Introduction

Quantum Schur–Weyl theory refers to a three-level duality relation. At Level I, it investigates a certain double centralizer property, *the quantum Schur– Weyl reciprocity*, associated with some bimodules of quantum $\mathfrak{gl}_n$ and the Hecke algebra (of type A)—the tensor spaces of the natural representation of quantum $\mathfrak{gl}_n$ (see [**43**], [**21**], [**27**]). This is the quantum version of the well-known Schur–Weyl reciprocity which was beautifully used in H. Weyl's influential book [**77**]. The key ingredient of the reciprocity is a class of important finite dimensional endomorphism algebras, the *quantum Schur algebras* or *q-Schur algebras*, whose classical version was introduced by I. Schur over a hundred years ago (see [**69**], [**70**]). At Level II, it establishes a certain *Morita equivalence* between quantum Schur algebras and Hecke algebras. Thus, quantum Schur algebras are used to bridge representations of quantum $\mathfrak{gl}_n$ and Hecke algebras. More precisely, they link polynomial representations of quantum $\mathfrak{gl}_n$ with representations of Hecke algebras via the Morita equivalence. The third level of this duality relation is motivated by two simple questions associated with the structure of (associative) algebras. If an algebra is defined by generators and relations, the *realization problem* is to reconstruct the algebra as a vector space with hopefully explicit multiplication formulas on elements of a basis; while, if an algebra is defined in terms of a vector space such as an endomorphism algebra, it is natural to seek their generators and defining relations.

As one of the important problems in quantum group theory, the realization problem is to construct a quantum group in terms of a vector space and certain multiplication rules on basis elements. This problem is crucial to understand their structure and representations (see [**47**, p. xiii] for a similar problem for Kac–Moody Lie algebras and [**60**] for a solution in the symmetrizable case). Though the Ringel–Hall algebra realization of the ±-part of quantum enveloping algebras associated with symmetrizable Cartan matrices was an important breakthrough in the early 1990s, especially for the introduction of the geometric approach to the theory, the same problem for the entire quantum groups is far from completion. However, Beilinson–Lusztig–MacPherson (BLM) [**4**] solved the problem for quantum $\mathfrak{gl}_n$ by exploring further properties coming from the quantum Schur–Weyl reciprocity. On the other hand, as endomorphism algebras and as homomorphic images of quantum $\mathfrak{gl}_n$, it is natural to look for presentations for quantum Schur algebras via the presentation of quantum $\mathfrak{gl}_n$. This problem was first considered in [**18**] (see also [**26**]). Thus, as a particular feature in the type A theory, realizing quantum $\mathfrak{gl}_n$ and presenting quantum Schur algebras form Level III of this duality relation. For a complete account of the quantum Schur–Weyl theory and further references, see Parts 3 and 5 of [**12**] (see also [**17**] for more applications).

There are several developments in the establishment of an affine analogue of the quantum Schur–Weyl theory. Soon after BLM's work, Ginzburg and Vasserot [**32, 75**] used a geometric and K-theoretic approach to investigate *affine quantum Schur algebras*[1] as homomorphic images of *quantum loop algebra* $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ of $\mathfrak{gl}_n$ in the sense of Drinfeld's new presentation [**20**], called quantum affine $\mathfrak{gl}_n$ (at level 0) in this book. This establishes at Level I the first centralizer property for the affine analogue of the quantum Schur–Weyl reciprocity. Six years later, investigations around affine quantum Schur algebras focused on their different definitions and, hence, different applications. For example, Lusztig [**56**] generalized the fundamental multiplication formulas [**4**, 3.4] for quantum Schur algebras to the affine case and showed that the "extended" quantum affine $\mathfrak{sl}_n$, $\mathbf{U}_\triangle(n)$, does not map onto affine quantum Schur algebras; Varagnolo–Vasserot [**73**] investigated Ringel–Hall algebra actions on tensor spaces and described the geometrically defined affine quantum Schur algebras in terms of the endomorphism algebras of tensor spaces. Moreover, they proved that the tensor space definition coincides with Green's definition [**35**] via q-permutation modules. Some progress on the second centralizer property has also been made recently by Pouchin [**61**]. The approaches used in these works are mainly geometric. However, like the non-affine case, there would be more favorable algebraic and combinatorial approaches.

At Level II, representations at non-roots-of-unity of quantum affine $\mathfrak{sl}_n$ and $\mathfrak{gl}_n$ over the complex number field $\mathbb{C}$, including classifications of finite dimensional simple modules, have been thoroughly investigated by Chari–Pressley [**6, 7, 8**], and Frenkel–Mukhin [**28**] in terms of Drinfeld polynomials. Moreover, an equivalence between the module category of the Hecke algebra

[1] Perhaps they should be called quantum affine Schur algebras. Since our purpose is to establish an affine analogue of the quantum Schur–Weyl theory, this terminology seems more appropriate to reflect this.

$\mathcal{H}_\triangle(r)_\mathbb{C}$ and a certain full subcategory of quantum affine $\mathfrak{sl}_n$ (resp., $\mathfrak{gl}_n$) has also been established algebraically by Chari–Pressley [**9**] (resp., geometrically by Ginzburg–Reshetikhin–Vasserot [**31**]) under the condition $n > r$ (resp., $n \geqslant r$). Note that the approach in [**31**] uses intersection cohomology complexes. It would be interesting to know how affine quantum Schur algebras would play a role in these works.

Much less progress has been made at Level III. When $n > r$, Doty–Green [**18**] and McGerty [**58**] have found a presentation for affine quantum Schur algebras, while the last two authors of this book have investigated the realization problem in [**24**], where they first developed an approach without using the stabilization property, a key property used in the BLM approach, and presented an ideal candidate for the realization of quantum affine $\mathfrak{gl}_n$.

This book attempts to establish the affine quantum Schur–Weyl theory as a whole and is an outcome of *algebraically* understanding the works mentioned above.

First, building on Schiffmann [**67**] and Hubery [**40**], our starting point is to present the double Ringel–Hall algebra $\mathfrak{D}_\triangle(n)$ of the cyclic quiver with n vertices in terms of Chevalley type generators together with infinitely many central generators. Thus, we obtain a central subalgebra $\mathbf{Z}_\triangle(n)$ such that $\mathfrak{D}_\triangle(n) = \mathbf{U}_\triangle(n)\mathbf{Z}_\triangle(n) \cong \mathbf{U}_\triangle(n) \otimes \mathbf{Z}_\triangle(n)$. We then establish an isomorphism between $\mathfrak{D}_\triangle(n)$ and Drinfeld's quantum affine $\mathfrak{gl}_n$ in the sense of [**20**]. In this way, we easily obtain an action on the tensor space which upon restriction coincides with the Ringel–Hall algebra action defined geometrically by Varagnolo–Vasserot [**73**] and commutes with the affine Hecke algebra action.

Second, by a thorough investigation of a BLM type basis for affine quantum Schur algebras, we introduce certain triangular relations for the corresponding structure constants and, hence, a triangular decomposition for affine quantum Schur algebras. With this decomposition, we establish explicit algebra epimorphisms $\xi_r = \xi_{r,\mathbb{Q}(v)}$ from the double Ringel–Hall algebra $\mathfrak{D}_\triangle(n)$ to affine quantum Schur algebras $\boldsymbol{\mathcal{S}}_\triangle(n,r) := \mathcal{S}_\triangle(n,r)_{\mathbb{Q}(v)}$ for all $r \geqslant 0$. This algebraic construction has several nice applications, especially at Levels II and III. For example, the homomorphic image of commutator formulas for semisimple generators gives rise to a beautiful polynomial identity whose combinatorial proof remains mysterious.

Like the quantum Schur algebra case, we will establish for $n \geqslant r$ a Morita equivalence between affine quantum Schur algebras $\mathcal{S}_\triangle(n,r)_\mathbb{F}$ and affine Hecke algebras $\mathcal{H}_\triangle(r)_\mathbb{F}$ of type A over a field $\mathbb{F}$ with a non-root-of-unity parameter. As a by-product, we prove that every simple $\mathcal{S}_\triangle(n,r)_\mathbb{F}$-module is finite dimensional. Thus, applying the classification of simple $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules by Zelevinsky [**81**] and Rogawski [**66**] yields a classification of simple

$\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. Hence, inflation via the epimorphisms $\xi'_{r,\mathbb{C}}$ gives many finite dimensional simple $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-modules. We will also use $\xi'_{r,\mathbb{C}}$ together with the action on tensor spaces and a result of Chari–Pressley to prove that finite dimensional simple polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$ are all inflations of simple $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. In this way, we can see the bridging role played by affine quantum Schur algebras between representations of quantum affine $\mathfrak{gl}_n$ and those of affine Hecke algebras. Moreover, we obtain a classification of simple $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules in terms of Drinfeld polynomials and, when $n > r$, we identify them with those arising from simple $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules.

Our findings also show that, if we regard the category $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-Mod of $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules as a full subcategory of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$-modules, this category is quite different from the category $\mathcal{C}^{hi} \cap \mathcal{C}'$ considered in [**54**, §6.2]. For example, the latter is completely reducible and simple objects are usually infinite dimensional, while $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-Mod is not completely reducible and all simple objects are finite dimensional. As observed in [**23**, Rem. 9.4(2)] for quantum $\mathfrak{gl}_\infty$ and infinite quantum Schur algebras, this is another kind of phenomenon of infinite type in contrast to the finite type case.

The discussion of the realization and presentation problems is also based on the algebra epimorphisms ξ_r and relies on the use of semisimple generators and indecomposable generators for $\mathfrak{D}_\triangle(n)$ which are crucial to understand the integral structure and multiplication formulas. We first use the new presentation for $\mathfrak{D}_\triangle(n)$ to give a decomposition for $\boldsymbol{\mathcal{S}}_\triangle(n,r) = \mathbf{U}_\triangle(n,r)\mathbf{Z}_\triangle(n,r)$ into a product of two subalgebras, where $\mathbf{Z}_\triangle(n,r)$ is a central subalgebra and $\mathbf{U}_\triangle(n,r)$ is the homomorphic image of $\mathbf{U}_\triangle(n)$, the extended quantum affine $\mathfrak{sl}_n$. By taking a close look at this structure, we manage to get a presentation for $\boldsymbol{\mathcal{S}}_\triangle(r,r)$ for all $r \geqslant 1$ and acknowledge that the presentation problem is very complicated in the $n < r$ case. On the other hand, we formulate a realization conjecture suggested by the work [**24**] and prove the conjecture in the classical ($v = 1$) case.

We remark that, unlike the geometric approach in which the ground ring must be a field or mostly the complex number field $\mathbb{C}$, the algebraic, or rather, the representation-theoretic approach we use in this book works largely over a ring or mostly the integral Laurent polynomial ring $\mathbb{Z}[v, v^{-1}]$.

We have organized the book as follows.

In the first preliminary chapter, we introduce in §1.4 three different types of generators and their associated monomial bases for the Ringel–Hall algebras of cyclic quivers, and display in §1.5 the Green–Xiao Hopf structure on the extended version of these algebras.

Chapter 2 introduces a new presentation using Chevalley generators for Drinfeld's quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ of $\mathfrak{gl}_n$. This is achieved by constructing the presentation for the double Ringel–Hall algebra $\mathfrak{D}_\triangle(n)$ associated with cyclic quivers (Theorem 2.3.1), based on the work of Schiffmann and Hubery, and by lifting Beck's algebra monomorphism from the quantum $\widehat{\mathfrak{sl}}_n$ with a Drinfeld–Jimbo presentation into $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ to obtain an isomorphism between $\mathfrak{D}_\triangle(n)$ and $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$ (Theorem 2.5.3).

Chapter 3 investigates the structure of affine quantum Schur algebras. We first recall the geometric definition by Ginzburg–Vasserot and Lusztig, the Hecke algebra definition by R. Green, and the tensor space definition by Varagnolo–Vasserot. Using the Chevalley generators of $\mathfrak{D}_\triangle(n)$, we easily obtain an action on the $\mathbb{Q}(v)$-space $\boldsymbol{\Omega}$ with a basis indexed by $\mathbb{Z}$ and, hence, an action of $\mathfrak{D}_\triangle(n)$ on $\boldsymbol{\Omega}^{\otimes r}$ (§3.5). We prove that this action commutes with the affine Hecke algebra action defined in [**73**]. Moreover, we show that the restriction of the action to the negative part of $\mathfrak{D}_\triangle(n)$ (i.e., to the corresponding Ringel–Hall algebra) coincides with the Ringel–Hall algebra action geometrically defined by Varagnolo–Vasserot (Theorem 3.6.3). As an application of this coincidence, the commutator formula associated with semisimple generators, arising from the skew-Hopf pairing, gives rise to a certain polynomial identity associated with a pair of elements $\lambda, \mu \in \mathbb{N}^n_\triangle$ (Corollary 3.9.6). The main result of the chapter is an elementary proof of the surjective homomorphism ξ_r from the double Ringel–Hall algebra $\mathfrak{D}_\triangle(n)$, i.e., the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{gl}}_n)$, onto the affine quantum Schur algebra $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ (Theorem 3.8.1). The approach we used is the establishment of a triangular decomposition of $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ (Theorem 3.7.7) through an analysis of the BLM type bases.

In Chapter 4, we discuss the representation theory of affine quantum Schur algebras over $\mathbb{C}$ and its connection to polynomial representations of quantum affine $\mathfrak{gl}_n$ and representations of affine Hecke algebras. We first establish a category equivalence between the module categories $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-Mod and $\mathcal{H}_\triangle(r)_\mathbb{C}$-Mod for $n \geqslant r$ (Theorem 4.1.3). As an application, we will reinterpret Chari–Pressley's category equivalence ([**9**, Th. 4.2]) between (level r) representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{sl}}_n)$ and those of affine Hecke algebras $\mathcal{H}_\triangle(r)_\mathbb{C}$, where $n > r$, in terms of representations of $\mathcal{S}_\triangle(n,r)_\mathbb{C}$ (Proposition 4.2.1). We then develop two approaches to the classification of simple $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules. In the so-called upward approach, we use the classification of simple $\mathcal{H}_\triangle(r)_\mathbb{C}$-modules of Zelevinsky and Rogawski to classify simple $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules (Theorems 4.3.4 and 4.5.3), while in the downward approach, we determine the classification of simple $\mathcal{S}_\triangle(n,r)_\mathbb{C}$-modules (Theorem 4.6.8) in terms of simple polynomial representations of $\mathrm{U}_\mathbb{C}(\widehat{\mathfrak{gl}}_n)$. When $n > r$, we prove an identification theorem (Theorem 4.4.2) for the two classifications. Finally, in §4.7, a classification of finite dimensional simple $U_\triangle(n,r)_\mathbb{C}$-modules is also completed and its connections to finite dimensional simple $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{sl}}_n)$-modules and finite dimensional simple (polynomial) $\mathrm{U}_{\mathbb{C}}(\widehat{\mathfrak{gl}}_n)$-modules are also discussed.

We move on to look at the presentation and realization problems in Chapter 5. We first observe $\boldsymbol{\mathcal{S}}_\triangle(n,r) = \mathbf{U}_\triangle(n,r)\mathbf{Z}_\triangle(n,r)$, where $\mathbf{U}_\triangle(n,r)$ and $\mathbf{Z}_\triangle(n,r)$ are homomorphic images of $\mathbf{U}_\triangle(n)$ and $\mathbf{Z}_\triangle(n)$, respectively, and that $\mathbf{Z}_\triangle(n,r) \subseteq \mathbf{U}_\triangle(n,r)$ if and only if $n > r$. A presentation for $\mathbf{U}_\triangle(n,r)$ is given in [**58**] (see also [**19**] for the $n > r$ case). Building on McGerty's presentation, we first give a Drinfeld–Jimbo type presentation for the subalgebra $\mathbf{U}_\triangle(n,r)$ (Theorem 5.1.3). We then describe a presentation for the central subalgebra $\mathbf{Z}_\triangle(n,r)$ as a Laurent polynomial ring in one indeterminate over a polynomial ring in $r-1$ indeterminates over $\mathbb{Q}(v)$. We manage to describe a presentation for $\boldsymbol{\mathcal{S}}_\triangle(r,r)$ for all $r \geqslant 1$ (Theorem 5.3.5) by adding an extra generator (and its inverse) together with an additional set of relations on top of the relations given in Theorem 5.1.3. What we will see from this case is that the presentation for $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ with $r > n$ can be very complicated.

We discuss the realization problem from §5.4 onwards. We first describe the modified BLM approach developed in [**24**]. With some supporting evidence, we then formulate the realization conjecture (Conjecture 5.4.2) as suggested in [**24**, 5.5(2)], and state its classical ($v = 1$) version. We end the chapter with a closer look at Lusztig's transfer maps [**57**] by displaying some explicit formulas for their action on the semisimple generators for $\boldsymbol{\mathcal{S}}_\triangle(n,r)$ (Corollary 5.5.2). These formulas also show that the homomorphism from $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$ to $\varprojlim \boldsymbol{\mathcal{S}}_\triangle(n,n+m)$ induced by the transfer maps cannot either be extended to the double Ringel–Hall algebra ${}'\mathfrak{D}_\triangle(n)$. (Lusztig already pointed out that it cannot be extended to $\mathbf{U}_\triangle(n)$.) This somewhat justifies why a direct product is used in the realization conjecture.

In the final Chapter 6, we prove the realization conjecture for the classical ($v = 1$) case. The key step in the proof is the establishment of more multiplication formulas (Proposition 6.2.3) between homogeneous indecomposable generators and an arbitrary BLM type basis element. As a by-product, we display a basis for the universal enveloping algebra of the loop algebra of $\mathfrak{gl}_n$ (Theorem 6.3.4) together with explicit multiplication formulas between generators and arbitrary basis elements (Corollary 6.3.5).

There are two appendices in §§3.10 and 6.4 which collect a number of lengthy calculations used in some proofs.

Conjectures and problems. There are quite a few conjectures and problems throughout the book. The conjectures are mostly natural generalizations to the affine case, for example, the realization conjecture 5.4.2 and the conjectures in §3.8 on an integral form for double Ringel–Hall algebras and the second centralizer property in the affine quantum Schur–Weyl reciprocity. Some problems are designed to seek further solutions to certain questions such as "quantum Serre relations" for semisimple generators (Problem 2.6.4), the Affine Branching Rule (Problem 4.3.6), and further identification of simple modules from different classifications (Problem 4.6.11). There are also problems for seeking different proofs. Problems 3.4.3 and 6.4.2 form a key step towards the proof of the realization conjecture.

Notational scheme. For most of the notation used throughout the book, if it involves a subscript $_\triangle$ or a superscript $^\triangle$, it indicates that the same notation without $\triangle$ has been used in the non-affine case, say, in [**4**], [**12**], [**33**], etc. Here the triangle $\triangle$ depicts the cyclic Dynkin diagram of affine type A.

For a ground ring $\mathcal{Z}$ and a $\mathcal{Z}$-module (or a $\mathcal{Z}$-algebra) $\mathcal{A}$, we often use the notation $\mathcal{A}_{\mathbb{F}} := \mathcal{A} \otimes \mathbb{F}$ to represent the object obtained by *base change* to a field $\mathbb{F}$, which itself is a $\mathcal{Z}$-module. In particular, if $\mathcal{Z} = \mathbb{Z}[v, v^{-1}]$, then we write $\boldsymbol{\mathcal{A}}$ for $\mathcal{A}_{\mathbb{Q}(v)}$.

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- Southeastern Lie Theory Workshop: Finite and Algebraic Groups and Leonard Scott Day, Charlottesville, June 2011;
- 55th Annual Meeting of the Australian Mathematical Society, Wollongong, September 2011.

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1

Preliminaries

We start with the loop algebra of $\mathfrak{gl}_n(\mathbb{C})$ and its interpretation in terms of matrix Lie algebras. We use the subalgebra of integer matrices of the latter to introduce several important index sets which will be used throughout the book. Ringel–Hall algebras $\mathfrak{H}_\triangle(n)$ associated with cyclic quivers $\triangle(n)$ and their geometric construction are introduced in §1.2. In §1.3, we discuss the composition subalgebra $\mathfrak{C}_\triangle(n)$ of $\mathfrak{H}_\triangle(n)$ and relate it to the quantum loop algebra $\mathbf{U}(\widehat{\mathfrak{sl}}_n)$. We then describe in §1.4 three types of generators for $\mathfrak{H}_\triangle(n)$, which consist of all simple modules together with, respectively, the Schiffmann–Hubery central elements, homogeneous semisimple modules, and homogeneous indecomposable modules, and their associated monomial bases (Corollaries 1.4.2 and 1.4.6). These generating sets will play different roles in what follows. Finally, extended Ringel–Hall algebras and their Hopf structure are discussed in §1.5.

1.1. The loop algebra $\widehat{\mathfrak{gl}}_n$ and some notation

For a positive integer n, let $\mathfrak{gl}_n(\mathbb{C})$ be the complex general linear Lie algebra, and let

$$\widehat{\mathfrak{gl}}_n(\mathbb{C}) := \mathfrak{gl}_n(\mathbb{C}) \otimes \mathbb{C}[t, t^{-1}]$$

be the loop algebra of $\mathfrak{gl}_n(\mathbb{C})$; see [**47**]. Thus, $\widehat{\mathfrak{gl}}_n(\mathbb{C})$ is spanned by $E_{i,j} \otimes t^m$ for all $1 \leqslant i, j \leqslant n$, and $m \in \mathbb{Z}$, where $E_{i,j}$ is the matrix $(\delta_{k,i}\delta_{j,l})_{1\leqslant k,l\leqslant n}$. The (Lie) multiplication is the bracket product associated with the multiplication

$$(E_{i,j} \otimes t^m)(E_{k,l} \otimes t^{m'}) = \delta_{j,k} E_{i,l} \otimes t^{m+m'}.$$

We may interpret the Lie algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$ as a matrix Lie algebra. Let $M_{\triangle,n}(\mathbb{C})$ be the set of all $\mathbb{Z}\times\mathbb{Z}$ complex matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}}$ with $a_{i,j} \in \mathbb{C}$

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such that

(a) $a_{i,j} = a_{i+n,j+n}$ for $i, j \in \mathbb{Z}$, and
(b) for every $i \in \mathbb{Z}$, the set $\{j \in \mathbb{Z} \mid a_{i,j} \neq 0\}$ is finite.

Clearly, conditions (a) and (b) imply that there are only finitely many non-zero entries in each column of A. For $A, B \in M_{\triangle,n}(\mathbb{C})$, let $[A, B] = AB - BA$. Then $(M_{\triangle,n}(\mathbb{C}), [\ ,\ ])$ becomes a Lie algebra over $\mathbb{C}$.

Denote by $M_{n,\bullet}(\mathbb{C})$ the set of $n \times \mathbb{Z}$ matrices $A = (a_{i,j})$ over $\mathbb{C}$ satisfying (b) with $i \in [1, n] := \{1, 2, \ldots, n\}$. Then there is a bijection

$$\flat_1 : M_{\triangle,n}(\mathbb{C}) \longrightarrow M_{n,\bullet}(\mathbb{C}), \quad (a_{i,j})_{i,j\in\mathbb{Z}} \longmapsto (a_{i,j})_{1\leqslant i\leqslant n, j\in\mathbb{Z}}. \quad (1.1.0.1)$$

For $i, j \in \mathbb{Z}$, let $E^{\triangle}_{i,j} \in M_{\triangle,n}(\mathbb{C})$ be the matrix $(e^{i,j}_{k,l})_{k,l\in\mathbb{Z}}$ defined by

$$e^{i,j}_{k,l} = \begin{cases} 1, & \text{if } k = i + sn, l = j + sn \text{ for some } s \in \mathbb{Z}; \\ 0, & \text{otherwise.} \end{cases}$$

The set $\{E^{\triangle}_{i,j} | 1 \leqslant i \leqslant n, j \in \mathbb{Z}\}$ is a $\mathbb{C}$-basis of $M_{\triangle,n}(\mathbb{C})$. Since

$$E^{\triangle}_{i,j+ln} E^{\triangle}_{p,q+kn} = \delta_{j,p} E^{\triangle}_{i,q+(l+k)n},$$

for all $i, j, p, q, l, k \in \mathbb{Z}$ with $1 \leqslant j, p \leqslant n$, it follows that the map

$$M_{\triangle,n}(\mathbb{C}) \longrightarrow \widehat{\mathfrak{gl}}_n(\mathbb{C}), \quad E^{\triangle}_{i,j+ln} \longmapsto E_{i,j} \otimes t^l, \ 1 \leqslant i, j \leqslant n, l \in \mathbb{Z}$$

is a Lie algebra isomorphism. We will identify the loop algebra $\widehat{\mathfrak{gl}}_n(\mathbb{C})$ with $M_{\triangle,n}(\mathbb{C})$ in the sequel.

In Chapter 6, we will consider the loop algebra $\widehat{\mathfrak{gl}}_n := \widehat{\mathfrak{gl}}_n(\mathbb{Q}) = M_{\triangle,n}(\mathbb{Q})$ defined over $\mathbb{Q}$ and its universal enveloping algebra $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ and triangular parts $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^+, \mathcal{U}(\widehat{\mathfrak{gl}}_n)^-$, and $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^0$. Here $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^+$ (resp., $\mathcal{U}(\widehat{\mathfrak{gl}}_n)^-, \mathcal{U}(\widehat{\mathfrak{gl}}_n)^0$) is the subalgebra of $\mathcal{U}(\widehat{\mathfrak{gl}}_n)$ generated by $E^{\triangle}_{i,j}$ for all $i < j$ (resp., $E^{\triangle}_{i,j}$ for all $i > j$, $E^{\triangle}_{i,i}$ for all i). We will also relate these algebras in §6.1 with the specializations at $v = 1$ of the Ringel–Hall algebra $\mathfrak{H}_{\triangle}(n)$ and the double Ringel–Hall algebra $\mathfrak{D}_{\triangle}(n)$.

We now introduce some **notation** which will be used throughout the book.

Consider the subset $M_{\triangle,n}(\mathbb{Z})$ of $M_{\triangle,n}(\mathbb{C})$ consisting of matrices with integer entries. For each $A \in M_{\triangle,n}(\mathbb{Z})$, let

$$\mathrm{ro}(A) = \Big(\sum_{j\in\mathbb{Z}} a_{i,j}\Big)_{i\in\mathbb{Z}} \quad \text{and} \quad \mathrm{co}(A) = \Big(\sum_{i\in\mathbb{Z}} a_{i,j}\Big)_{j\in\mathbb{Z}}.$$

We obtain functions

$$\mathrm{ro}, \mathrm{co} : M_{\triangle,n}(\mathbb{Z}) \longrightarrow \mathbb{Z}^n_{\triangle},$$