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Introduction

This volume collects lecture notes from talks given in the Appalachian Set Theory workshop series (supported by the National Science Foundation) during the period 2006–2012.

This workshop series grew out of an informal series of expository lectures held at Carnegie Mellon University and attended by set theorists from universities in Appalachian states before 2006. The success of these earlier gatherings inspired the editors to formalize the series and seek funding to help more people attend. Participants from other universities were invited to host workshops as well. Typically there are three meetings a year with one taking place at Carnegie Mellon University and the remaining two elsewhere. Several of the workshops have been held in neighbouring regions but the series retains its Appalachian flavour.

At each workshop a leading researcher lectures for six hours on an important topic or technique in modern set theory. Students are engaged to assist in writing notes based on the lectures, and these notes are disseminated on the web. This provides a learning opportunity for the students and makes the notes universally available.

The papers collected here represent more polished versions of the lecture notes from most of the workshops to date. They were prepared collaboratively by the lecturers and the student assistants. The lecturers are the principal authors and their names appear first, followed by the names of the assistants. One workshop (represented in Chapter 7) had two lecturers, and two workshops (represented in Chapters 1 and 13) had two assistants. All the papers in this volume were refereed, several by two referees; many of the referee reports were outstandingly detailed and helpful.

The main goals of the series are:

- To disseminate important ideas (some of which have been known to experts for a long time) to a wider audience, and in particular to make them accessible to graduate students.
- To bring researchers in set theory into contact with each other, and strengthen the set-theoretic research community.

The workshops have covered a wide range of topics including forcing and large cardinals, descriptive set theory, and applications of set-theoretic ideas in group theory and analysis. In line with our goals, about half of the workshop audiences have been students and postdoctoral researchers. We have also attracted regular faculty mathematicians at all levels of seniority. Most of the participants work in various parts of set theory but a significant number are experts in other areas of mathematics.

Information about past and future workshops in the series can be found at the URL

`http://www.math.cmu.edu/~eschimme/Appalachian/Index.html`

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Pittsburgh, June 2012

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Many thanks are also due to the local organisers of the meetings: Dietmar Bisch, Elizabeth Brown, John Clemens, Alan Dow, Todd Eisworth, George Elliott, Ilijas Farah, Paul Larson, Justin Moore, Christian Rosendal, Steve Simpson, and Juris Steprāns.

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1

An introduction to $\mathbb{P}_{\max}$ forcing

Paul B. Larson[a], Peter Lumsdaine and Yimu Yin

The first Appalachian Set Theory workshop was held at Carnegie Mellon University in Pittsburgh on September 9, 2006. The lecturer was Paul Larson. As graduate students Peter Lumsdaine and Yimu Yin assisted in writing this chapter, which is based on the workshop lectures.

1 Introduction

The forcing construction $\mathbb{P}_{\max}$ was invented by W. Hugh Woodin in the early 1990's in the wake of his result that the saturation of the nonstationary ideal on ω_1 (NS_{ω_1}) plus the existence of a measurable cardinal implies that there is a definable counterexample to the Continuum Hypothesis (in particular, it implies that $\boldsymbol{\delta}^1_2 = \omega_2$, which implies $\neg$CH). These notes outline a proof of the Π_2 maximality of the $\mathbb{P}_{\max}$ extension, which we can state as follows.

Theorem 1.1 ([9]) *Suppose that there exist proper class many Woodin cardinals, $A \subseteq \mathbb{R}$, $A \in L(\mathbb{R})$, φ is Π_2 in the extended language containing two additional unary predicates, and in some set forcing extension*

$$\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A^* \rangle \models \varphi$$

(where A^* is the reinterpretation of A in this extension). Then

$$L(\mathbb{R})^{\mathbb{P}_{\max}} \models [\langle H(\omega_2), \in, \mathrm{NS}_{\omega_1}, A \rangle \models \varphi]\,.$$

[a] Research supported by NSF grants DMS-0401603 and DMS-0801009.

Forcing with $\mathbb{P}_{\max}$ does not add reals, so there is no need to reinterpret A in the last line of the theorem. The theorem says that any such Π_2 statement that we can force in any extension must hold in the $\mathbb{P}_{\max}$ extension of $L(\mathbb{R})$, so $H(\omega_2)$ of $L(\mathbb{R})^{\mathbb{P}_{\max}}$ is maximal, or complete, in a certain sense; among other things, it models ZFC, Martin's Axiom, certain fragments of Martin's Maximum [1], and the negation of the Continuum Hypothesis. The reinterpretation A^* will be defined later, in terms of tree representations for sets of reals. We will not give the definition of Woodin cardinals (but see [4]).

We have reworked the standard proof of Theorem 1.1 in order to minimize the prerequisites. In particular, the need for (mentioning) sharps has been eliminated. However, they and much more will need to be reintroduced to go any further than the material presented here.

Woodin's book on $\mathbb{P}_{\max}$, *The axiom of determinacy, forcing axioms and the nonstationary ideal* [9] runs to around 1000 pages. The first author's article for the Handbook of Set Theory [6], introducing $\mathbb{P}_{\max}$, has about 65. The advance notes for these lectures were about 30 pages, and previous lecture courses have taken about 12-15 hours to cover $\mathbb{P}_{\max}$. Today we have about six hours, so we will, of course, have to be brief…

2 Setup: iterations and the definition of $\mathbb{P}_{\max}$

Suppose that $M \models$ ZFC and that $I \in M$ is a normal ideal on ω_1^M in M. Force over M with $((\mathcal{P}(\omega_1) \setminus I)/I)^M$. The resulting generic G is now an M-normal ultrafilter on ω_1^M; so we may form the corresponding ultrapower and elementary embedding

$$j : M \to \mathrm{Ult}(M, G) := \{f : \omega_1^M \to M \mid f \in M\}/ =_G .$$

(We will use this construction many times today.) Via Fodor's Lemma, we get that $\mathrm{crit}(j) = \omega_1^M$, and for $A \in \mathcal{P}(\omega_1)^M$,

$$A \in G \iff \omega_1^M \in j(A).$$

Via consideration of surjections from ω_1^M onto each ordinal below ω_2^M, we get that $j(\omega_1^M) \geq \omega_2^M$. There is no need to assume that M is well-founded, though it will be in the cases we are interested in. When the model $\mathrm{Ult}(M, G)$ is well-founded, we identify it with its Mostowski collapse. It is not hard to see that in this case $\mathrm{Ord}^{\mathrm{Ult}(M,G)} = \mathrm{Ord}^M$.

Definition 2.1 I is *precipitous* if $\mathrm{Ult}(M, G)$ thus constructed is well-founded from the point of view of $M[G]$, for all M-generic G. (N.B. this is definable in M via forcing.)

We need a pair of theories satisfying the following conditions.

- T_0, a theory consistent with ZFC and strong enough to make sense of the generic ultrapower construction above and prove that $j : M \to \mathrm{Ult}(M, G)$ is elementary.
- T_1, a theory consistent with ZFC and at least as strong as T_0 + "every set lies in some $H(\kappa) \models T_0$."

In [6], we take T_0 (which we call ZFC°) to be ZFC - Replacement - Powerset plus "$\mathcal{P}(\mathcal{P}(\omega_1))$ exists" plus the scheme saying that definable trees of height ω_1 have maximal branches. Then we let

$$T_1 = T_0 + \text{Powerset} + \text{Choice} + \Sigma_1\text{–Replacement}$$

(though we don't express it in these terms). In [9], Woodin has an even weaker fragment of ZFC (which he calls ZFC*) playing the role of T_0. Today we may as well let T_0 be ZFC and T_1 be ZFC plus the existence of a proper class of strongly inaccessible cardinals. From now on we will just use the terms T_0 and T_1.

We now extend the generic ultrapower construction given above to iterated ultrapowers. Suppose we have (M_0, I_0), $G_0 \subseteq (\mathcal{P}(\omega_1)/I_0)^{M_0}$,

$$j_0 : (M_0, I_0) \to \mathrm{Ult}(M_0, G_0),$$

all as before; let $M_1 = \mathrm{Ult}(M_0, G_0)$, $I_1 = j_0(I_0)$. Now we can take the generic ultrapower of M_1 by I_1, and iterate. At limit stages, we have a directed system of elementary embeddings, so we can just take the direct limit, so we can keep going up to length ω_1. (No further, as if we force again there, we collapse ω_1^V, so are back to countable length!) Note that the final model of the iteration, M_{ω_1}, is an element of $H(\omega_2)$.

Definition 2.2 An *iteration* of (M, I) (as above, with $\mathcal{P}(\mathcal{P}(\omega_1))^M$ countable) of length γ consists of M_α, I_α ($\alpha \leq \gamma$), G_η ($\eta < \gamma$), and $j_{\alpha,\beta}$ ($\alpha \leq \beta \leq \gamma$), satisfying

- $M_0 = M$, $I_0 = I$.
- G_η is M_η-generic for $(P(\omega_1)/I_\eta)^{M_\eta}$.
- $j_{\eta,\eta+1}$ is the canonical embedding of M_η into $\mathrm{Ult}(M_\eta, G_\eta) = M_{\eta+1}$.
- $j_{\alpha,\beta} : M_\alpha \to M_\beta$ are a commuting family of elementary embeddings.

- $I_\beta = j_{0,\beta}(I)$.
- For limit β, M_β is the direct limit of $\{M_\alpha \mid \alpha < \beta\}$ under the embeddings $j_{\alpha,\eta}$ ($\alpha \leq \eta < \beta$).

In practice, we almost always have $\gamma = \omega_1^N$ for some larger $N \supseteq M$. We will generally write $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ for the iteration, or just "j is an iteration" to mean that j is the $j_{0,\gamma}$ of an iteration, with $j(M)$ for M_γ. (As we will see, in some circumstances, if we know M_0, M_γ, $j_{0,\gamma}$, we can (with slight assumptions) recover the full iteration.) We say that the M_α's are *iterates* of (M, I); (M, I) is *iterable* if all iterates are well-founded; and (M, I) is an *iterable pair* if M is a transitive model of T_0 with $\mathcal{P}(\mathcal{P}(\omega_1))^M$ countable, I a normal ideal on $\mathcal{P}(\omega_1)$ in M, and (M, I) is iterable.

If M is well-founded and $M \models$ "I is precipitous", then certainly (M, I) is finitely iterable (i.e., its finite-length iterations produce wellfounded models); and in fact, we will show that in this case (M, I) is iterable to any $\alpha \in \mathrm{Ord}^M$.

The proof of the following lemma is left as an exercise (the proof is by induction on the length of the iteration). In a typical application, M is $H(\kappa)^N$ for some suitable κ.

Lemma 2.3 *Suppose that $M \in N$ are models of T_0, M is closed under ω_1-sequences from N, and $\mathcal{P}(\mathcal{P}(\omega_1))^M = \mathcal{P}(\mathcal{P}(\omega_1))^N$. Let $I \in M$ be an M-normal ideal on ω_1^M. Then the following hold.*

- *For each iteration $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ of (M, I) there is a unique iteration $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ of (N, I) such that for all $\beta \leq \gamma$: $j^*_{0,\beta}(M) = M_\beta$, M_β is closed under ω_1-sequences from N_β, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, and $j^*_{\alpha,\beta} \restriction_{M_\alpha} = j_{\alpha,\beta}$.*
- *For each iteration $\langle N_\alpha, I_\alpha, G_\eta, j^*_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ of (N, I) there is a unique iteration $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \gamma,\ \eta < \gamma \rangle$ of (M, I) such that for all $\beta \leq \gamma$: $j^*_{0,\beta}(M) = M_\beta$, M_β is closed under ω_1-sequences from N_β, $\mathcal{P}(\mathcal{P}(\omega_1))^{M_\beta} = \mathcal{P}(\mathcal{P}(\omega_1))^{N_\beta}$, and $j^*_{\alpha,\beta} \restriction_{M_\alpha} = j_{\alpha,\beta}$.*

Corollary 2.4 *In the context of Lemma 2.3, if (M, I) has an ill-founded iterate by an iteration of length α, then so does (N, I).*

Lemma 2.5 below then shows that (M, I) is iterable if N contains ω_1 (recall that iterations can have length at most ω_1, and note that an illfounded iteration of length ω_1 must be illfounded at some countable stage).

First we fix a coding of elements of $H(\omega_1)$ by reals. Fix a recursive bijection $\pi : \omega \times \omega \to \omega$, and say $X \subseteq \omega$ *codes* $a \in H(\omega_1)$ if

$$(\mathrm{tc}(\{a\}), \in) \cong (\omega, \{(i, j) \mid \pi(i, j) \in X\}),$$

where $tc(b)$ denotes the transitive closure of b. Then $\in$ and $=$ are Σ^1_1 (as permutations of ω induce different codes for the same object).

Lemma 2.5 *Suppose that N is a transitive model of T_1, $\gamma \in \mathrm{Ord}^N$, and I is a normal precipitous ideal on ω_1^N in N. Then any iterate of (N, I) by an iteration of length γ is well-founded.*

Proof It suffices to prove that iterations of the form $(H(\kappa)^N, I)$ produce wellfounded models for all $\kappa \in N$ such that $H(\kappa)^N \models T_0$; for if any iterate of N is ill-founded, then some ordinal in N is large enough to witness this (i.e., $\sup(\mathrm{rge}(f))$, where f witnesses ill-foundedness) and by assumption (as $N \models T_1$) this is contained in some $H(\kappa)^N$ that models T_0.

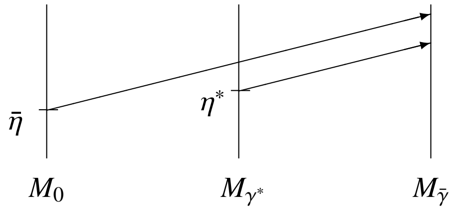
Let $(\bar{\gamma}, \bar{\kappa}, \bar{\eta})$ be the lexicographically minimal triple (γ, κ, η) satisfying (with N) the formula $\varphi(N, \gamma, \kappa, \eta)$ defined by "$H(\kappa)^N \models T_0$ and there exists an iteration of $(H(\kappa)^N, I)$ of length γ which is ill-founded below the image of η".

Using our fixed coding of elements of $H(\omega_1)$ by reals there is a Σ^1_1 formula $\varphi'(x, y, z)$ saying "x codes a model of T_0 and a normal ideal in the model on the ω_1 of the model and there exists an iteration of this pair whose length is coded by y and which is illfounded below the image of the element of this model coded by z."

For all cardinals $\kappa, \rho \in N$ and all ordinals $\gamma, \eta \in N$, if $\rho \in N$ is larger than $|H(\kappa)|^N$, $|\eta|^N$ and $|\gamma|^N$, then there exist reals coding $H(\kappa)^N$, η, and γ in any forcing extension of N by $\mathrm{Coll}(\omega, \rho)$. Such an extension is correct about whether these reals satisfy φ'. However, this is a homogeneous forcing extension of N; so there is a formula $\psi(\gamma, \kappa, \eta)$ saying that in every forcing extension in which $H(\kappa)$ (of the ground model), η and γ are all countable there exist reals coding $H(\kappa)$, η and γ which satisfy φ'. It follows that that $N \models \psi(\gamma, \kappa, \eta)$ if and only if $\varphi(N, \gamma, \kappa, \eta)$ holds, and furthermore, since φ' is Σ^1_1, for all well-founded iterates N^* of N, and all $\gamma, \kappa, \eta \in N^*$, $N^* \models \psi(\gamma, \kappa, \eta)$ if and only if $\varphi(N^*, \gamma, \kappa, \eta)$ holds.

Since I is precipitous in N, $\bar{\gamma}$ is a limit ordinal, and clearly $\bar{\eta}$ is a limit ordinal as well. Fix an iteration $\langle M_\alpha, G_\beta, j_{\alpha,\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ of $(H(\bar{\kappa})^N, I)$ such that $j_{0,\bar{\gamma}}(\bar{\eta})$ is not wellfounded, and let $\langle N_\alpha, G_\beta, j'_{\alpha,\delta} : \beta < \alpha \leq \delta \leq \bar{\gamma}\rangle$ be the corresponding iteration of N as in Lemma 2.3. By the minimality of $\bar{\gamma}$ we have that N_α is wellfounded for all $\alpha < \bar{\gamma}$. Since

$M_{\bar{\gamma}}$ is the direct limit of the iteration leading up to it, we may fix $\gamma^* < \bar{\gamma}$ and $\eta^* < j_{0,\gamma^*}(\bar{\eta})$ such that $j_{\gamma^*,\bar{\gamma}}(\eta^*)$ is not wellfounded. By Lemma 2.3, $j'_{\gamma^*,\bar{\gamma}}(\eta^*) = j_{\gamma^*,\bar{\gamma}}(\eta^*)$ and $j'_{\gamma^*,\bar{\gamma}}(\bar{\eta}) = j_{\gamma^*,\bar{\gamma}}(\bar{\eta})$.



But now, $N_{\gamma^*} \models \psi(\bar{\gamma} - \gamma^*, j_{0,\gamma^*}(\bar{\kappa}), \eta^*)$, $\bar{\gamma} - \gamma^* \leq \bar{\gamma}$, and $\eta^* < j_{0,\gamma^*}(\bar{\eta})$, contradicting the minimality of $(j_{0,\gamma^*}(\bar{\gamma}), j_{0,\gamma^*}(\bar{\kappa}), j_{0,\gamma^*}(\bar{\eta}))$ in N_{γ^*}. □

We note that ZFC does not imply the existence of iterable pairs. However, by Lemma 2.5, if there exist a normal, precipitous ideal J on ω_1, and a measurable cardinal κ with a κ-complete ultrafilter μ, then there exist iterable pairs. The main point here is that if $\theta > \kappa$ is a regular cardinal and X is a countable elementary submodel of $H(\theta)$ with $\kappa, J \in X$, then X can be end-extended below κ by taking γ to be any member of $\bigcap(X \cap \mu)$, and letting $X[\gamma]$ be the set of values $f(\gamma)$ for all functions f in X with domain κ. Applying this fact ω_1 many times, we get that the transitive collapse M of $X \cap V_\kappa$ is a countable model which is a rank initial segment of a model containing ω_1. Letting I be the image of J under the transitive collapse, then, (M, I) is an iterable pair. This is a special case of the proof of Lemma 4.6, and a key point in Woodin's proof (which appears in Chapter 3 of [9]) that if there exists a measurable cardinal and the nonstationary ideal on ω_1 is saturated, then CH fails.

If there is a precipitous ideal on ω_1, then sharps exist for subsets of ω_1, and a countable iterable model will be correct about these sharps. We will work around this today to avoid having to talk about sharps.

Lemma 2.6 *If (M, I) is an iterable pair and A is an element of $\mathcal{P}(\omega_1)^M$, then $(\omega_1^{L[A]})^M = \omega_1^{L[A]}$.*

Proof Let $\langle M_\alpha, I_\alpha, G_\eta, j_{\alpha,\beta} \mid \alpha \leq \beta \leq \omega_1,\ \eta < \omega_1 \rangle$ be an iteration of (M, I). The ordinals of M and M_1 are the same, so $L[A]^M = L[A]^{M_1}$. The critical point of j_{1,ω_1} is greater than ω_1^M, and thus greater than the ω_1 of $L[A]^M$. The restriction of j_{1,ω_1} to $L[A]^M$ embeds $L[A]^M$ elementarily into $L[A]^{M_{\omega_1}}$, which means that $L[A]^M$ and $L[A]^{M_{\omega_1}}$ have the same ω_1. Since $\omega_1 \subset M_{\omega_1}$, $(\omega_1^{L[A]})^{M_{\omega_1}} = \omega_1^{L[A]}$. □

Now we can define $\mathbb{P}_{\max}$.

Definition 2.7 The partial order $\mathbb{P}_{\max}$ is the set of pairs $\langle (M, I), a\rangle$ such that

1. M is a countable transitive model of $T_0 + \mathrm{MA}_{\aleph_1}$,
2. (M, I) is an iterable pair,
3. $a \in \mathcal{P}(\omega_1)^M$ and there exists $x \in \mathcal{P}(\omega)^M$ such that $\omega_1^{L[x,a]} = \omega_1^M$,

ordered by: $p < q$ (where $p = \langle (M, I), a\rangle$, $q = \langle (N, J), b\rangle$) if there is some iteration $j : (N, J) \to (N^*, J^*)$ such that

1. $j \in M$,
2. $j(b) = a$,
3. $J^* = N^* \cap I$ (and hence $j(\omega_1^N) = \omega_1^M$),
4. $q \in H(\omega_1)^M$.

Note that since $j \in M$ in definition of the $\mathbb{P}_{\max}$ order above, N and N^* are both in M as well.

Definition 2.8 We say that (M, I) is a $\mathbb{P}_{\max}$ *precondition* if there exists an a such that $\langle (M, I), a\rangle \in \mathbb{P}_{\max}$, or equivalently just if (M, I) satisfies conditions 1 and 2 in the definition of $\mathbb{P}_{\max}$ conditions above.

Suppose that (M, I) is an iterable pair, and $j\colon (M, I) \to (M', I')$ is an iteration. Then for any $A \in \mathcal{P}(\omega_1)^M$ which is bounded in ω_1^M, $j(A) = A$. By Lemma 2.6, it follows then that $\omega_1^{L[A]} < \omega_1^M$, since $j(\omega_1^M) > \omega_1^M$ if j is nontrivial. Therefore, the set a from a $\mathbb{P}_{\max}$ condition $\langle (M, I), a\rangle$ must always be unbounded in ω_1^M to make $\omega_1^{L[x,a]} = \omega_1^M$ possible.

If $p_0 < p_1 < p_2$ ($p_i = \langle (M_i, I_i), a_i\rangle$), and these are witnessed by $j_{1,0}$, $j_{2,1}$, then $p_0 < p_2$ is witnessed by $j_{1,0}(j_{2,1})$: $j_{1,0} \in H(\omega_2)^{M_0}$, $j_{2,1} \in H(\omega_2)^{M_1}$; $j_{2,1}$ is an iteration of (M_2, I_2), and $j_{1,0}((M_2, I_2)) = (M_2, I_2)$.

Under our fixed coding, "(M, I) is iterable" is Π^1_2 in a code for (M, I): roughly, "for anything satisfying the first-order properties of being an iteration, either there is no infinite descending sequence in the ordinals of the final model, or there is an infinite descending sequence in the indices of the iteration." Since iterable models embed elementarily into models containing ω_1, they are $\mathbf{\Pi}^1_2$-correct. It follows that "(M, I) is iterable" is absolute to iterable models containing a code for (M, I).

So now we see that $\mathbb{P}_{\max} \in L(\mathbb{R})$ — all constructions involved are nicely codable.