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Javier Bonet , Antonio J. Gil , Richard D. Wood
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CHAPTER ONE

INTRODUCTION

In this chapter a number of very simple rigid link-spring structures are considered which illustrate many features of nonlinear behavior often associated with more complex structures.

EXAMPLE 1.1

The structure shown in Figure 1.1 comprises a rigid weightless rod $a-b$ supported by a spring of stiffness k. The force F is positive downward as is the vertical displacement v.
(a) Find the equilibrium equation relating F to the slope angle θ and then plot F against the vertical displacement v.
(b) Also determine the directional derivative of F with respect to a change β in θ.

This simple example illustrates the phenomenon known as *snap-through behavior*.

Solution

(a) When considering a finite deformation problem, the equilibrium equation must be established in the deformed position. Consideration of the vertical equilibrium of joint a gives

$$F = T\sin\theta\ ; \qquad T = \frac{F}{\sin\theta}, \tag{1.1}$$

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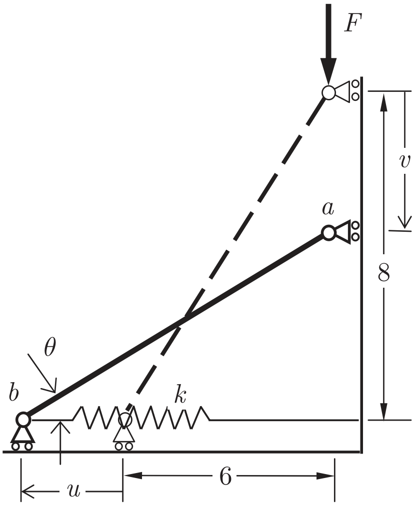

FIGURE 1.1 Rod spring structure

where T is the compressive force in the rod ab. The horizontal equilibrium equation for joint b is found in terms of the tensile force in the spring $S = ku$ as

$$T\cos\theta = S. \tag{1.2}$$

Geometrical considerations yield the displacements u and v in terms of the deformed angle θ as

$$u = 10\cos\theta - 6\,;\; v = 8 - 10\sin\theta. \tag{1.3}$$

Substituting Equation (1.1) into Equation (1.2) gives F as a function of θ as

$$F = k\tan\theta(10\cos\theta - 6), \tag{1.4}$$

which together with Equation (1.3) enables Figure 1.2 to be drawn.

Note that all points on the plot in Figure 1.2 represent positions of equilibrium. Also observe that for some values of F three equilibrium positions are possible. Indeed, it will be seen in Chapter 3 that multiple positions often are in equilibrium for a given load.

Although not demonstrated here, the positions between the upper and lower peaks are positions of unstable equilibrium. The plot is drawn by

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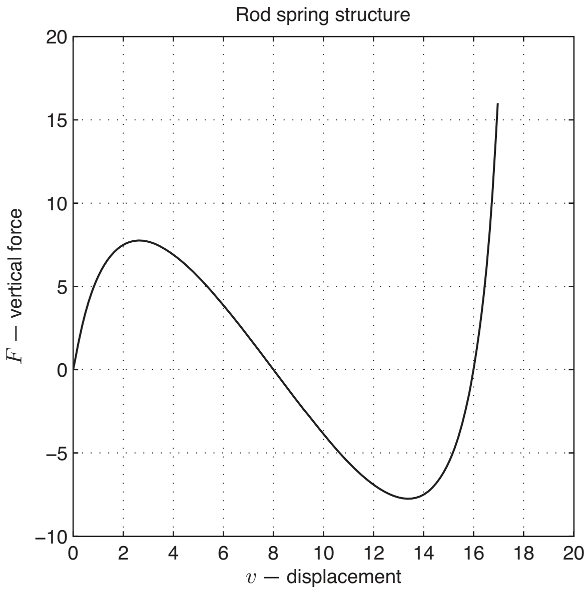

FIGURE 1.2 Rod spring–equilibrium path

choosing a value of θ and calculating F and v; however in practice it is more likely that the force F will determine the value of θ and hence v. The resulting nonlinear solution technique will involve the determination of v as F is gradually incremented at least until the first peak is reached at about $v \approx 2.5$ when a small increase in F will result in a solution in the region of $v \approx 16.5$. This sudden movement (dynamic in reality) is called *snap-through behavior*. Although shallow domes can exhibit such undesirable behavior, snap-through behavior is beneficially employed in everyday life, in light switches, bottle caps, children's hair clips, and many other applications.

(b) The single degree of freedom nature of this example enables nonlinear solutions to be discovered extremely easily, however in reality this is not the case and nonlinear solution techniques need to be devised. Here the Newton–Raphson iterative procedure predominates, which requires the concept of linearization of a nonlinear function which involves the directional derivative. Examples of the directional derivative are presented in some detail in the next chapter, but a brief example is included here.

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The directional derivative is a general concept involving the change in a mathematical entity, for example, an integral, matrix, or tensor due to a change in a variable upon which that entity depends. The notion of a "directed" change is illustrated by inquiring about the change of the determinant of a matrix $\boldsymbol{A}$ as $\boldsymbol{A}$ changes in the "direction" $\boldsymbol{U}$, where $\boldsymbol{U}$ is a matter of choice, i.e., direction. Unfortunately, in this single degree of freedom example the directional derivative loses this generality.

The formal definition of the directional derivative of F in the direction β, is given from Equation (1.4) as

$$D(F)[\beta] = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} k\tan(\theta+\epsilon\beta)(10\cos(\theta+\epsilon\beta)-6), \tag{1.5}$$

giving

$$D(F)[\beta] = k\bigg[(10\cos(\theta+\epsilon\beta)-6)\sec^2(\theta+\epsilon\beta)\beta$$
$$+ k\left(\tan(\theta+\epsilon\beta)(-10\sin(\theta+\epsilon\beta))\right)\beta\bigg]\bigg|_{\epsilon=0} \tag{1.6a}$$
$$= K(\theta)\beta, \tag{1.6b}$$
$$\text{where } K(\theta) = k\left((10\cos\theta-6)\sec^2\theta - 10\tan\theta\sin\theta\right). \tag{1.6c}$$

Insofar as Equation (1.6b) is the change in F at some position θ due to a change β, then $K(\theta)$ is the stiffness at position θ.

EXAMPLE 1.2

Figure 1.3 shows a weightless rigid column supported by a torsion spring at the base. In the unloaded position the column has an initial imperfection of θ_0. The length of the column is 10, the torsional stiffness 10, and the initial imperfection is $\theta_0 = 0.01$ rads. This example is a simple model of the nonlinear behavior of a vertical column under the action of an axial load.
(a) Find the rotational equilibrium equation and plot the force P against the lateral displacement u.
(b) Linearize the equilibrium equation and set out in outline a Newton–Raphson procedure to solve the equilibrium equation.
(c) Write a computer program to implement the Newton–Raphson solution.

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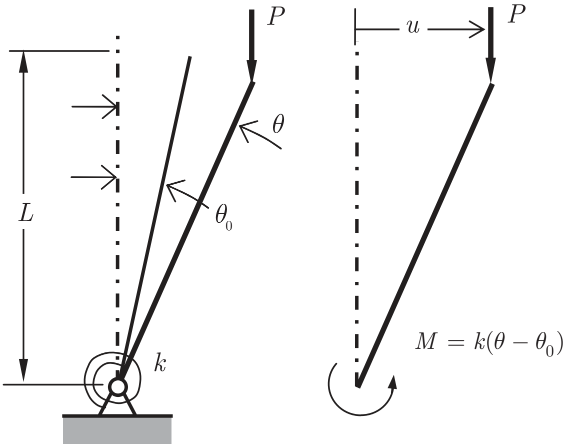

FIGURE 1.3 Imperfect column

Solution

(a) The equilibrium equation is easily found by taking moments about the base to give

$$k(\theta - \theta_0) = PL\sin\theta. \tag{1.7}$$

To plot P against u for the given values of k, L, and θ_0, simply choose a value of θ and solve for P to give

$$P = \frac{k}{L\sin\theta}(\theta - \theta_0)\ ;\ u = L\sin\theta. \tag{1.8}$$

This is shown in Figure 1.4. For a perfect column $\theta_0 = 0$ and for small angles θ the equilibrium equation approximates as $(k - PL)\theta = 0$. Since $\theta \neq 0$, then $P = k/L$, which for this structure is the classical buckling load $P_{\text{critical}} = 1$. Observe that the exact nonlinear solution clearly shows that in the region of P_{critical} a small increase in load produces a large increase in deflection.
(b) In order to conform with the nomenclature used in the program given below, the internal (resisting) moment and external (applied) moment are written as

$$T(\theta) = k(\theta - \theta_0)\ ;\ F = PL\sin\theta. \tag{1.9}$$

This enables the equilibrium Equation (1.7) to be rewritten in terms of a residual moment $R(\theta)$ suitable for the development of the Newton–Raphson

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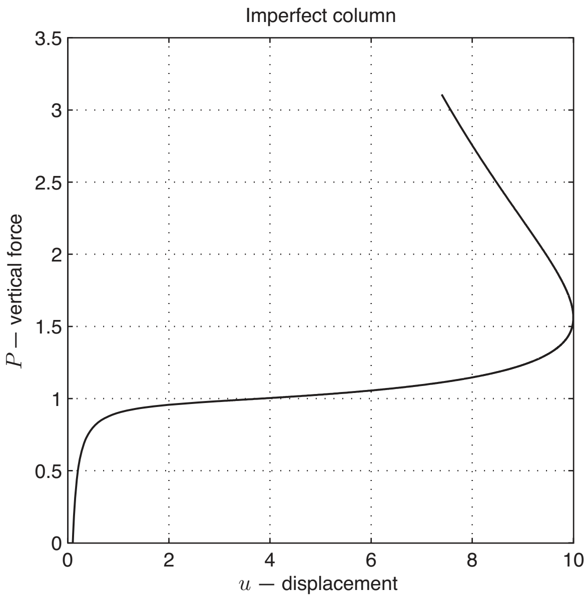

FIGURE 1.4 Imperfect column–equilibrium path

procedure as

$$R(\theta) = T(\theta) - F = 0. \tag{1.10}$$

For a given applied force P, the residual moment $R(\theta)$ is the error in the equilibrium equation due to an incorrect choice of θ. The Newton–Raphson procedure seeks to systematically correct this incorrect choice in order to satisfy the equilibrium equation. To this end, Equation (1.10) is linearized as follows:

$$R(\theta + \Delta\theta) \approx R(\theta) + K(\theta)\Delta\theta = 0. \tag{1.11}$$

where $K(\theta) = DR(\theta)[\Delta\theta]$ is the directional derivative of $R(\theta)$ in the direction $\Delta\theta$ which is found as

$$DR(\theta)[\Delta\theta] = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} [k(\theta + \epsilon\Delta\theta - \theta_0) - PL\sin(\theta + \epsilon\Delta\theta)] \tag{1.12a}$$

$$= [k\Delta\theta - PL\cos(\theta + \epsilon\Delta\theta)\Delta\theta]\Big|_{\epsilon=0} \tag{1.12b}$$

$$= (k - PL\cos\theta)\Delta\theta = K(\theta)\Delta\theta. \tag{1.12c}$$

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A schematic nonlinear solution can now be established using the Newton–Raphson procedure. The force P is applied in a series of increments in order to trace out the complete equilibrium path. The Newton–Raphson iteration is contained within the DO WHILE loop.*

BOX 1.1: Newton-Raphson Algorithm for column problem

- INPUT L, k, θ_0, ΔP and solution parameters (tolerance)
- INITIALIZE $P = 0$, $\theta = \theta_0$ (initial geometry), $R(\theta) = 0$
- FIND initial $K(\theta)$
- LOOP over load increments
  - SET $P = P + \Delta P$
  - SET $R = T(\theta) - F$
  - DO WHILE ($\|R(\theta)\|/\|F\| >$ tolerance)
    - SOLVE $K(\theta)\Delta\theta = -R(\theta)$
    - UPDATE $\theta = \theta + \Delta\theta$
    - FIND $T(\theta)$ and $K(\theta)$
    - FIND $R(\theta) = T(\theta) - F$
  - ENDDO
- ENDLOOP

(c) The schematic nonlinear solution procedure given above is expanded into the FORTRAN program that follows. The nomenclature is largely self-explanatory and follows the symbols used in Equations (1.9) to (1.12c) with the exception that `stiff`$=K$.

Computer Program for column problem

```
      program Newton Raphson
c     NR program for eccentric simple column
      implicit real*8(a-h,o-z)
      open(10,file='column.out',status='unknown',form='formatted')
c     data c
      tolerance=1.0e-06
      spring=10.0
```

* In Box 1.1 in the textbook, any remaining residual $R(\theta)$ less than the tolerance is carried over into the next load increment.

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```
      slength=10.0
      theta0=0.01
      finc=0.02
      ninc=200
      miter=20
c     initialization
      force=0.0
      residual=0.0
      theta=theta0
      stiff=spring-force*slength*cos(theta)
      write(10,'(i5,4f10.5,i5)')0,
     &         theta,force,slength*sin(theta),force,0
c     load loop
      do incrm=1,ninc
         force=force+finc
         t_internal=spring*(theta-theta0)
         f_external=force*slength*sin(theta)
         residual=t_internal-f_external
c     N-R iteration loop
         niter=0
         do while((abs(residual).gt.tolerance).and.(niter.lt.miter))
            niter=niter+1
            dtheta=-residual/stiff
            theta=theta+dtheta
            t_internal=spring*(theta-theta0)
            f_external=force*slength*sin(theta)
            residual=t_internal-f_external
            stiff=spring-force*slength*cos(theta)
         enddo
         if(niter.ge.miter)then
             print *, incrm,' no convergence'
         endif
         write(10,'(i5,4f10.5,i5)')incrm,
     &         theta,force,slength*sin(theta),force,niter
      enddo
      close (10)
      stop
      end
```

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CHAPTER TWO

MATHEMATICAL PRELIMINARIES

This chapter presents worked solutions to problems involving vector and tensor algebra, linearization, and the concept of the directional derivative and basic tensor analysis expressions.

Equation summary

Scalar (dot) product [2.5]

$$\boldsymbol{u}\cdot\boldsymbol{v} = \left(\sum_{i=1}^{3} u_i\,\boldsymbol{e}_i\right)\cdot\left(\sum_{j=1}^{3} v_j\,\boldsymbol{e}_j\right)$$

$$= \sum_{i,j=1}^{3} u_i v_j\,(\boldsymbol{e}_i\cdot\boldsymbol{e}_j)$$

$$= \sum_{i=1}^{3} u_i v_i = \boldsymbol{v}\cdot\boldsymbol{u}. \tag{2.1}$$

Transformation of Cartesian axes [2.10, 11a]

$$Q_{ij} = \boldsymbol{e}_i\cdot\boldsymbol{e}'_j, \tag{2.2a}$$

$$\boldsymbol{e}'_j = \sum_{i=1}^{3}(\boldsymbol{e}'_j\cdot\boldsymbol{e}_i)\,\boldsymbol{e}_i = \sum_{i=1}^{3} Q_{ij}\,\boldsymbol{e}_i. \tag{2.2b}$$

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Identity tensor [2.30a,b]

$$\boldsymbol{I} = \sum_{i=1}^{3} \boldsymbol{e}_i \otimes \boldsymbol{e}_i \quad \text{or} \quad \boldsymbol{I} = \sum_{i,j=1}^{3} \delta_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{2.3}$$

Tensor product components [Example 2.3]

$$(\boldsymbol{u} \otimes \boldsymbol{v}) = \left(\sum_{i=1}^{3} u_i\, \boldsymbol{e}_i \right) \otimes \left(\sum_{j=1}^{3} v_j\, \boldsymbol{e}_j \right) = \sum_{i,j=1}^{3} u_i v_j\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j. \tag{2.4}$$

Alternative Cartesian basis for second-order tensor [2.41]

$$\boldsymbol{S} = \sum_{i,j=1}^{3} S_{ij}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \sum_{i,j=1}^{3} S'_{ij}\, \boldsymbol{e}'_i \otimes \boldsymbol{e}'_j. \tag{2.5}$$

Alternative tensor components [2.42]

$$[\boldsymbol{S}]' = [\boldsymbol{Q}]^T [\boldsymbol{S}][\boldsymbol{Q}] \quad \text{or} \quad S'_{ij} = \sum_{k,l=1}^{3} Q_{ki} S_{kl} Q_{lj}. \tag{2.6}$$

Properties of the double product and trace of two tensors [2.51]

$$\boldsymbol{A} : \boldsymbol{B} = \text{tr}(\boldsymbol{A}^T \boldsymbol{B}) = \text{tr}(\boldsymbol{B}\boldsymbol{A}^T) = \text{tr}(\boldsymbol{B}^T \boldsymbol{A}) = \text{tr}(\boldsymbol{A}\boldsymbol{B}^T) \tag{2.7a}$$

$$= \sum_{i,\,j=1}^{3} A_{ij} B_{ij}. \tag{2.7b}$$

Symmetric second-order tensor in principal directions [2.59]

$$\boldsymbol{S} = \sum_{i=1}^{3} \lambda_i\, \boldsymbol{n}_i \otimes \boldsymbol{n}_i. \tag{2.8}$$

Cartesian basis for third-order tensor [2.64]

$$\boldsymbol{\mathcal{A}} = \sum_{i,j,k=1}^{3} \mathcal{A}_{ijk}\, \boldsymbol{e}_i \otimes \boldsymbol{e}_j \otimes \boldsymbol{e}_k. \tag{2.9}$$

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