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1 Introduction

Gravity

As a first approximation the Earth's gravity is given by that of a rotating sphere. The gravitational potential of a sphere of mass M is:

$$V = \frac{GM}{r}$$

where r is the position vector (Fig. A) and G the universal gravitational constant.

If the sphere is rotating with angular velocity ω the centrifugal potential at a point on the surface is given by

$$\Phi = \frac{1}{2}\omega^2 r^2 \sin^2\theta$$

where θ is the angle that r forms with the axis of rotation.

The gravity potential is their sum $U = V + \Phi$.

The value of the acceleration due to gravity (the gravity 'force') is given by the gradient of the potential:

$$\boldsymbol{g} = \nabla U$$

The radial component of the gravity force is given by

$$g_r = -\frac{GM}{r^2} + r\omega^2 \sin^2\theta$$

The potential of the Earth to a first-order approximation corresponds to that of a rotating ellipsoid, and is given by

$$U = \frac{GM}{a}\left[\frac{a}{r} - \frac{J_2}{2}\left(\frac{a}{r}\right)^3 \left(3\sin^2\varphi - 1\right) + \frac{m}{2}\left(\frac{r}{a}\right)^2 \cos^2\varphi\right]$$

where $\varphi = 90^{\circ} - \theta$ is the geocentric latitude and a the equatorial radius.

The coefficient m is the ratio between the centrifugal and gravitational forces on the sphere of radius a at the equator:

$$m = \frac{a^3\omega^2}{GM}$$

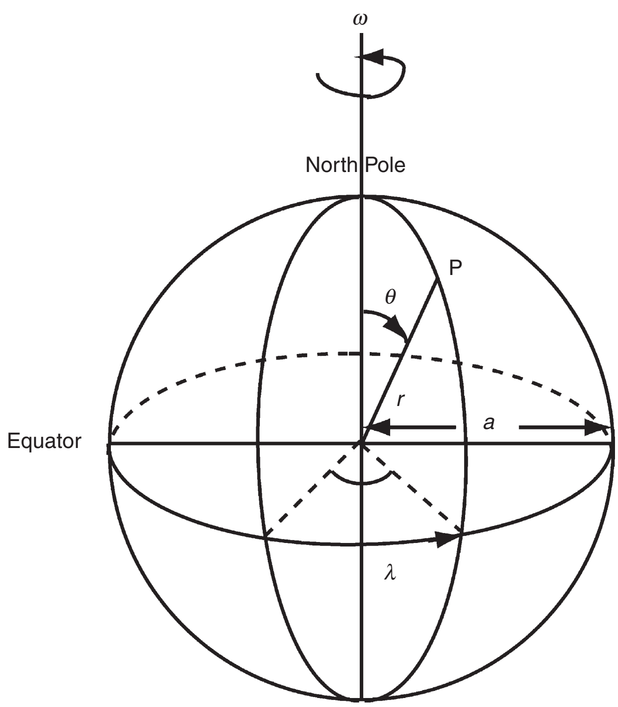

Fig. A

The dynamic form factor J_2 is defined as

$$J_2 = \frac{C - A}{a^2 M}$$

where C and A are the moments of inertia about the axis of rotation and an equatorial axis.

The flattening of the ellipsoid (the shape of the Earth to a first-order approximation) of equatorial and polar radius a and c is:

$$\alpha = \frac{a - c}{a}$$

In terms of J_2 and m,

$$\alpha = \frac{3}{2} J_2 + \frac{m}{2}$$

The dynamic ellipticity is

$$H = \frac{C - A}{C}$$

The gravity flattening is

$$\beta = \frac{\gamma_p - \gamma_e}{\gamma_e}$$

where γ_p and γ_e are the normal values of gravity at the pole and the equator, respectively.

The gravity at a point of geocentric latitude $\varphi = 90° - \theta$ is

$$\gamma = \gamma_e\left(1 + \beta \sin^2\varphi\right)$$

The geocentric latitude of a point is the angle between the equator and the radius vector of the point. The geodetic latitude is defined as the angle between the equatorial plane and the normal to the ellipsoid surface at a point. Astronomical latitude is the angle between the equatorial plane and the observed vertical at a point.

The normal or theoretical gravity at a point of geocentric latitude φ referred to the GRS1980 reference ellipsoid is

$$\gamma = 9.780327\,(1 + 0.0053024\sin^2\varphi - 0.0000059\sin^2 2\varphi)\ \mathrm{m\,s^{-2}}$$

The effect of the Sun and Moon on the Earth is to produce the phenomenon of the tides. If one considers more generally the tidal effect due to an astronomical body of mass M at a distance R from the centre of the Earth, one must add the corresponding potential, which, in the first-order approximation, is given by

$$\psi = \frac{GMr^2}{2R^3}\left(3\cos^2\vartheta - 1\right)$$

where r is the geocentric radius vector of the point, and ϑ is the angle the position vector r forms with the distance vector R.

Gravity anomalies, defined as $\Delta g = g - \gamma$, are the effects of the existence of anomalous masses inside the Earth. The gravity anomaly along the Z (vertical) axis at a point distance x along the horizontal axis produced by a sphere of radius R, density contrast $\Delta\rho$, and buried at a depth d, is given by

$$\Delta g(x,z) = \frac{\partial V_a}{\partial z} = \frac{G\Delta M(z+d)}{\left[x^2 + (z+d)^2\right]^{3/2}}$$

where V_a is the potential produced by the anomalous spherical mass $\Delta M = 4/3\pi R^3\,\Delta\rho$.

For problems in two dimensions, one uses the anomaly produced by an infinite horizontal cylinder at depth d, perpendicular to the plane under consideration. The anomalous potential is given by

$$V_a = 2\pi G\Delta\rho a^2 \ln\left(\frac{1}{\sqrt{x^2 + (z+d)^2}}\right)$$

and the anomaly by

$$\Delta g(x,z) = -\frac{\partial V_a}{\partial z} = \frac{2\pi G\Delta\rho a^2(z+d)}{x^2 + (z+d)^2}$$

To correct for the height above sea level at which measurements are made, one uses the concepts of the free-air and Bouguer anomalies. The free-air anomaly is

$$\Delta g^{FA} = g - \gamma + 3.086h$$

where g is the observed gravity, h the height in metres, and the anomaly is obtained in gu (gravity units) $\mu\mathrm{m\,s^{-2}}$.

The Bouguer anomaly is

$$\Delta g^{\mathrm{B}} = g - \gamma + (3.086 - 0.419\rho)h$$

with ρ being the density of the plate of thickness h.

To account for isostatic compensation at height in mountainous areas, one adds an isostatic correction which can be calculated assuming either the Airy or Pratt hypotheses.

With the Airy hypothesis, the root t of a mountain is given by

$$t = \frac{\rho_c}{\rho_M - \rho_c} h$$

where ρ_c and ρ_M are the densities of the crust and mantle, and h is the height of the mountain. For an ocean zone, with water density ρ_a, the anti-root is

$$t' = \frac{\rho_c - \rho_a}{\rho_M - \rho_a} h'$$

With the Pratt hypothesis, the density contrast in a mountainous area is

$$\Delta\rho = \rho - \rho_0 = \frac{-h}{D + h}\rho_0$$

where D is the level of compensation, h the height of the mountain, and ρ_0 the density at sea level. For an oceanic zone of depth h':

$$\rho' = \frac{\rho_0 D - \rho_a h'}{D - h'}$$
$$\Delta\rho = \rho' - \rho_0$$

The isostatic correction can be calculated using a cylinder of radius a and height b, whose base is located at a distance c beneath the point, and with density contrast $\Delta\rho$:

$$C^I = 2\pi G\Delta\rho\left(b + \sqrt{a^2 + (c - b)^2} - \sqrt{a^2 + c^2}\right)$$

For mountainous zones, with the Airy hypothesis: $b = t$, $c = h + H + t$ ($H =$ crustal thickness, $h =$ height of the point); and with the Pratt hypothesis: $b = D$, $c = D + h$.

Geomagnetism

To a first approximation, the internal magnetic field of the Earth can be approximated by a centred dipole inclined at 11.5° to the axis of rotation. The potential created by a magnetic dipole at a point distant r from its centre and forming an angle θ with the axis of the dipole is

$$\Phi = \frac{-Cm\cos\theta}{r^2}$$

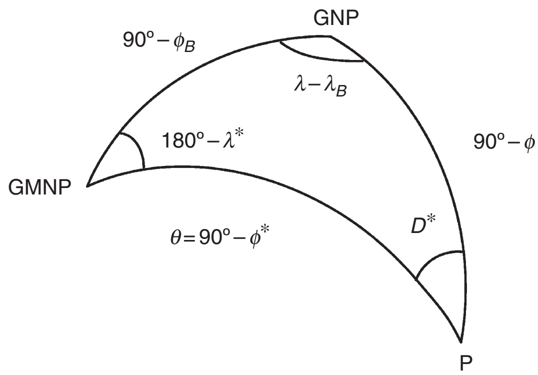

Fig. B

where $C = \mu_0/4\pi$ with $\mu_0 = 4\pi \times 10^{-7}$ H m^{-1}, and m is the dipole moment in units of A m². The product Cm is given in T m^3.

The components of the magnetic dipole field $\boldsymbol{B}$ are:

$$B_r = -\frac{\partial \Phi}{\partial r} = -\frac{2Cm\cos\theta}{r^3}$$
$$B_\theta = -\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = -\frac{2Cm\sin\theta}{r^3}$$

In the centred dipole approximation for the Earth's magnetic field, the geomagnetic coordinates (ϕ^*, λ^*) of a point ($\theta = 90° - \phi^*$) in terms of its geographic coordinates (ϕ, λ) and those of the Geomagnetic North Pole (GMNP) (ϕ_B, λ_B) can be calculated using the expressions of spherical trigonometry (Fig. B):

$$\sin\phi^* = \sin\phi_B \sin\phi + \cos\phi_B \cos\phi \cos(\lambda - \lambda_B)$$
$$\sin\lambda^* = \frac{\sin(\lambda - \lambda_B)\cos\phi}{\cos\phi^*}$$

The vertical and horizontal components of the field, the geomagnetic constant B_0, and the total field are given by:

$$Z^* = 2B_0 \sin\phi^*$$
$$H^* = B_0 \cos\phi^*$$
$$B_0 = \frac{Cm}{a^3}$$
$$F^* = \sqrt{H^{*2} + Z^{*2}} = B_0\sqrt{1 + 3\sin^2\phi^*}$$

The units used for the components of the magnetic field are the tesla T and the nanotesla nT $= 10^{-9}$ T. The NS (X^*) and EW (Y^*) components are

$$X^* = H\cos D^*$$
$$Y^* = H\sin D^*$$

and the declination and inclination are given by

$$\sin D^* = \frac{-\cos\phi_B \sin(\lambda - \lambda_B)}{\cos\phi^*}$$

$$\tan I^* = 2\tan\phi^*$$

The radius vector at each point of the line of force is:

$$r = r_0 \cos^2\phi^* = r_0 \sin^2\theta$$

where r_0 is the radius vector of the point of the line of force located at the geomagnetic equator.

Magnetic anomalies are produced by magnetic materials within the Earth. The anomalous potential due to a vertical dipole buried at depth d is

$$\Phi_A = \frac{Cm\cos\theta}{r^2} = \frac{Cm(z+d)}{\left[x^2 + (z+d)^2\right]^{3/2}}$$

The vertical (z) and the horizontal (x) components of the magnetic anomaly at the surface ($z = 0$) produced by a vertical magnetic dipole at depth d are:

$$\Delta Z = \frac{Cm(2d^2 - x^2)}{(x^2 + d^2)^{5/2}}$$

$$\Delta X = \frac{3Cmxd}{(x^2 + d^2)^{5/2}}$$

The Earth is affected by an external magnetic field produced mainly by the activity of the Sun. This field is variable in time, with distinct periods of variation. The most noticeable is the diurnal variation (*Sq*) with a maximum at 12 noon local time. The most important non-periodic variations are the so-called magnetic storms.

Seismology

Earthquakes produce elastic waves which propagate through the interior and along the surface of the Earth. Using the plane-wave approximation, the displacements of the internal P- and S-waves (u_i^{P} and u_i^{S}) can be obtained from a scalar potential and a vector potential:

$$u_i = u_i^{\mathrm{P}} + u_i^{\mathrm{S}} = (\nabla\varphi)_i + \left(\nabla \times \psi_j\right)_i$$

$$\varphi = A \exp ik_\alpha\left(\gamma_j x_j - \alpha t\right)$$

$$\psi_j = B_j \exp ik_\beta\left(\gamma_j x_j - \beta t\right)$$

where A and B_j are the amplitudes, x_j the coordinates of the observation point, k_α and k_β the wavenumbers, γ_j are the direction cosines defined from the azimuth a_z and angle of incidence i of the ray as:

$$\gamma_1 = \sin i \cos a_z$$
$$\gamma_2 = \sin i \ \sin a_z$$
$$\gamma_3 = \cos i$$

and α and β are the P- and S-wave velocities of propagation, respectively, defined from the Lamé coefficients (λ and shear modulus μ) and the density ρ:

$$v^{\mathrm{P}} = \alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

$$v^{\mathrm{S}} = \beta = \sqrt{\frac{\mu}{\rho}}$$

Units used are: displacement amplitudes (u) in μm; potential amplitudes (A, B_i) in 10^{-3} m^2; wavenumber (k) in km^{-1}; and wave velocity (α, β) in km s^{-1}.

Poisson's ratio is defined in terms of the Lamé coefficients as

$$\sigma = \frac{\lambda}{2(\lambda + \mu)}$$

The angle of polarization of S-wave ε is defined as

$$\varepsilon = \tan^{-1} \frac{u^{\mathrm{SH}}}{u^{\mathrm{SV}}}$$

where u^{SH} is the amplitude of the SH component, and u^{SV} that of the SV component. SH and SV are the horizontal and vertical components of the S-wave on the wavefront plane.

The coefficients of reflection V and transmission W are given by the respective ratios between the amplitudes of the reflected or transmitted potentials and the incident potential:

$$V = \frac{A}{A_0}$$
$$W = \frac{A'}{A_0}$$

where A_0 is the amplitude of the incident wave potential, A that of the reflected potential, and A' of the transmitted potential.

Snell's law for plane media is expressed as

$$p = \frac{\sin i}{v}$$

and for spherical media

$$p = \frac{r \sin i}{v}$$

where p is the ray parameter, i the angle of incidence, v the propagation velocity of the medium, and r the position vector along the ray.

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In the case of plane media with propagation velocity varying with depth $v(z)$, the epicentral distance and the travel time of a ray for a surface focus are given by

$$x = 2\int_0^h \frac{p\,dz}{\sqrt{\eta^2 - p^2}}$$

$$t = 2\int_0^h \frac{\eta^2\,dz}{\sqrt{\eta^2 - p^2}}$$

where $\eta = v^{-1}$ and h is the depth of maximum penetration of the ray. The variation of the epicentral distance x with the ray parameter p is given by

$$\frac{dx}{dp} = -\frac{2}{\varsigma_0\sqrt{\eta_0^2 - p^2}} + 2\int_0^{\varsigma} \frac{\frac{d\varsigma}{dz}dz}{\varsigma^2\sqrt{\eta^2 - p^2}}$$

where

$$\varsigma = \frac{1}{v}\frac{dv}{dz}$$

In spherical media with velocity varying with depth $v(r)$, the epicentral distance, trajectory along the ray, and travel time are given by

$$\Delta = 2\int_{r_p}^{r_0} \frac{p}{r}\frac{dr}{\sqrt{\eta^2 - r^2}}$$

$$s = 2\int_{r_p}^{r_0} \frac{\eta dr}{\sqrt{\eta^2 - r^2}}$$

$$t = 2\int_{r_p}^{r_0} \frac{dr}{v\sqrt{\eta^2 - r^2}}$$

where $\eta = rv^{-1}$, r_0 is the radius at the surface of the Earth, and r_p is the radius at the point of maximum penetration of the ray.

The variation of the distance from the epicentre Δ with the ray parameter p in a spherical medium is

$$\frac{d\Delta}{dp} = -\frac{2}{(1 - \varsigma_0)\sqrt{\eta_0^2 - p^2}} + 2\int_0^{\varsigma} \frac{\frac{d\varsigma}{dr}dr}{(1 - \varsigma^2)\sqrt{\eta^2 - p^2}}$$

where

$$\varsigma = \frac{r}{v}\frac{dv}{dr}$$

The radial and vertical components (u_1 and u_3) of surface waves can be obtained from the potentials φ and ψ. The transverse component (u_2) is kept apart

$$u_1 = \frac{\partial\varphi}{\partial x_1} - \frac{\partial\psi}{\partial x_3} = \varphi_{,1} - \psi_{,3}$$

$$u_2 = C\exp[-iksx_3 + ik(x_1 - ct)]$$

$$u_3 = \frac{\partial\varphi}{\partial x_3} + \frac{\partial\psi}{\partial x_1} = \varphi_{,3} + \psi_{,1}$$

where c is the wave propagation velocity and

$$\varphi = A \exp[-ikrx_3 + ik(x_1 - ct)]$$
$$\psi = B \exp[-iksx_3 + ik(x_1 - ct)]$$
$$r = \sqrt{\frac{c^2}{\alpha^2} - 1}$$
$$s = \sqrt{\frac{c^2}{\beta^2} - 1}$$

For surface waves, $c < \beta < \alpha$, and hence r and s are imaginary.

For dispersive waves, the relationship between the phase velocity c and the group velocity U is

$$U = c + k\frac{dc}{dk}$$

where k is the wavenumber.

The position of the seismic focus is given by the coordinates of the epicentre (φ_0, λ_0) and the depth h. The time is that of the origin of the earthquake t_0. The size is given by the magnitude which is proportional to the logarithm of the amplitude of the recorded waves. For surface waves this is:

$$M_s = \log\frac{A}{T} + 1.66 \log \Delta + 3.3$$

where A is the amplitude of ground motion in microns, T is the period in seconds, and Δ the epicentral distance in degrees.

The magnitude of the moment is given by

$$M_w = \frac{2}{3}\log M_0 - 6.1$$

where M_0 is the seismic moment in N m (newton metres). The seismic moment is related to the displacement of the fault Δu and its area S:

$$M_0 = \mu \Delta u S$$

The mechanism of earthquakes is given by the orientation of the fracture plane (fault) defined by the angles φ (azimuth), δ (dip), and λ (slip angle or rake), or by the vectors $\boldsymbol{n}$ (the normal to the fault plane) and $\boldsymbol{l}$ (the direction of slip).

The elastic displacement of the waves produced by a point shear fault is

$$u_k(x_s, t) = \mu \Delta u(t) S\left(l_i n_j + l_j n_i\right) \frac{\partial G_{ki}}{\partial x_j}$$

where G_{ki} is the medium's Green's function which, for an isotropic, homogeneous, infinite medium, and P-waves in the far-field regime, is given by

$$G^{P}_{ki} = \frac{1}{4\pi\rho\alpha^2 r}\gamma_i\gamma_k\delta\left(t - \frac{r}{\alpha}\right)$$

The P-wave displacements are given by:

$$u_k^{\mathrm{P}}(x_{\mathrm{s}}, t) = \frac{\Delta\dot{u}(t)S}{4\pi\rho\alpha^3 r}\mu\left(l_i n_j + l_j n_i\right)\gamma_i\gamma_j\gamma_k$$

This equation can be expressed also in terms of the moment tensor M_{ij}

$$u_k^{\mathrm{P}}(x_s, t) = \frac{\dot{M}_{ij}(t)}{4\pi\rho\alpha^3 r}\gamma_i\gamma_j\gamma_k$$

M_{ij} is a more general representation of a point source.

Heat flow

The Fourier law of heat transfer by diffusion states that the heat flux $\dot{q}$ is proportional to the gradient of the temperature T:

$$\dot{q} = -K\nabla T$$

where K is the thermal conductivity coefficient. The units of heat flow are W m^{-2}.

The heat diffusion equation, assuming that K is constant, is given by

$$\kappa\nabla^2 T + \frac{\varepsilon}{\rho C_v} = \frac{\partial T}{\partial t}$$

where C_v is the specific heat, ρ the density, ε the heat generated per unit volume and unit time (heat sources), and κ the thermal diffusivity:

$$\kappa = \frac{K}{\rho C_v}$$

If there are no heat sources, the diffusion equation is

$$\kappa\nabla^2 T = \frac{\partial T}{\partial t}$$

In the case of one-dimensional flow with periodic variation of temperature over time, one has:

$$T(z, t) = T_0 \exp\left[-\sqrt{\frac{\omega}{2\kappa}}z + i\left(-\sqrt{\frac{\omega}{2\kappa}}z + \omega t\right)\right]$$

where z is the vertical direction (positive towards the nadir) and ω the angular frequency.

In the case of stationary one-dimensional solutions (T constant in time) one obtains from the diffusion equation:

$$T = -\frac{\varepsilon}{2K}z^2 + \frac{\dot{q}_0}{K}z + T_0$$