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Part I

Introduction to variational inequalities

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Preliminaries on functional analysis

This chapter presents preliminary material from functional analysis which will be used in subsequent chapters. Some of the results are stated without proofs, since they are standard and can be found in many references. Nevertheless, we pay particular attention to the results which are repeatedly used in the following chapters of the book, for which we present the details in proofs. They include the Banach fixed point theorem, the projection lemma, the Riesz representation theorem and the Weiersterass theorem, among others. All the linear spaces considered in this book including abstract normed spaces, Banach spaces, Hilbert spaces and various function spaces are assumed to be real linear spaces.

1.1 Normed spaces

We start this section with basic definitions, notation and results concerning the normed spaces. Then we recall two main fixed point results: the Banach fixed point theorem and the Schauder fixed point theorem.

1.1.1 Basic definitions

Given a linear space X, we recall that a *norm* $\|\cdot\|_X$ is a function from X to $\mathbb{R}$ with the following properties.

(1) $\|u\|_X \geq 0 \;\; \forall\, u \in X$, and $\|u\|_X = 0$ iff $u = 0_X$.

(2) $\|\alpha\, u\|_X = |\alpha|\, \|u\|_X \quad \forall\, u \in X, \forall\, \alpha \in \mathbb{R}.$

(3) $\|u + v\|_X \le \|u\|_X + \|v\|_X \quad \forall\, u, v \in X.$

The pair $(X, \|\cdot\|_X)$ is called a *normed space*. Here and everywhere in this book 0_X will denote the zero element of X. Also, we will simply say X is a normed space when the definition of the norm is understood from the context.

On a linear space various norms can be defined. Sometimes it is desirable to know if two norms are related. Let $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ be two norms over a linear space X. The two norms are said to be *equivalent* if there exist two constants c_1, $c_2 > 0$ such that

$$c_1\, \|u\|^{(1)} \le \|u\|^{(2)} \le c_2\, \|u\|^{(1)} \quad \forall\, u \in X. \tag{1.1}$$

We recall that a sequence $\{u_n\} \subset X$ is said to *converge* (strongly) to $u \in X$ if

$$\|u_n - u\|_X \to 0 \quad \text{as } n \to \infty. \tag{1.2}$$

In this case u is called the (strong) *limit* of the sequence $\{u_n\}$ and we write

$$u = \lim_{n\to\infty} u_n \quad \text{or} \quad u_n \to u \quad \text{in } X.$$

It is straightforward to verify that the limit of a sequence, if it exists, is unique. The adjective “strong” is introduced in the previous definition to distinguish this convergence from the weak convergence which will be introduced on page 6.

A sequence $\{u_n\} \subset X$ is said to be *bounded* if there exists $M > 0$ such that

$$\|u_n\|_X \le M \qquad \forall\, n \in \mathbb{N} \tag{1.3}$$

or, equivalently, if

$$\sup_n \|u_n\|_X < \infty.$$

To test the convergence of a sequence without knowing its limit, it is usually convenient to refer to the notion of a Cauchy sequence. Let X be a normed space. A sequence $\{u_n\} \subset X$ is called a *Cauchy sequence* if

$$\|u_m - u_n\|_X \to 0 \quad \text{as } m, n \to \infty.$$

Obviously, a convergent sequence is a Cauchy sequence, but in a general infinite dimensional space, a Cauchy sequence may fail to converge. This justifies the following definition.

Definition 1.1 A normed space is said to be *complete* if every Cauchy sequence from the space converges to an element in the space. A complete normed space is called a *Banach space*.

From this definition it follows that $(X, \|\cdot\|_X)$ is a Banach space iff for every Cauchy sequence $\{u_n\} \subset X$ there exists an element $u \in X$ such that $u_n \to u$ in X. Moreover, using (1.1) and (1.2) it is easy to see that, for two equivalent norms, convergence in one norm implies convergence in the other norm. As a consequence, if $\|\cdot\|^{(1)}$ and $\|\cdot\|^{(2)}$ are equivalent norms on the linear space X then $(X, \|\cdot\|^{(1)})$ is a Banach space iff $(X, \|\cdot\|^{(2)})$ is a Banach space.

We introduce in what follows a particular type of normed space, in which the norm is defined in a special way. Given a linear space X we recall that an *inner product* $(\cdot,\cdot)_X$ is a function from $X \times X$ to $\mathbb{R}$ with the following properties.

(1) $(u,u)_X \geq 0 \quad \forall\, u \in X$, and $(u,u)_X = 0$ iff $u = 0_X$.

(2) $(u,v)_X = (v,u)_X \quad \forall\, u,v \in X$.

(3) $(\alpha\, u + \beta\, v, w)_X = \alpha\,(u,w)_X + \beta\,(v,w)_X \qquad \forall\, u,v,w \in X,\ \forall\, \alpha,\beta \in \mathbb{R}$.

The pair $(X, (\cdot,\cdot)_X)$ is called an *inner product space*. When the definition of the inner product $(\cdot,\cdot)_X$ is clear from the context, we simply say X is an inner product space.

Next, it is well known that an inner product $(\cdot,\cdot)_X$ induces a norm through the formula

$$\|u\|_X = \sqrt{(u,u)_X} \qquad \forall\, u \in X \tag{1.4}$$

and we note that everywhere in this book the norm in an inner product space is the one induced by the inner product through the above formula. For an inner product space we have the *Cauchy–Schwarz inequality*:

$$|(u,v)_X| \leq \|u\|_X \|v\|_X \qquad \forall\, u,v \in X, \tag{1.5}$$

with the equality holding iff u and v are linearly dependent. Moreover, the following identity holds:

$$\|u+v\|_X^2 + \|u-v\|_X^2 = 2\left(\|u\|_X^2 + \|v\|_X^2\right) \qquad \forall\, u,v \in X. \tag{1.6}$$

Identity (1.6) is called the *parallelogram identity* or the *parallelogram law*.

Among the inner product spaces, of particular importance are the Hilbert spaces.

Definition 1.2 A complete inner product space is called a *Hilbert space*.

From the definition, we see that an inner product space X is a Hilbert space if X is a Banach space under the norm induced by the inner product.

1.1.2 *Linear continuous operators*

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces and let $L : X \to Y$ be an operator. We recall that $L : X \to Y$ is *linear* if

$$L(\alpha_1 v_1 + \alpha_2 v_2) = \alpha_1 L(v_1) + \alpha_2 L(v_2) \quad \forall\, v_1, v_2 \in X,\ \forall\, \alpha_1, \alpha_2 \in \mathbb{R}.$$

The operator L is said to be *continuous* if

$$u_n \to u \ \text{ in } X \implies L(u_n) \to L(u) \ \text{ in } Y.$$

It can be proved that, if L is linear, then L is continuous if and only if it is *bounded*, i.e., there exists $M > 0$ such that

$$\|L(v)\|_Y \le M\|v\|_X \quad \forall\, v \in X.$$

We will use the notation $\mathcal{L}(X, Y)$ for the set of all linear continuous operators from X to Y. For $L \in \mathcal{L}(X, Y)$, the quantity

$$\|L\|_{\mathcal{L}(X,Y)} = \sup_{0 \ne v \in X} \frac{\|Lv\|_Y}{\|v\|_X} \tag{1.7}$$

is called the *operator norm* of L, and $L \mapsto \|L\|_{\mathcal{L}(X,Y)}$ defines a norm on the space $\mathcal{L}(X, Y)$. Moreover, if Y is a Banach space then $\mathcal{L}(X, Y)$ is also a Banach space. For a linear operator L, we usually write $L(v)$ as Lv, but sometimes we also write Lv even when L is not linear.

For a normed space X, the space $\mathcal{L}(X, \mathbb{R})$ is called the *dual* space of X and is denoted by X'. The elements of X' are linear continuous functionals on X. Recall that a linear functional $\ell : X \to \mathbb{R}$ belongs to X' iff it is *continuous*, i.e.

$$u_n \to u \ \text{ in } X \implies \ell(u_n) \to \ell(u) \ \text{ in } \mathbb{R}$$

or, equivalently, if it is *bounded*, i.e. there exists $M > 0$ such that

$$|\ell(v)| \le M\|v\|_X \quad \forall\, v \in X.$$

The duality pairing between X' and X is usually denoted by $\ell(v)$ or $\langle v', v\rangle$ or $\langle v', v\rangle_{X' \times X}$ for $\ell, v' \in X'$ and $v \in X$. It follows from (1.7) that a norm on X' is

$$\|\ell\|_{X'} = \sup_{0 \ne v \in X} \frac{|\ell(v)|}{\|v\|_X}\,. \tag{1.8}$$

Moreover, $(X', \|\cdot\|_{X'})$ is always a Banach space.

We can now introduce another kind of convergence in a normed space. A sequence $\{u_n\} \subset X$ is said to *converge weakly* to $u \in X$ if for each $\ell \in X'$,

$$\ell(u_n) \to \ell(u) \quad \text{as } n \to \infty.$$

In this case u is called the *weak limit* of $\{u_n\}$ and we write

$$u_n \rightharpoonup u \quad \text{in } X.$$

It follows from the Hahn–Banach theorem that the weak limit of a sequence, if it exists, is unique. Moreover, it is easy to see that strong convergence implies weak convergence, i.e., if $u_n \to u$ in X, then $u_n \rightharpoonup u$ in X. The converse of this property is not true in general.

Assume now that $(X, (\cdot,\cdot)_X)$ is an inner product space and note that in this case $u \mapsto (u, v)_X$ is a linear continuous functional on X, for all $v \in X$. Therefore, it follows from the definition of the weak convergence that

$$u_n \rightharpoonup u \quad \text{in } X \implies (u_n, v)_X \to (u, v)_X \quad \text{as } n \to \infty, \quad \forall\, v \in X. \tag{1.9}$$

We shall see in Section 1.2 that, in the case when $(X, (\cdot,\cdot)_X)$ is a Hilbert space, the converse of (1.9) is true.

The convergence of sequences is used to define closed subsets in a normed space.

Definition 1.3 Let X be a normed space. A subset $K \subset X$ is called:

(i) *(strongly) closed* if the limit of each convergent sequence of elements of K belongs to K, that is

$$\{u_n\} \subset K, \quad u_n \to u \quad \text{in } X \implies u \in K;$$

(ii) *weakly closed* if the limit of each weakly convergent sequence of elements of K belongs to K, that is

$$\{u_n\} \subset K, \quad u_n \rightharpoonup u \quad \text{in } X \implies u \in K.$$

Evidently, every weakly closed subset of X is (strongly) closed, but the converse is not true, in general. An exception is provided by the class of convex subsets of a Banach space, as shown in the following result.

Theorem 1.4 (Mazur's theorem) *A convex subset of a Banach space is (strongly) closed if and only if it is weakly closed.*

Here, for the convenience of the reader, we recall that a subset K of a linear space is said to be *convex* if it has the property

$$u, v \in K \implies (1-t)\,u + t\,v \in K \quad \forall\, t \in [0,1].$$

Let X and Y be linear spaces. A mapping $a : X \times Y \to \mathbb{R}$ is called a *bilinear form* if it is linear in each argument, that is for any u_1, u_2, $u \in X$, v_1, v_2, $v \in Y$, and α_1, $\alpha_2 \in \mathbb{R}$,

$$a(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1\, a(u_1, v) + \alpha_2\, a(u_2, v),$$
$$a(u, \alpha_1 v_1 + \alpha_2 v_2) = \alpha_1\, a(u, v_1) + \alpha_2\, a(u, v_2).$$

For the case $X = Y$ we say that a bilinear form is *symmetric* if

$$a(u, v) = a(v, u) \qquad \forall\, u\,, v \in X.$$

Let now $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces. A bilinear form $a : X \times Y \to \mathbb{R}$ is said to be *continuous* if there exists a constant $M > 0$ such that

$$a(u, v) \le M\, \|u\|_X \|v\|_Y \quad \forall\, u \in X,\ \forall\, v \in Y.$$

For the case $X = Y$ we say that a bilinear form is *positive* if

$$a(u, u) \ge 0 \qquad \forall\, u \in X$$

and X*-elliptic* if there exists a constant $m > 0$ such that

$$a(u, u) \ge m\, \|u\|_X^2 \qquad \forall\, u \in X.$$

It is easy to see that if the bilinear form a is X-elliptic then it is positive. The converse of this property is not true, in general.

1.1.3 Fixed point theorems

Let X be a Banach space with the norm $\|\cdot\|_X$, and K a subset of X. Also, let $\Lambda : K \to X$ be an operator defined on K. We are interested in the existence of a solution $u \in K$ of the operator equation

$$\Lambda u = u. \tag{1.10}$$

An element $u \in K$ which satisfies (1.10) is called a *fixed point* of the operator Λ.

We introduce in what follows two main theorems which state the existence of the fixed points of nonlinear operators: the Banach fixed point theorem and the Schauder fixed point theorem. Both of them represent major results of functional analysis.

Theorem 1.5 (The Banach fixed point theorem) *Let K be a nonempty closed subset of a Banach space $(X, \|\cdot\|_X)$. Assume that $\Lambda : K \to K$ is a* contraction*, i.e. there exists a constant $\alpha \in [0, 1)$ such that*

$$\|\Lambda u - \Lambda v\|_X \le \alpha\, \|u - v\|_X \quad \forall\, u, v \in K. \tag{1.11}$$

Then there exists a unique $u \in K$ such that $\Lambda u = u$.

Proof Let $u_0 \in K$ be an arbitrary element of K and let $\{u_n\}$ be the sequence defined by

$$u_{n+1} = \Lambda u_n \qquad \forall\, n = 0, 1, 2, \dots$$

Since $\Lambda : K \to K$, the sequence $\{u_n\}$ is well-defined. Let us first prove that $\{u_n\}$ is a Cauchy sequence. Using the contractivity of the operator Λ, we have

$$\|u_{n+1} - u_n\|_X \le \alpha \,\|u_n - u_{n-1}\|_X \le \cdots \le \alpha^n \|u_1 - u_0\|_X.$$

Then for any $m > n \ge 1$,

$$\begin{aligned}\|u_m - u_n\|_X &\le \sum_{j=0}^{m-n-1} \|u_{n+j+1} - u_{n+j}\|_X \\ &\le \sum_{j=0}^{m-n-1} \alpha^{n+j}\|u_1 - u_0\|_X \\ &\le \frac{\alpha^n}{1-\alpha}\,\|u_1 - u_0\|_X.\end{aligned}$$

Since $\alpha \in [0,1)$, it follows from the previous inequalities that

$$\|u_m - u_n\|_X \to 0 \quad \text{as} \quad m, n \to \infty.$$

Thus, $\{u_n\}$ is a Cauchy sequence and has a limit $u \in K$, since K is a closed subset of the Banach space X. Moreover, since $u_n \to u$ in X, it follows from (1.11) that $\Lambda u_n \to \Lambda u$ in X. Therefore, taking the limit in $u_{n+1} = \Lambda u_n$ it follows that $u = \Lambda u$, which concludes the existence part of the theorem.

Suppose now that $u_1, u_2 \in K$ are fixed points of Λ. Then from $u_1 = \Lambda u_1$ and $u_2 = \Lambda u_2$ we obtain

$$u_1 - u_2 = \Lambda u_1 - \Lambda u_2.$$

Hence, (1.11) yields

$$\|u_1 - u_2\|_X = \|\Lambda u_1 - \Lambda u_2\|_X \le \alpha \,\|u_1 - u_2\|_X,$$

which implies $\|u_1 - u_2\|_X = 0$, since $\alpha \in [0,1)$. So, if Λ has a fixed point it follows that this fixed point is unique, which concludes the proof. ■

We also need a version of the Banach fixed point theorem which we recall in what follows. To this end, for an operator Λ, we define its powers inductively by the formula $\Lambda^m = \Lambda(\Lambda^{m-1})$ for $m \ge 2$.

Theorem 1.6 *Assume that K is a nonempty closed subset of a Banach space X and let $\Lambda : K \to K$. Assume also that $\Lambda^m : K \to K$ is a contraction for some positive integer m. Then Λ has a unique fixed point.*

Proof By Theorem 1.5, the mapping Λ^m has a unique fixed point $u \in K$. From

$$\Lambda^m(u) = u,$$

we obtain

$$\Lambda^m(\Lambda(u)) = \Lambda(\Lambda^m(u)) = \Lambda u.$$

Thus $\Lambda u \in K$ is also a fixed point of Λ^m. Since Λ^m has a unique fixed point, we must have

$$\Lambda u = u,$$

i.e. u is a fixed point of Λ. The uniqueness of a fixed point of Λ follows easily from that of Λ^m. ■

We turn now to the Schauder fixed point theorem. To introduce it we need the following preliminaries.

Definition 1.7 Let X be a normed space. A subset $K \subset X$ is called:

(i) *bounded* if there exists $M > 0$ such that

$$\|u\|_X \le M \qquad \forall\, u \in K;$$

(ii) *relatively sequentially compact* if each sequence in K has a convergent subsequence in X.

It follows from Definition 1.7 (i) that if $K \subset X$ is bounded, then every sequence $\{u_n\} \subset K$ satisfies (1.3) and, therefore, is bounded. Moreover, Definition 1.7 (ii) shows that a subset $K \subset X$ is relatively sequentially compact if for each $\{u_n\} \subset K$ there exists a subsequence $\{u_{n_k}\}$ and an element $u \in X$ such that $u_{n_k} \to u$ in X. Note also that, for simplicity, everywhere below we shall use the terminology *relatively compact* instead of *relatively sequentially compact*.

Definition 1.8 Let X and Y be normed spaces. The operator $\Lambda : K \subset X \to Y$ is called:

(i) *continuous at the point* $u \in K$ if for each sequence $\{u_n\} \subset K$ which converges in X to u, the sequence $\{\Lambda u_n\} \subset Y$ converges to Λu in Y, that is

$$\{u_n\} \subset K, \quad u_n \to u \quad \text{in } X \implies \Lambda u_n \to \Lambda u \quad \text{in } Y;$$

(ii) *continuous* if it is continuous at each point $u \in K$;

(iii) *compact* if it is continuous and maps bounded sets into relatively compact sets.

It follows from Definitions 1.8 (iii) and 1.7 (ii) that a continuous operator $\Lambda : K \subset X \to Y$ is compact if and only if for each bounded sequence $\{u_n\} \subset K$ there exists a subsequence $\{u_{n_k}\}$ such that the sequence $\{\Lambda u_{n_k}\}$ is convergent in Y.

We proceed with the following result.

Theorem 1.9 (The Schauder fixed point theorem) *Let K be a nonempty closed convex bounded subset of a Banach space X and let $\Lambda : K \to X$ be a compact operator such that $\Lambda(K) \subset K$. Then Λ has at least one fixed point.*

The proof of Theorem 1.9 requires a number of preliminary results and, therefore, we skip it. Nevertheless, we indicate that such a proof can be found in [155, p. 56]. Also, note that, unlike the Banach fixed point theorem, the Schauder fixed point theorem does not provide the uniqueness of the fixed point of the operator Λ. We shall use Theorem 1.9 in Section 2.3 in order to prove the solvability of a class of elliptic quasivariational inequalities.

1.2 Hilbert spaces

We introduce in what follows some useful results which are valid in Hilbert spaces. This concerns the projection operators, some properties related to orthogonality and the Riesz representation theorem, together with its consequences.

1.2.1 Projection operators

The projection operators represent an important class of nonlinear operators defined in a Hilbert space; to introduce them we need the following existence and uniqueness result.

Theorem 1.10 (The projection lemma) *Let K be a nonempty closed convex subset of a Hilbert space X. Then, for each $f \in X$ there exists a unique element $u \in K$ such that*

$$\|u - f\|_X = \min_{v \in K} \|v - f\|_X. \tag{1.12}$$

Proof Let $f \in X$, denote

$$d = \inf_{v \in K} \|v - f\|_X, \tag{1.13}$$

and let $\{u_n\}$ be a sequence of elements of K such that

$$\|u_n - f\|_X \to d \qquad \text{as} \quad n \to \infty. \tag{1.14}$$

Since K is a convex subset of X we deduce that $\dfrac{u_m + u_n}{2} \in K$ for all $m,\ n \in \mathbb{N}$ and, therefore, (1.13) implies that

$$\left\| \frac{u_m + u_n}{2} - f \right\|_X \ge d \qquad \forall\, m,\ n \in \mathbb{N}. \tag{1.15}$$