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John Williams Calkin: a short biography

Seppo Hassi, Henk de Snoo and Franciszek Hugon Szafraniec

All employees at Los Alamos had Project Y security badges. Calkin's photograph[1] is one of these badge photos.

John Williams Calkin (October 11, 1909–August 5, 1964). A native of New Rochelle, N.Y., he graduated with honors in mathematics from Columbia University in 1933. He was awarded his MA in 1934 and his PhD in June 1937 by Harvard University. His PhD dissertation, *Applications of the Theory of Hilbert Space to Partial Differential Equations; the Self-Adjoint Transformations in Hilbert Space Associated with a Formal Partial Differential Operator of the Second Order and Elliptic Type*, was written under the direction and at the suggestion of M.H. Stone; fruitful conversations with J. von Neumann were acknowledged. In the fall of 1937 Calkin went to the Institute for Advanced Study in Princeton on a year's fellowship to work with O. Veblen and J. von Neumann. He later served as an assistant professor at the University of New Hampshire and the Illinois Institute of Technology in Chicago. During World War II Calkin was part of a mine warfare operations analysis group with J.L. Doob, J. von Neumann, and M.H. Stone[2]. Von Neumann and Calkin worked on shock waves and damage by explosives; they were sent to England to learn of the progress under way there. When it appeared that their special knowledge would be useful for the Manhattan Project, they moved to Los Alamos. The Los Alamos Laboratory, or Project Y, had come into existence in early 1943 for a single purpose: to design and build an atomic bomb. Calkin remained there until 1946 when he accepted a Guggenheim fellowship at the California Institute of Technology. He later taught at the Rice Institute in Houston before returning to Los Alamos in 1949 as a member of the theoretical division. In 1958, he accepted a consulting appointment at New York University and at Brookhaven National Laboratory and, in 1961, was named head, and then chairman of the Applied Mathematics Department. He died[3 4] in Westhampton, N.Y., at the age of 54.

Calkin's publications date back to the time before he joined the war effort. We have not been able to find other published mathematical work. The list consists of the following papers:

- Abstract self-adjoint boundary conditions, *Proc. Nat. Acad. Sci.*, **24** (1938), 38–42.
- General self-adjoint boundary conditions for certain partial differential

operators, *Proc. Nat. Acad. Sci.*, **25** (1939), 201–206.

- Abstract symmetric boundary conditions, *Trans. Amer. Math. Soc.*, **45** (1939), 369–442.
- Abstract definite boundary value problems, *Proc. Nat. Acad. Sci.*, **26** (1940), 708–712.
- Functions of several variables and absolute continuity I, *Duke Math. J.*, **6** (1940), 170–186.
- Symmetric transformations in Hilbert space, *Duke Math. J.*, **7** (1940), 504–508.
- Two-sided ideals and congruences in the ring of bounded operators in Hilbert space. *Ann. Math.*, **42** (1941), 839–873.

Today, Calkin is mostly remembered for the algebras bearing his name; the relevant work dates back to 1941. However, it is clear that this work was directly influenced by his earlier work on boundary value problems. After World War II Calkin did not return to his earlier work, but remained involved in applied mathematics and physics both at Los Alamos and at Brookhaven National Laboratory.

The environmentalist Brant Calkin[5] (1934) writes us about his father: "My most vivid memories of him were while we were living at Los Alamos, the atomic bomb laboratory which we first went to in 1943. There was of course no opportunity to visit him at work, since all the offices were behind security fences patrolled by the army. It was there that I can remember the long lasting and raucous poker games which we often hosted. What games those must have been. Players included Stan Ulam, Nick Metropolis, Carson Mark, and occasionally, I think, Enrico Fermi and John von Neumann. The latter, someone my dad had worked with at Princeton. Much earlier, we lived in Chicago, where my father haunted obscure and mostly black night clubs where he loved the jazz, and where he met singers who later became famous. His love of jazz resulted in the music which we mostly heard in our house at that time. His contacts were esoteric, and in the late 50s he invited me to join him at a New York City jazz club. There, he greeted a famous drummer by first name and with whom he had played chess. He was not a car fan, but he had fondness for them. In Chicago, he once walked out to buy some groceries and came back with a used car."

Another glimpse of the man Calkin is provided by S.M. Ulam[6]. When Ulam and his wife traveled to New Mexico by train he was infinitely surprised to see Calkin, whom he had known from Chicago, waiting at the whistle-stop to drive them to Los Alamos: "He was a tall, pleasant-looking, man with more savoir-faire than most mathematicians", and Ulam continues describing life in Los Alamos, including discussions with von Neumann and Calkin.

[1]

[2] M.H. Stone, Review: Steve J. Heims, John von Neumann and Norbert Wiener, from mathematics to the technologies of life and death, *Bull. A.M.S.* (*N.S.*), **8**, (1983), 395–399.

[3] John W. Calkin, a mathematician, *The New York Times*, August 6, 1964.

[4] John W. Calkin, *Phys. Today*, **17**, 1964, 102–105.

[5] Brant Calkin, personal communication.

[6] S.M. Ulam, *Adventures of a mathematician*, University of California Press, 1991.

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On Calkin's abstract symmetric boundary conditions

Seppo Hassi and Rudi Wietsma

Abstract J.W. Calkin introduced in the late 1930s the concept of a reduction operator in order to investigate (maximal) symmetric extensions of symmetric operators. A precise interpretation of reduction operators can be given using Kreĭn space terminology; this simultaneously connects his work with recent investigations on boundary triplets. Here an overview of his main results on reduction operators is given, with emphasis on the more involved unbounded case. In particular, by extending a domain decomposition of reduction operators from [Calkin, 1939a] into a graph decomposition and using Kreĭn space methods, simplified proofs for most of his main results are given.

2.1 Introduction

In order to investigate boundary value problems appearing in connection with ordinary and partial differential equations, see e.g. [Calkin, 1939b], Calkin introduced in the end of the 1930s the concept of a reduction operator in [Calkin, 1939a]; Definition 2.1 below contains a slightly reformulated form of his notion.

Definition 2.1 Let S be a closed linear operator in a Hilbert space $(\mathfrak{H}, (\cdot, -)_{\mathfrak{H}})$ and assume that S is densely defined. Then a closed linear operator U with domain in the graph of S^* and with range in a Hilbert space $(\mathfrak{K}, (\cdot, -)_{\mathfrak{K}})$ is said to be a *reduction operator* for S^* if:

(i) $\overline{\operatorname{dom} U} = \operatorname{gr} S^*$;
(ii) there exists a unitary operator W in $(\mathfrak{K}, (\cdot, -)_{\mathfrak{K}})$ such that

$$(\operatorname{gr} U)^{\perp} = \{\{S^* f, -f, WU\{f, S^* f\}\} : \{f, S^* f\} \in \operatorname{dom} U\}.$$

Here $\mathfrak{K}$ is called the *range space* of U and W is called the *rotation* associated with U.

The assumption that S is densely defined in Definition 2.1 means that S^* exists as an operator; this prevents the use of linear relations. Without this condition all the theory that follows holds without essential changes. However, as in Calkin's original paper, here only the densely defined case is considered.

Next observe that condition (ii) in Definition 2.1 implies the following (abstract) Green's (or Lagrange's) identity:

$$(f, S^*g)_{\mathfrak{H}} - (S^*f, g)_{\mathfrak{H}} = (WU\{f, S^*f\}, U\{g, S^*g\})_{\mathfrak{K}}, \tag{2.1}$$

where $\{f, S^*f\}, \{g, S^*g\} \in \operatorname{dom} U$. This equality shows in particular that $\ker U = \operatorname{gr} S$ and, hence, S is a closed symmetric operator.

The main problems investigated in [Calkin, 1939a] can be roughly formulated as follows:

(i) investigate the cardinality and type of maximal symmetric extensions of S whose graph is contained in the domain of a reduction operator for S^*;
(ii) determine necessary and sufficient conditions on a subspace belonging to the range of the reduction operator for its pre-image under the reduction operator to be maximal symmetric;
(iii) describe the pathology of unbounded reduction operators.

It is the purpose of this chapter to give an overview of Calkin's answers to the above questions. It should be noted that concepts which are essentially special cases of Calkin's reduction operator have been introduced in later literature. For instance, the widely known concept of a boundary triplet (or boundary value space), see [Gorbachuk and Gorbachuk, 1991] and the references therein, can be considered as a bounded reduction operator with a suitable choice of the rotation W. In particular, this means that Calkin's results for bounded reduction operators, i.e. for reduction operators with $\operatorname{dom} U = S^*$, have in special cases been rediscovered in later works; for further details see Chapter 7 and the references therein. In this overview the main focus will be on the more complicated unbounded case. In order to make Calkin's results more accessible, they are here reformulated using Kreĭn space terminology, which is a natural framework for his investigation. Furthermore, simplified proofs are included for the main statements by extending the central result from [Calkin, 1939a]. In addition, it is shown how the concept of a reduction operator is connected to modern day equivalent objects used in the extension theory of symmetric operators.

The contents of the chapter are now outlined. Section 2 contains some preliminary results on operators (relations) in Kreĭn spaces. In Section 3 reduction operators are investigated: their connection to (unitary) boundary triplets is made explicit, they are characterized and a classification of them is introduced. In Section 4, Calkin's answers to the above mentioned questions are presented.

The idea for writing this overview of Calkin's 1939 paper entitled "Abstract symmetric boundary conditions" came from Vladimir Derkach's lecture "Survey of boundary triplets and boundary relations" given during a workshop at the Lorentz Center in Leiden, the Netherlands (December 14–18, 2009). That lecture included a short survey of Calkin's work as can be found in Chapter 7.

2.2 Preliminaries

The notion of a Kreĭn space is recalled including a short introduction to relations in such spaces, see [Azizov and Iokhvidov, 1989] for details. In particular, definitions of (nonstandard) unitary operators in Kreĭn spaces and unitary boundary triplets are given; these objects, which are used in the extension theory of symmetric operators, are later shown to be closely related to reduction operators.

Kreĭn spaces Let $\mathfrak{j}$ be a *fundamental symmetry*, also called a *signature operator*, in the Hilbert space $(\mathfrak{H}, (\cdot, -))$, i.e., $\mathfrak{j}$ is an everywhere defined operator satisfying $\mathfrak{j}^* = \mathfrak{j} = \mathfrak{j}^{-1}$. Then the space $\mathfrak{H}$ equipped with the indefinite inner product $(\mathfrak{j}\,\cdot, -)$, i.e. $(\mathfrak{H}, (\mathfrak{j}\,\cdot, -))$, is called a *Kreĭn space*. Note that the fundamental symmetry $\mathfrak{j}$ induces an orthogonal decomposition $\mathfrak{H}^+ + \mathfrak{H}^-$ of $\mathfrak{H}$ by $\mathfrak{H}^\pm = \ker(I \mp \mathfrak{j})$ and that $(\mathfrak{H}^\pm, \pm(\mathfrak{j}\cdot, -))$ are Hilbert spaces; this orthogonal decomposition of $\mathfrak{H}$ is called the *canonical decomposition of* $(\mathfrak{H}, (\cdot, -))$ *induced by* $\mathfrak{j}$.

Example 2.2 For a Hilbert space $(\mathfrak{H}, (\cdot, -))$ define $\mathfrak{j}_{\mathfrak{H}}$ on $\mathfrak{H}^2$ as

$$\mathfrak{j}_{\mathfrak{H}}\{f, f'\} = \{-\mathrm{i}f', \mathrm{i}f\}, \quad \{f, f'\} \in \mathfrak{H}^2. \tag{2.2}$$

Then $\mathfrak{j}_{\mathfrak{H}} = (\mathfrak{j}_{\mathfrak{H}})^{-1} = (\mathfrak{j}_{\mathfrak{H}})^*$, i.e., $\mathfrak{j}_{\mathfrak{H}}$ is a fundamental symmetric in $(\mathfrak{H}^2, (\cdot, -))$. Consequently, $(\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\cdot, -))$ is a Kreĭn space. Note that if $\mathfrak{H}^+ + \mathfrak{H}^-$ is the canonical decomposition of $\mathfrak{H}^2$ induced by $\mathfrak{j}_{\mathfrak{H}}$, then

$$\mathfrak{H}^+ = \ker(I - \mathfrak{j}_{\mathfrak{H}}) = \{\{f, \mathrm{i}f\} : f \in \mathfrak{H}\},$$
$$\mathfrak{H}^- = \ker(I + \mathfrak{j}_{\mathfrak{H}}) = \{\{f, -\mathrm{i}f\} : f \in \mathfrak{H}\}.$$

Let $\mathfrak{L}$ be a subspace of the Hilbert space $(\mathfrak{H}, (\cdot, -))$ with fundamental symmetry $\mathfrak{j}$ and let $[\cdot, -] := (\mathfrak{j}\cdot, -)$. Then $\mathfrak{L}$ is called a *positive*, *negative*, *nonnegative*, *nonpositive* or *neutral* subspace of $(\mathfrak{H}, (\mathfrak{j}\cdot, -))$ if $[f, f] > 0$, $[f, f] < 0$, $[f, f] \geq 0$, $[f, f] \leq 0$ or $[f, f] = 0$ for every $f \in \mathfrak{L} \setminus \{0\}$, resp. (Note that Calkin uses the notion of W-symmetry for neutrality, see [Calkin, 1939a, Definition 1.3].) A neutral, nonnegative or nonpositive subspace is called *maximal neutral*, *maximal nonnegative* or *maximal nonpositive* if it has no neutral, nonnegative or nonpositive extension, resp. In particular, maximal semi-definite (i.e., neutral, nonnegative or nonpositive) subspaces are closed. If $P^{\pm}$ denotes the orthogonal projection onto $\mathfrak{H}^{\pm}$ in $\mathfrak{H}$ with respect to $(\cdot, -)$ for a canonical decomposition $\mathfrak{H}^+ + \mathfrak{H}^-$ of $(\mathfrak{H}, (\mathfrak{j}\cdot, -))$, then a nonnegative or nonpositive subspace $\mathfrak{L}$ is maximal if and only if $P^+\mathfrak{L} = \mathfrak{H}^+$ or $P^-\mathfrak{L} = \mathfrak{H}^-$, resp.

The *orthogonal complements* of $\mathfrak{L}$ with respect to $(\cdot, -)$ and $[\cdot, -] := (\mathfrak{j}\,\cdot, -)$ are denoted by $\mathfrak{L}^{\perp}$ and $\mathfrak{L}^{[\perp]}$: they are the closed subspaces defined as

$$\mathfrak{L}^{\perp} = \{f \in \mathfrak{H} : (f, g) = 0 \text{ for all } g \in \mathfrak{L}\},$$
$$\mathfrak{L}^{[\perp]} = \{f \in \mathfrak{H} : [f, g] = 0 \text{ for all } g \in \mathfrak{L}\}.$$

Clearly, $\mathfrak{L}^{[\perp]} = \mathfrak{j}\mathfrak{L}^{\perp} = (\mathfrak{j}\mathfrak{L})^{\perp}$. With the above notation, a subspace $\mathfrak{L}$ is neutral if and only if $\mathfrak{L} \subseteq \mathfrak{L}^{[\perp]}$. For a neutral subspace $\mathfrak{L}$ of $(\mathfrak{H}, (\mathfrak{j}\,\cdot, -))$ the *abstract first von Neumann formula* holds:

$$\mathfrak{L}^{[\perp]} = \operatorname{clos}(\mathfrak{L}) \dotplus (\mathfrak{L}^{[\perp]} \cap \mathfrak{H}^+) \dotplus (\mathfrak{L}^{[\perp]} \cap \mathfrak{H}^-), \tag{2.3}$$

see [Azizov and Iokhvidov, 1989, Chapter 1 : 4.20]. Note that the first von Neumann formula is nothing else than the canonical decomposition for the Kreĭn space $(\mathfrak{L}^{[\perp]} \ominus \operatorname{clos}\mathfrak{L}, (\mathfrak{j}\,\cdot, -))$ induced by the canonical decomposition $\mathfrak{H}^+ + \mathfrak{H}^-$ of $(\mathfrak{H}, (\mathfrak{j}\cdot, -))$. For a neutral subspace $\mathfrak{L}$ its defect numbers $n_+(\mathfrak{L})$ and $n_-(\mathfrak{L})$ are defined as

$$\begin{aligned} n_+(\mathfrak{L}) &= \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{H}^-) = \dim(\mathfrak{H}^- \ominus P^-\mathfrak{L}); \\ n_-(\mathfrak{L}) &= \dim(\mathfrak{L}^{[\perp]} \cap \mathfrak{H}^+) = \dim(\mathfrak{H}^+ \ominus P^+\mathfrak{L}). \end{aligned} \tag{2.4}$$

A neutral subspace is called *hyper-maximal neutral* if it is maximal nonnegative and maximal nonpositive, see [Azizov and Iokhvidov, 1989, Chapter 1: Definition 4.15]. Equivalently, a neutral subspace $\mathfrak{L}$ of the Kreĭn space $(\mathfrak{H}, (\mathfrak{j}\,\cdot, -))$ is hyper-maximal neutral if and only if $\mathfrak{L} = \mathfrak{L}^{[\perp]}$. In other words, $\mathfrak{L}$ is hyper-maximal neutral if and only if it induces the following orthogonal decomposition of the Hilbert space $(\mathfrak{H}, (\cdot, -))$:

$$\mathfrak{H} = \mathfrak{L} \oplus \mathfrak{j}\,\mathfrak{L}. \tag{2.5}$$

Operators in Hilbert spaces Let $(\mathfrak{H}_1, (\cdot, -)_1)$ and $(\mathfrak{H}_2, (\cdot, -)_2)$ be Hilbert spaces. Then H is called a (*linear*) *relation* from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ if it is a subspace of $\mathfrak{H}_1 \times \mathfrak{H}_2$; here and in what follows, linear operators are usually identified with their graphs. In particular, H is *closed* if and only if its graph, $\operatorname{gr} H$, is closed (as a subspace of $\mathfrak{H}_1 \times \mathfrak{H}_2$). The symbols $\operatorname{dom} H$, $\operatorname{ran} H$, $\ker H$, and $\operatorname{mul} H$ stand for the domain, range, kernel, and the multi-valued part of H, respectively. In particular, $\operatorname{mul} H = \{0\}$ if and only if H is an operator. For a subspace $\mathfrak{L}$ of $\operatorname{dom} H$, $H(\mathfrak{L})$ denotes the subspace

$$H(\mathfrak{L}) = \{f' \in \mathfrak{H}_2 : \ \{f, f'\} \in H \ \text{ for some } f \in \mathfrak{L}\}.$$

The inverse H^{-1} of a relation H is defined as

$$H^{-1} = \{\{f, f'\} \in \mathfrak{H}_2 \times \mathfrak{H}_1 : \{f', f\} \in H\}$$

and its adjoint H^* is defined via

$$H^* = \{\{f, f'\} \in \mathfrak{H}_2 \times \mathfrak{H}_1 : (f', g)_1 = (f, g')_2, \quad \{g, g'\} \in H\}. \tag{2.6}$$

From this definition it follows that $\ker H^* = (\operatorname{ran} H)^{\perp_2}$ and $\operatorname{mul} H^* = (\operatorname{dom} H)^{\perp_1}$. In particular, this shows that H^* is an operator if and only if $\overline{\operatorname{dom}}\, H = \mathfrak{H}_1$.

An operator U from the Hilbert space $(\mathfrak{H}_1, (\cdot, -)_1)$ to the Hilbert space $(\mathfrak{H}_2, (\cdot, -)_2)$ is called *isometric* or *unitary* if $U \subseteq U^{-*}$ or $U = U^{-*}$, respectively [1]. An isometric operator is said to be *maximal isometric* if it has no isometric extension. Similarly, an operator (or relation) S in the Hilbert space $(\mathfrak{H}, (\cdot, -))$ is called *symmetric* or *selfadjoint* if

$$S \subseteq S^* \quad \text{or} \quad S = S^*,$$

respectively, and a symmetric operator (or relation) is said to be *maximal symmetric* if it has no symmetric extension. For a symmetric operator (or relation) S in the Hilbert space $(\mathfrak{H}, (\cdot, -))$ the notation $\widehat{\mathfrak{N}}_\lambda(S^*)$ is used to denote its defect spaces:

$$\widehat{\mathfrak{N}}_\lambda(S^*) = \{\{f_\lambda, \lambda f_\lambda\} : f_\lambda \in \ker (S^* - \lambda)\}, \quad \lambda \in \mathbb{C}.$$

With this notation the *first von Neumann formula* holds:

$$S^* = S \dotplus \widehat{\mathfrak{N}}_\lambda(S^*) \dotplus \widehat{\mathfrak{N}}_{\bar\lambda}(S^*), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, \tag{2.7}$$

cf. (2.3). In fact, the direct sums in (2.7) are orthogonal for $\lambda = \pm\, \mathrm{i}$. The defect numbers $n_+(S)$ and $n_-(S)$ for S are defined as

$$n_+(S) = \dim \widehat{\mathfrak{N}}_{\bar\lambda}(S^*) \quad \text{and} \quad n_-(S) = \dim \widehat{\mathfrak{N}}_\lambda(S^*), \quad \lambda \in \mathbb{C}_+.$$

[1] For a relation U, the symbol U^{-*} is a shorthand notation for $(U^*)^{-1} = (U^{-1})^*$.

There exist connections between (maximal) symmetric or selfadjoint relations, (maximal) isometric or unitary relations and (maximal) neutral or hyper-maximal neutral subspaces. In Calkin's paper the connection between neutral subspaces of a Kreĭn space and isometric operators, by means of what are now called angular operators (see e.g. [Azizov and Iokhvidov, 1989, Chapter 1: §8]), is used very frequently. Also the connection between symmetric operators and neutral subspaces of Kreĭn spaces was used, see in particular [Calkin, 1939a, Theorem 2.7]. Since this connection is also frequently used here, it is formulated below.

Proposition 2.3 *Let $(\mathfrak{H},(\cdot,-))$ be a Hilbert space and let $\mathfrak{j}_{\mathfrak{H}}$ be the fundamental symmetry in $\mathfrak{H}^2$ as in Example 2.2. Then S is a (closed, maximal) symmetric or selfadjoint operator (or relation) in $(\mathfrak{H},(\cdot,-))$ if and only if $\operatorname{gr} S$ is a (closed, maximal) neutral or hyper-maximal neutral subspace of the Kreĭn space $(\mathfrak{H}^2,(\mathfrak{j}_{\mathfrak{H}}\,\cdot,-))$, respectively. Moreover,*

$$S^* = (\operatorname{gr} S)^{[\perp]},$$

where $[\perp]$ stands for the orthogonal complement with respect to $(\mathfrak{j}_{\mathfrak{H}}\,\cdot,-)$.

Unitary operators and boundary triplets Let H be an operator (or relation) from the Hilbert space $(\mathfrak{H}_1,(\cdot,-)_1)$ to the Hilbert space $(\mathfrak{H}_2,(\cdot,-)_2)$ and let $\mathfrak{j}_i$ be a fundamental symmetry in $(\mathfrak{H}_i,(\cdot,-)_i)$, $i = 1,2$. Then the adjoint $H^{[*]}$ of H as a mapping from the Kreĭn space $(\mathfrak{H}_1,(\mathfrak{j}_1\,\cdot,-)_1)$ to the Kreĭn space $(\mathfrak{H}_2,(\mathfrak{j}_2\,\cdot,-)_2)$ is defined via

$$H^{[*]} = \{\{f,f'\} \in \mathfrak{H}_2 \times \mathfrak{H}_1 : (\mathfrak{j}_1 f',g)_1 = (\mathfrak{j}_2 f,g')_2, \quad \{g,g'\} \in H\}\,.$$

This Kreĭn space adjoint $H^{[*]}$ of H and the Hilbert space adjoint H^* of H in (2.6) are connected by

$$H^{[*]} = \mathfrak{j}_1 H^* \mathfrak{j}_2. \tag{2.8}$$

By means of the Kreĭn space adjoint an operator (or relation) U from the Kreĭn space $(\mathfrak{H}_1,(\mathfrak{j}_1\,\cdot,-)_1)$ to the Kreĭn space $(\mathfrak{H}_2,(\mathfrak{j}_2\,\cdot,-)_2)$ is called an *isometric* or a *unitary* operator (or relation) if

$$U^{-1} \subseteq U^{[*]} \quad \text{or} \quad U^{-1} = U^{[*]}, \tag{2.9}$$

respectively. Different from the Hilbert space situation, there exist isometric and unitary relations (with non-trivial kernels). Here, however, only unitary operators are encountered. In this connection note that combining the equalities $\ker H^* = (\operatorname{ran} H)^{\perp_2}$ and $\operatorname{mul} H^* = (\operatorname{dom} H)^{\perp_1}$

with (2.8) and (2.9) yields that for a unitary relation U between Kreĭn spaces the following equalities hold:

$$\ker U = (\operatorname{dom} U)^{[\perp]_1} \quad \text{and} \quad \operatorname{mul} U = (\operatorname{ran} U)^{[\perp]_2}, \tag{2.10}$$

where $[\perp]_i$ is the orthogonal complement with respect to $(\mathfrak{j}_i \cdot, -)_i$, for $i = 1, 2$. In particular, a unitary operator has a dense range and a unitary operator has a trivial kernel if and only if it is densely defined.

By the definitions in (2.9) a unitary operator is always closed, and an operator and its inverse are simultaneously isometric or unitary. Clearly, an operator U is isometric in the sense of (2.9) if and only if

$$(\mathfrak{j}_1 f, g)_1 = (\mathfrak{j}_2 Uf, Ug)_2, \quad f, g \in \operatorname{dom} U. \tag{2.11}$$

Furthermore, an isometric operator U is unitary if and only if for $g \in \mathfrak{K}_1$ and $g' \in \mathfrak{K}_2$, $(\mathfrak{j}_1 f, g)_1 = (\mathfrak{j}_2 Uf, g')_2$ for all $f \in \operatorname{dom} U$, implies that $g \in \operatorname{dom} U$ and $g' = Ug$. In other words, unitary operators are a special type of maximal isometric operators.

Equation (2.11) shows that isometric operators map neutral, nonnegative and nonpositive subspaces of $(\mathfrak{H}_1, (\mathfrak{j}_1 \cdot, -)_1)$ onto neutral, nonnegative and nonpositive subspaces of $(\mathfrak{H}_2, (\mathfrak{j}_2 \cdot, -)_2)$, respectively. In general, isometric or unitary operators do not map closed subspaces onto closed subspaces or maximal semi-definite subspaces onto maximal semi-definite subspaces. However, if a unitary operator has a closed domain, or, equivalently, a closed range, then it preserves all such properties. In fact, unitary operators with closed domain, i.e. bounded unitary operators, also preserve the defect numbers of neutral subspaces, cf. Theorem 2.20 below.

Next the general definition of a unitary boundary triplet is recalled, cf. [Derkach et al., 2006, Definition 3.1], and its connection to unitary operators between Kreĭn spaces is explained.

Definition 2.4 Let S be a closed symmetric relation in a Hilbert space $(\mathfrak{H}, (\cdot, -)_\mathfrak{H})$ and let $(\mathcal{H}, (\cdot, -)_\mathcal{H})$ be an auxiliary Hilbert space. Then the triplet $\{\mathcal{H}, \Gamma_0, \Gamma_1\}$, where $\Gamma_i : S^* \subseteq \mathfrak{H}^2 \to \mathcal{H}$ is a linear operator for $i = 0, 1$, is called a *unitary boundary triplet* for S^*, if

(i) $\overline{\operatorname{dom}}\, \Gamma = S^*$ and $\overline{\operatorname{ran}}\, \Gamma = \mathcal{H}^2$, where $\Gamma = \Gamma_0 \times \Gamma_1$;
(ii) Green's identity holds: for every $\{f, f'\}, \{g, g'\} \in \operatorname{dom} \Gamma$,

$$(f', g)_\mathfrak{H} - (f, g')_\mathfrak{H} = (\Gamma_1\{f, f'\}, \Gamma_0\{g, g'\})_\mathcal{H} - (\Gamma_0\{f, f'\}, \Gamma_1\{g, g'\})_\mathcal{H};$$

(iii) if $g, g' \in \mathfrak{H}$ and $k, k' \in \mathcal{H}$ are such that for all $\{f, f'\} \in \operatorname{dom}\Gamma$

$$(f', g)_{\mathfrak{H}} - (f, g')_{\mathfrak{H}} = (\Gamma_1\{f, f'\}, k)_{\mathcal{H}} - (\Gamma_0\{f, f'\}, k')_{\mathcal{H}},$$

then $\{g, g'\} \in \operatorname{dom}\Gamma$ and $\{k, k'\} = \Gamma\{g, g'\}$.

If S is a densely defined operator, so that its adjoint S^* is also an operator, then $\{f, f'\}$ and $\{g, g'\}$ appearing in Definition 2.4 can be replaced by f and g, respectively. Green's identity in that case becomes:

$$(S^* f, g)_{\mathfrak{H}} - (f, S^* g)_{\mathfrak{H}} = (\Gamma_1 f, \Gamma_0 g)_{\mathcal{H}} - (\Gamma_0 f, \Gamma_1 g)_{\mathcal{H}},$$

cf. (2.1). Note that, with the notation as in (2.2), Definition 2.4 means that Γ is a unitary operator from $(\mathfrak{H}^2, (\mathfrak{j}_{\mathfrak{H}}\,\cdot, -))$ to $(\mathcal{H}^2, (\mathfrak{j}_{\mathcal{H}}\,\cdot, -))$. This, in particular, implies that $\ker\Gamma = S$, cf. (2.10) and Proposition 2.3.

Remark 2.5 If S is a symmetric operator and there exists a unitary boundary triplet for S^*, then $n_\pm(S) = \dim\mathcal{H}^\mp$, where $\mathcal{H}^+ + \mathcal{H}^-$ is the canonical decomposition of $\mathcal{H}^2$ induced by $\mathfrak{j}_{\mathcal{H}}$. Since it is easy to see that $\dim\mathcal{H}^+ = \dim\mathcal{H}^-$, unitary boundary triplets only exist for the adjoints of symmetric relations with equal defect numbers. For symmetric relations with unequal defect numbers either D-boundary triplets or boundary relations have to be used, see [Mogilevskiĭ, 2006] or [Derkach et al., 2006, Proposition 3.7], respectively.

2.3 Reduction operators

In the first subsection the basic properties of reduction operators are given and it is shown how reduction operators can be interpreted as unitary operators and/or unitary boundary triplets. In the second subsection a graph decomposition of reduction operators is presented which will be the central tool in Section 2.4. In the third subsection a classification of reduction operators is introduced and characterized. In the fourth and final subsection bounded reduction operators are shortly considered.

Reduction operators, unitary operators and boundary triplets
Let U be a reduction operator for S^* as in Definition 2.1, then as a direct consequence of its definition U has the following three basic properties:

(i) $\ker U = S$ and, hence, S is a symmetric operator in $(\mathfrak{H}, (\cdot, -))$;
(ii) $\overline{\operatorname{ran}}\, U = \mathfrak{K}$;
(iii) $I + W^2 = 0$,