γ_M is invariant under M, and for any point $p \in \gamma$ the distance to its image has the same value $\ell_M := |s|$.

Definition 1.14. In the above situation we call γ_M the *axis of* M and ℓ_M its *displacement length.*

Exercise 1.15.

$$\operatorname{trace}(M) = 1 + 2\mathbf{C}_\varepsilon(\ell_M).$$

In a similar way, if $p \in \mathrm{H}^2_\varepsilon$, $\phi \in \mathrm{O}_\varepsilon(3)$ such that $\phi(p) = e_3$, and $M = \phi^{-1} R_{\alpha_M} \phi$ for some $\alpha_M \in \,]-\pi, \pi]$, then we have

Exercise 1.16.

$$\operatorname{trace}(M) = 1 + 2\cos(\alpha_M).$$

We call, in this case, M a *rotation* with center p and $|\alpha_M|$ is the (absolute) *angle of rotation* of M.

Exercise 1.17. If M is a rotation with angle α_M and center p, then for any unit vector $v \in T_p\mathrm{H}^2_\varepsilon$,

$$g_\varepsilon(v, Mv) = \cos(\alpha_M).$$

The preceding exercise shows that our definition of an angle of rotation is compatible with the usual definition of the angle between vectors relatively to a given scalar product:

Definition 1.18. For any $p \in \mathrm{H}^2_\varepsilon$ and any pair of unit vectors $v, w \in T_p\mathrm{H}^2_\varepsilon$ the *angle* $\sphericalangle(v, w)$ is defined as the unique real number in the interval $[0, \pi]$ such that

$$\cos \sphericalangle(v, w) = g_\varepsilon(v, w).$$

If M is the rotation with center p and angle α_M sending v to w, then the comparison of this definition with Exercise 1.17 shows that indeed $|\alpha_M| = \sphericalangle(v,w)$.

1.2 The Case $\varepsilon = 0$

Interpreting the terms suitably, it is also possible to include the case $\varepsilon = 0$ into our considerations. The set

$$\mathbb{S}_0^2 = \{x \in \mathbb{R}^3 \mid x_3^2 = 1\}$$

is a pair of parallel planes, and H_0^2 becomes the plane $x_3 = 1$. For arbitrary vectors in $\mathbb{R}^3$ the definition of h_0 as given by (1) does not make sense, but for $p \in \mathbb{S}_0^2$ the tangent vectors have the third coordinate equal to zero, and we may well define

$$g_0(v,w) = v_1w_1 + v_2w_2, \quad v,w \in T_p\mathbb{S}_0^2.$$

Thus, H_0^2 together with g_0 becomes a model of the Euclidean plane.

The planes through O intersect H_0^2 in straight lines, and we get all straight lines like this. Hence, on H_0^2 the geodesics as described in Theorem 1.8 are precisely the straight lines of Euclidean geometry.

It is interesting to see how the group $\mathrm{O}_0(3)$ comes out. Since $h_\varepsilon(e_3,e_3) \to \infty$ as $\varepsilon \to 0$, the definition of $\mathrm{O}_\varepsilon(3)$ as given by (3) cannot be extended to $\varepsilon = 0$. But the matrices S_1, S_2, S_3, R_α in Example 1.1 are still defined, and the N_t too: inspecting the Taylor series expansion (4) we may extend the definitions of the functions $\mathbf{C}_\varepsilon$ and $\mathbf{S}_\varepsilon$ to $\varepsilon = 0$ as follows.

$$\mathbf{C}_0(t) = 1, \quad \mathbf{S}_0(t) = t, \quad t \in \mathbb{R}.$$

The matrices N_t then become

$$N_t = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and for $x \in \mathrm{H}_0^2$ we get

$$N_t \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + t \\ x_2 \\ 1 \end{bmatrix}.$$

i.e. N_t operates as a translation. Guided by Exercise 1.2 we now define $\mathrm{O}_0(3)$ as the group generated by S_1, S_2, S_3, the R_α's and the N_t's in the above form. The reader may then easily check the following, where $\mathrm{O}(2)$ is the group of all orthogonal 2×2-matrices.

$$\mathrm{O}_0(3) = \left\{ \begin{pmatrix} a_{11} & a_{12} & t_1 \\ a_{21} & a_{22} & t_2 \\ 0 & 0 & \pm 1 \end{pmatrix} \,\middle|\, \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \in \mathbb{R}^2, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathrm{O}(2) \right\} \tag{7}$$

Exercise 1.19. The subgroup $O_0^+(3)$ formed by all $M \in O_0(3)$ with last component equal to $+1$ preserves H_0^2 and coincides with the group of Euclidean motions in the plane.

1.3 Curvature

In Riemannian geometry of dimension 2 the *Gauss curvature* $K(p)$ at any point p measures the deviation from Euclidean geometry by comparing the lengths of small distance circles $c_\rho(p)$ of radius ρ and center p with the lengths of the corresponding circles in the Euclidean plane. The definition is as follows, where ℓ denotes the length,

$$K(p) \overset{\text{def}}{=} -\frac{3}{\pi} \lim_{\rho \to 0} \frac{1}{\rho^3} \Big(\ell(c_\rho(p)) - 2\pi\rho \Big). \tag{8}$$

For H_ε^2 the length of $c_\rho(p)$ may be computed in polar coordinates via (6) and is equal to $2\pi \mathbf{S}_\varepsilon(\rho)$. The Taylor series expansion (4) then yields

$$\mathbb{S}_\varepsilon^2 \text{ has constant curvature } \varepsilon. \tag{9}$$

Volume Growth

Expression (8) for the deviation from Euclidean geometry may give the impression that we have here merely a negligible third order effect. Yet, the contrary is true! Let us consider, e.g. the area of large discs. By (6), the area $d\omega_\varepsilon$ of an infinitesimal region whose description in polar coordinates (ρ, σ) is that of a rectangle of width $d\rho$ and height $d\sigma$, is given by

$$d\omega_\varepsilon = \mathbf{S}_\varepsilon(\rho)\, d\rho\, d\sigma. \tag{10}$$

For the circular disc $B_r(p) = \{x \in H_\varepsilon^2 \mid d(x,p) < r\}$ of radius $r < d_\varepsilon$ (see (5)), the area becomes

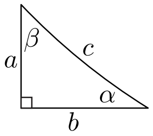
$$\text{area} B_r(p) = \int_0^{2\pi} \int_0^r \mathbf{S}_\varepsilon(\rho)\, d\rho\, d\sigma = \frac{-2\pi}{\varepsilon} (\mathbf{C}_\varepsilon(r) - 1). \tag{11}$$

We see here that in the case of constant negative curvature the area grows exponentially. This exponential growth, so radically different from the Euclidean case, is responsible for many unexpected spectral phenomena in hyperbolic geometry.

Trigonometry and the Gauss-Bonnet Formula for Triangles

In the following we consider geodesic triangles i.e. triangles formed by geodesic arcs of lengths $< d_\varepsilon$. By abuse of notation, sides and their lengths are named by the same symbol.

Theorem 1.20 (Right-angled triangles). *For any geodesic triangle in* H^2_ε *with sides* a, b, c*, opposite angles* α, β *and* a *orthogonal to* b *the following hold.*



(i) $\mathbf{C}_\varepsilon(c) = \mathbf{C}_\varepsilon(a)\,\mathbf{C}_\varepsilon(b)$,
(ii) $\cos(\beta) = \mathbf{C}_\varepsilon(b)\sin(\alpha)$.

Proof. We show (i) and leave (ii) as an exercise. The strategy is to move the triangles using different isometries and obtain the formulas by comparing the traces.

Consider the common axis η of the isometries N_t (Lemma 1.7), and define $\mu = R_{\pi/2}(\eta)$. Then μ is the common axis of all

$$M_t = R_{\pi/2} N_t R_{-\pi/2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathbf{C}_\varepsilon(t) & \mathbf{S}_\varepsilon(t) \\ 0 & -\varepsilon\mathbf{S}_\varepsilon(t) & \mathbf{C}_\varepsilon(t) \end{pmatrix}. \tag{12}$$

Place the triangle such that a, b lie on μ, η as shown in Fig. 1 with vertices at $\mu(a)$ and $\eta(b)$, and consider a symmetric copy (shaded) across vertex $\mu(a)$.

Now shift the triangles along μ using M_{-2a} as shown in the figure so that in its new position the shaded one has side b on η, and then apply N_{2b}. Under this operation side c is shifted by $2c$ along the geodesic γ through $\mu(a), \eta(b)$, and so $N_{2b}M_{-2a}$ has axis γ and displacement length $2c$. Using Exercise 1.15 we compute

$$1 + 2\mathbf{C}_\varepsilon(2c) = \mathrm{trace}(N_{2b}M_{-2a}) = (1 + \mathbf{C}_\varepsilon(2a))(1 + \mathbf{C}_\varepsilon(2b)) - 1,$$

and (i) follows from the rule $\mathbf{C}_\varepsilon(2t) = 2\mathbf{C}_\varepsilon(t)^2 - 1$.

For the proof of (ii) the exercise is to find a position of the triangle such that applying $R_{2\alpha-\pi}M_{-2b}$ to it is the same as rotating it by -2β around its vertex that has angle β. □

Remark 1.21. It is interesting to observe what formula (i) yields in the limit $\varepsilon \to 0$. In paragraph 1.2 we defined $\mathbf{C}_0$ as the function which is constant equal

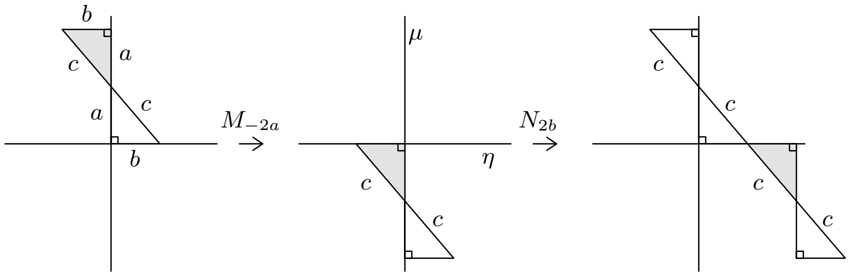


Fig. 1. The product $N_{2b}M_{-2a}$ of the vertical shift followed by the horizontal shift has displacement length $2c$.