

# Contents

---

<i>Preface to the Third Edition</i>	page xix
<i>Preface to the Second Edition</i>	xxi
<i>Preface to the Revised Printing</i>	xxiii
<i>Preface to the First Edition</i>	xxv
<b>Overview. An Informal Overview of Cartan’s Exterior Differential Forms, Illustrated with an Application to Cauchy’s Stress Tensor</b>	xxix
Introduction	xxix
0.a. Introduction	xxix
Vectors, 1-Forms, and Tensors	xxx
0.b. Two Kinds of Vectors	xxx
0.c. Superscripts, Subscripts, Summation Convention	xxxiii
0.d. Riemannian Metrics	xxxiv
0.e. Tensors	xxxvii
Integrals and Exterior Forms	xxxvii
0.f. Line Integrals	xxxvii
0.g. Exterior 2-Forms	xxxix
0.h. Exterior $p$ -Forms and Algebra in $\mathbb{R}^n$	xl
0.i. The Exterior Differential $d$	xli
0.j. The Push-Forward of a Vector and the Pull-Back of a Form	xlii
0.k. Surface Integrals and “Stokes’ Theorem”	xliv
0.l. Electromagnetism, or, Is it a Vector or a Form?	xlvi
0.m. Interior Products	xlvii
0.n. Volume Forms and Cartan’s Vector Valued Exterior Forms	xlviii
0.o. Magnetic Field for Current in a Straight Wire	1
Elasticity and Stresses	li
0.p. Cauchy Stress, Floating Bodies, Twisted Cylinders, and Strain Energy	li
0.q. Sketch of Cauchy’s “First Theorem”	lvii
0.r. Sketch of Cauchy’s “Second Theorem,” Moments as Generators of Rotations	lix
0.s. A Remarkable Formula for Differentiating Line, Surface, and . . . , Integrals	lxi

<b>I Manifolds, Tensors, and Exterior Forms</b>	
<b>1 Manifolds and Vector Fields</b>	<b>3</b>
1.1. Submanifolds of Euclidean Space	3
1.1a. Submanifolds of $\mathbb{R}^N$	4
1.1b. The Geometry of Jacobian Matrices: The “Differential”	7
1.1c. The Main Theorem on Submanifolds of $\mathbb{R}^N$	8
1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body	9
1.2. Manifolds	11
1.2a. Some Notions from Point Set Topology	11
1.2b. The Idea of a Manifold	13
1.2c. A Rigorous Definition of a Manifold	19
1.2d. Complex Manifolds: The Riemann Sphere	21
1.3. Tangent Vectors and Mappings	22
1.3a. Tangent or “Contravariant” Vectors	23
1.3b. Vectors as Differential Operators	24
1.3c. The Tangent Space to $M^n$ at a Point	25
1.3d. Mappings and Submanifolds of Manifolds	26
1.3e. Change of Coordinates	29
1.4. Vector Fields and Flows	30
1.4a. Vector Fields and Flows on $\mathbb{R}^n$	30
1.4b. Vector Fields on Manifolds	33
1.4c. Straightening Flows	34
<b>2 Tensors and Exterior Forms</b>	<b>37</b>
2.1. Covectors and Riemannian Metrics	37
2.1a. Linear Functionals and the Dual Space	37
2.1b. The Differential of a Function	40
2.1c. Scalar Products in Linear Algebra	42
2.1d. Riemannian Manifolds and the Gradient Vector	45
2.1e. Curves of Steepest Ascent	46
2.2. The Tangent Bundle	48
2.2a. The Tangent Bundle	48
2.2b. The Unit Tangent Bundle	50
2.3. The Cotangent Bundle and Phase Space	52
2.3a. The Cotangent Bundle	52
2.3b. The Pull-Back of a Covector	52
2.3c. The Phase Space in Mechanics	54
2.3d. The Poincaré 1-Form	56
2.4. Tensors	58
2.4a. Covariant Tensors	58
2.4b. Contravariant Tensors	59
2.4c. Mixed Tensors	60
2.4d. Transformation Properties of Tensors	62
2.4e. Tensor Fields on Manifolds	63

CONTENTS	ix
2.5. The Grassmann or Exterior Algebra	66
2.5a. The Tensor Product of Covariant Tensors	66
2.5b. The Grassmann or Exterior Algebra	66
2.5c. The Geometric Meaning of Forms in $\mathbb{R}^n$	70
2.5d. Special Cases of the Exterior Product	70
2.5e. Computations and Vector Analysis	71
2.6. Exterior Differentiation	73
2.6a. The Exterior Differential	73
2.6b. Examples in $\mathbb{R}^3$	75
2.6c. A Coordinate Expression for $d$	76
2.7. Pull-Backs	77
2.7a. The Pull-Back of a Covariant Tensor	77
2.7b. The Pull-Back in Elasticity	80
2.8. Orientation and Pseudoforms	82
2.8a. Orientation of a Vector Space	82
2.8b. Orientation of a Manifold	83
2.8c. Orientability and 2-Sided Hypersurfaces	84
2.8d. Projective Spaces	85
2.8e. Pseudoforms and the Volume Form	85
2.8f. The Volume Form in a Riemannian Manifold	87
2.9. Interior Products and Vector Analysis	89
2.9a. Interior Products and Contractions	89
2.9b. Interior Product in $\mathbb{R}^3$	90
2.9c. Vector Analysis in $\mathbb{R}^3$	92
2.10. Dictionary	94
<b>3 Integration of Differential Forms</b>	<b>95</b>
3.1. Integration over a Parameterized Subset	95
3.1a. Integration of a $p$ -Form in $\mathbb{R}^p$	95
3.1b. Integration over Parameterized Subsets	96
3.1c. Line Integrals	97
3.1d. Surface Integrals	99
3.1e. Independence of Parameterization	101
3.1f. Integrals and Pull-Backs	102
3.1g. Concluding Remarks	102
3.2. Integration over Manifolds with Boundary	104
3.2a. Manifolds with Boundary	105
3.2b. Partitions of Unity	106
3.2c. Integration over a Compact Oriented Submanifold	108
3.2d. Partitions and Riemannian Metrics	109
3.3. Stokes's Theorem	110
3.3a. Orienting the Boundary	110
3.3b. Stokes's Theorem	111
3.4. Integration of Pseudoforms	114
3.4a. Integrating Pseudo- $n$ -Forms on an $n$ -Manifold	115
3.4b. Submanifolds with Transverse Orientation	115

3.4c.	Integration over a Submanifold with Transverse Orientation	116
3.4d.	Stokes's Theorem for Pseudoforms	117
3.5.	Maxwell's Equations	118
3.5a.	Charge and Current in Classical Electromagnetism	118
3.5b.	The Electric and Magnetic Fields	119
3.5c.	Maxwell's Equations	120
3.5d.	Forms and Pseudoforms	122
<b>4</b>	<b>The Lie Derivative</b>	<b>125</b>
4.1.	The Lie Derivative of a Vector Field	125
4.1a.	The Lie Bracket	125
4.1b.	Jacobi's Variational Equation	127
4.1c.	The Flow Generated by $[X, Y]$	129
4.2.	The Lie Derivative of a Form	132
4.2a.	Lie Derivatives of Forms	132
4.2b.	Formulas Involving the Lie Derivative	134
4.2c.	Vector Analysis Again	136
4.3.	Differentiation of Integrals	138
4.3a.	The Autonomous (Time-Independent) Case	138
4.3b.	Time-Dependent Fields	140
4.3c.	Differentiating Integrals	142
4.4.	A Problem Set on Hamiltonian Mechanics	145
4.4a.	Time-Independent Hamiltonians	147
4.4b.	Time-Dependent Hamiltonians and Hamilton's Principle	151
4.4c.	Poisson brackets	154
<b>5</b>	<b>The Poincaré Lemma and Potentials</b>	<b>155</b>
5.1.	A More General Stokes's Theorem	155
5.2.	Closed Forms and Exact Forms	156
5.3.	Complex Analysis	158
5.4.	The Converse to the Poincaré Lemma	160
5.5.	Finding Potentials	162
<b>6</b>	<b>Holonomic and Nonholonomic Constraints</b>	<b>165</b>
6.1.	The Frobenius Integrability Condition	165
6.1a.	Planes in $\mathbb{R}^3$	165
6.1b.	Distributions and Vector Fields	167
6.1c.	Distributions and 1-Forms	167
6.1d.	The Frobenius Theorem	169
6.2.	Integrability and Constraints	172
6.2a.	Foliations and Maximal Leaves	172
6.2b.	Systems of Mayer–Lie	174
6.2c.	Holonomic and Nonholonomic Constraints	175

## CONTENTS

xi

<b>6.3.</b>	Heuristic Thermodynamics via Caratheodory	178
<b>6.3a.</b>	Introduction	178
<b>6.3b.</b>	The First Law of Thermodynamics	179
<b>6.3c.</b>	Some Elementary Changes of State	180
<b>6.3d.</b>	The Second Law of Thermodynamics	181
<b>6.3e.</b>	Entropy	183
<b>6.3f.</b>	Increasing Entropy	185
<b>6.3g.</b>	Chow's Theorem on Accessibility	187

**II Geometry and Topology**

<b>7</b>	<b><math>\mathbb{R}^3</math> and Minkowski Space</b>	<b>191</b>
<b>7.1.</b>	Curvature and Special Relativity	191
<b>7.1a.</b>	Curvature of a Space Curve in $\mathbb{R}^3$	191
<b>7.1b.</b>	Minkowski Space and Special Relativity	192
<b>7.1c.</b>	Hamiltonian Formulation	196
<b>7.2.</b>	Electromagnetism in Minkowski Space	196
<b>7.2a.</b>	Minkowski's Electromagnetic Field Tensor	196
<b>7.2b.</b>	Maxwell's Equations	198
<b>8</b>	<b>The Geometry of Surfaces in <math>\mathbb{R}^3</math></b>	<b>201</b>
<b>8.1.</b>	The First and Second Fundamental Forms	201
<b>8.1a.</b>	The First Fundamental Form, or Metric Tensor	201
<b>8.1b.</b>	The Second Fundamental Form	203
<b>8.2.</b>	Gaussian and Mean Curvatures	205
<b>8.2a.</b>	Symmetry and Self-Adjointness	205
<b>8.2b.</b>	Principal Normal Curvatures	206
<b>8.2c.</b>	Gauss and Mean Curvatures: The Gauss Normal Map	207
<b>8.3.</b>	The Brouwer Degree of a Map: A Problem Set	210
<b>8.3a.</b>	The Brouwer Degree	210
<b>8.3b.</b>	Complex Analytic (Holomorphic) Maps	214
<b>8.3c.</b>	The Gauss Normal Map Revisited: The Gauss–Bonnet Theorem	215
<b>8.3d.</b>	The Kronecker Index of a Vector Field	215
<b>8.3e.</b>	The Gauss Looping Integral	218
<b>8.4.</b>	Area, Mean Curvature, and Soap Bubbles	221
<b>8.4a.</b>	The First Variation of Area	221
<b>8.4b.</b>	Soap Bubbles and Minimal Surfaces	226
<b>8.5.</b>	Gauss's <i>Theorema Egregium</i>	228
<b>8.5a.</b>	The Equations of Gauss and Codazzi	228
<b>8.5b.</b>	The <i>Theorema Egregium</i>	230
<b>8.6.</b>	Geodesics	232
<b>8.6a.</b>	The First Variation of Arc Length	232
<b>8.6b.</b>	The Intrinsic Derivative and the Geodesic Equation	234
<b>8.7.</b>	The Parallel Displacement of Levi-Civita	236

<b>9 Covariant Differentiation and Curvature</b>	<b>241</b>
<b>9.1. Covariant Differentiation</b>	241
<b>9.1a. Covariant Derivative</b>	241
<b>9.1b. Curvature of an Affine Connection</b>	244
<b>9.1c. Torsion and Symmetry</b>	245
<b>9.2. The Riemannian Connection</b>	246
<b>9.3. Cartan's Exterior Covariant Differential</b>	247
<b>9.3a. Vector-Valued Forms</b>	247
<b>9.3b. The Covariant Differential of a Vector Field</b>	248
<b>9.3c. Cartan's Structural Equations</b>	249
<b>9.3d. The Exterior Covariant Differential of a Vector-Valued Form</b>	250
<b>9.3e. The Curvature 2-Forms</b>	251
<b>9.4. Change of Basis and Gauge Transformations</b>	253
<b>9.4a. Symmetric Connections Only</b>	253
<b>9.4b. Change of Frame</b>	253
<b>9.5. The Curvature Forms in a Riemannian Manifold</b>	255
<b>9.5a. The Riemannian Connection</b>	255
<b>9.5b. Riemannian Surfaces <math>M^2</math></b>	257
<b>9.5c. An Example</b>	257
<b>9.6. Parallel Displacement and Curvature on a Surface</b>	259
<b>9.7. Riemann's Theorem and the Horizontal Distribution</b>	263
<b>9.7a. Flat metrics</b>	263
<b>9.7b. The Horizontal Distribution of an Affine Connection</b>	263
<b>9.7c. Riemann's Theorem</b>	266
<b>10 Geodesics</b>	<b>269</b>
<b>10.1. Geodesics and Jacobi Fields</b>	269
<b>10.1a. Vector Fields Along a Surface in <math>M^n</math></b>	269
<b>10.1b. Geodesics</b>	271
<b>10.1c. Jacobi Fields</b>	272
<b>10.1d. Energy</b>	274
<b>10.2. Variational Principles in Mechanics</b>	275
<b>10.2a. Hamilton's Principle in the Tangent Bundle</b>	275
<b>10.2b. Hamilton's Principle in Phase Space</b>	277
<b>10.2c. Jacobi's Principle of "Least" Action</b>	278
<b>10.2d. Closed Geodesics and Periodic Motions</b>	281
<b>10.3. Geodesics, Spiders, and the Universe</b>	284
<b>10.3a. Gaussian Coordinates</b>	284
<b>10.3b. Normal Coordinates on a Surface</b>	287
<b>10.3c. Spiders and the Universe</b>	288
<b>11 Relativity, Tensors, and Curvature</b>	<b>291</b>
<b>11.1. Heuristics of Einstein's Theory</b>	291
<b>11.1a. The Metric Potentials</b>	291
<b>11.1b. Einstein's Field Equations</b>	293
<b>11.1c. Remarks on Static Metrics</b>	296

CONTENTS		xiii
<b>11.2.</b>	Tensor Analysis	298
<b>11.2a.</b>	Covariant Differentiation of Tensors	298
<b>11.2b.</b>	Riemannian Connections and the Bianchi Identities	299
<b>11.2c.</b>	Second Covariant Derivatives: The Ricci Identities	301
<b>11.3.</b>	Hilbert's Action Principle	303
<b>11.3a.</b>	Geodesics in a Pseudo-Riemannian Manifold	303
<b>11.3b.</b>	Normal Coordinates, the Divergence and Laplacian	303
<b>11.3c.</b>	Hilbert's Variational Approach to General Relativity	305
<b>11.4.</b>	The Second Fundamental Form in the Riemannian Case	309
<b>11.4a.</b>	The Induced Connection and the Second Fundamental Form	309
<b>11.4b.</b>	The Equations of Gauss and Codazzi	311
<b>11.4c.</b>	The Interpretation of the Sectional Curvature	313
<b>11.4d.</b>	Fixed Points of Isometries	314
<b>11.5.</b>	The Geometry of Einstein's Equations	315
<b>11.5a.</b>	The Einstein Tensor in a (Pseudo-)Riemannian Space-Time	315
<b>11.5b.</b>	The Relativistic Meaning of Gauss's Equation	316
<b>11.5c.</b>	The Second Fundamental Form of a Spatial Slice	318
<b>11.5d.</b>	The Codazzi Equations	319
<b>11.5e.</b>	Some Remarks on the Schwarzschild Solution	320
<b>12</b>	<b>Curvature and Topology: Synge's Theorem</b>	<b>323</b>
<b>12.1.</b>	Synge's Formula for Second Variation	324
<b>12.1a.</b>	The Second Variation of Arc Length	324
<b>12.1b.</b>	Jacobi Fields	326
<b>12.2.</b>	Curvature and Simple Connectivity	329
<b>12.2a.</b>	Synge's Theorem	329
<b>12.2b.</b>	Orientability Revisited	331
<b>13</b>	<b>Betti Numbers and De Rham's Theorem</b>	<b>333</b>
<b>13.1.</b>	Singular Chains and Their Boundaries	333
<b>13.1a.</b>	Singular Chains	333
<b>13.1b.</b>	Some 2-Dimensional Examples	338
<b>13.2.</b>	The Singular Homology Groups	342
<b>13.2a.</b>	Coefficient Fields	342
<b>13.2b.</b>	Finite Simplicial Complexes	343
<b>13.2c.</b>	Cycles, Boundaries, Homology and Betti Numbers	344
<b>13.3.</b>	Homology Groups of Familiar Manifolds	347
<b>13.3a.</b>	Some Computational Tools	347
<b>13.3b.</b>	Familiar Examples	350
<b>13.4.</b>	De Rham's Theorem	355
<b>13.4a.</b>	The Statement of de Rham's Theorem	355
<b>13.4b.</b>	Two Examples	357

<b>14 Harmonic Forms</b>	<b>361</b>
14.1. The Hodge Operators	361
14.1a. The $*$ Operator	361
14.1b. The Codifferential Operator $\delta = d^*$	364
14.1c. Maxwell's Equations in Curved Space–Time $M^4$	366
14.1d. The Hilbert Lagrangian	367
14.2. Harmonic Forms	368
14.2a. The Laplace Operator on Forms	368
14.2b. The Laplacian of a 1-Form	369
14.2c. Harmonic Forms on Closed Manifolds	370
14.2d. Harmonic Forms and de Rham's Theorem	372
14.2e. Bochner's Theorem	374
14.3. Boundary Values, Relative Homology, and Morse Theory	375
14.3a. Tangential and Normal Differential Forms	376
14.3b. Hodge's Theorem for Tangential Forms	377
14.3c. Relative Homology Groups	379
14.3d. Hodge's Theorem for Normal Forms	381
14.3e. Morse's Theory of Critical Points	382
<b>III Lie Groups, Bundles, and Chern Forms</b>	
<b>15 Lie Groups</b>	<b>391</b>
15.1. Lie Groups, Invariant Vector Fields and Forms	391
15.1a. Lie Groups	391
15.1b. Invariant Vector Fields and Forms	395
15.2. One Parameter Subgroups	398
15.3. The Lie Algebra of a Lie Group	402
15.3a. The Lie Algebra	402
15.3b. The Exponential Map	403
15.3c. Examples of Lie Algebras	404
15.3d. Do the 1-Parameter Subgroups Cover $G$ ?	405
15.4. Subgroups and Subalgebras	407
15.4a. Left Invariant Fields Generate Right Translations	407
15.4b. Commutators of Matrices	408
15.4c. Right Invariant Fields	409
15.4d. Subgroups and Subalgebras	410
<b>16 Vector Bundles in Geometry and Physics</b>	<b>413</b>
16.1. Vector Bundles	413
16.1a. Motivation by Two Examples	413
16.1b. Vector Bundles	415
16.1c. Local Trivializations	417
16.1d. The Normal Bundle to a Submanifold	419
16.2. Poincaré's Theorem and the Euler Characteristic	421
16.2a. Poincaré's Theorem	422
16.2b. The Stiefel Vector Field and Euler's Theorem	426



CONTENTS	xv
<b>16.3.</b> Connections in a Vector Bundle	428
<b>16.3a.</b> Connection in a Vector Bundle	428
<b>16.3b.</b> Complex Vector Spaces	431
<b>16.3c.</b> The Structure Group of a Bundle	433
<b>16.3d.</b> Complex Line Bundles	433
<b>16.4.</b> The Electromagnetic Connection	435
<b>16.4a.</b> Lagrange's Equations Without Electromagnetism	435
<b>16.4b.</b> The Modified Lagrangian and Hamiltonian	436
<b>16.4c.</b> Schrödinger's Equation in an Electromagnetic Field	439
<b>16.4d.</b> Global Potentials	443
<b>16.4e.</b> The Dirac Monopole	444
<b>16.4f.</b> The Aharonov–Bohm Effect	446
<b>17 Fiber Bundles, Gauss–Bonnet, and Topological Quantization</b>	<b>451</b>
<b>17.1.</b> Fiber Bundles and Principal Bundles	451
<b>17.1a.</b> Fiber Bundles	451
<b>17.1b.</b> Principal Bundles and Frame Bundles	453
<b>17.1c.</b> Action of the Structure Group on a Principal Bundle	454
<b>17.2.</b> Coset Spaces	456
<b>17.2a.</b> Cosets	456
<b>17.2b.</b> Grassmann Manifolds	459
<b>17.3.</b> Chern's Proof of the Gauss–Bonnet–Poincaré Theorem	460
<b>17.3a.</b> A Connection in the Frame Bundle of a Surface	460
<b>17.3b.</b> The Gauss–Bonnet–Poincaré Theorem	462
<b>17.3c.</b> Gauss–Bonnet as an Index Theorem	465
<b>17.4.</b> Line Bundles, Topological Quantization, and Berry Phase	465
<b>17.4a.</b> A Generalization of Gauss–Bonnet	465
<b>17.4b.</b> Berry Phase	468
<b>17.4c.</b> Monopoles and the Hopf Bundle	473
<b>18 Connections and Associated Bundles</b>	<b>475</b>
<b>18.1.</b> Forms with Values in a Lie Algebra	475
<b>18.1a.</b> The Maurer–Cartan Form	475
<b>18.1b.</b> $\mathfrak{g}$ -Valued $p$ -Forms on a Manifold	477
<b>18.1c.</b> Connections in a Principal Bundle	479
<b>18.2.</b> Associated Bundles and Connections	481
<b>18.2a.</b> Associated Bundles	481
<b>18.2b.</b> Connections in Associated Bundles	483
<b>18.2c.</b> The Associated $Ad$ Bundle	485
<b>18.3.</b> $r$ -Form Sections of a Vector Bundle: Curvature	488
<b>18.3a.</b> $r$ -Form sections of $E$	488
<b>18.3b.</b> Curvature and the $Ad$ Bundle	489
<b>19 The Dirac Equation</b>	<b>491</b>
<b>19.1.</b> The Groups $SO(3)$ and $SU(2)$	491
<b>19.1a.</b> The Rotation Group $SO(3)$ of $\mathbb{R}^3$	492
<b>19.1b.</b> $SU(2)$ : The Lie algebra $\mathfrak{su}(2)$	493

19.1c.	$SU(2)$ is Topologically the 3-Sphere	495
19.1d.	$Ad : SU(2) \rightarrow SO(3)$ in More Detail	496
19.2.	Hamilton, Clifford, and Dirac	497
19.2a.	Spinors and Rotations of $\mathbb{R}^3$	497
19.2b.	Hamilton on Composing Two Rotations	499
19.2c.	Clifford Algebras	500
19.2d.	The Dirac Program: The Square Root of the d'Alembertian	502
19.3.	The Dirac Algebra	504
19.3a.	The Lorentz Group	504
19.3b.	The Dirac Algebra	509
19.4.	The Dirac Operator $\not{\partial}$ in Minkowski Space	511
19.4a.	Dirac Spinors	511
19.4b.	The Dirac Operator	513
19.5.	The Dirac Operator in Curved Space–Time	515
19.5a.	The Spinor Bundle	515
19.5b.	The Spin Connection in $\mathcal{SM}$	518
<b>20</b>	<b>Yang–Mills Fields</b>	<b>523</b>
20.1.	Noether's Theorem for Internal Symmetries	523
20.1a.	The Tensorial Nature of Lagrange's Equations	523
20.1b.	Boundary Conditions	526
20.1c.	Noether's Theorem for Internal Symmetries	527
20.1d.	Noether's Principle	528
20.2.	Weyl's Gauge Invariance Revisited	531
20.2a.	The Dirac Lagrangian	531
20.2b.	Weyl's Gauge Invariance Revisited	533
20.2c.	The Electromagnetic Lagrangian	534
20.2d.	Quantization of the $A$ Field: Photons	536
20.3.	The Yang–Mills Nucleon	537
20.3a.	The Heisenberg Nucleon	537
20.3b.	The Yang–Mills Nucleon	538
20.3c.	A Remark on Terminology	540
20.4.	Compact Groups and Yang–Mills Action	541
20.4a.	The Unitary Group Is Compact	541
20.4b.	Averaging over a Compact Group	541
20.4c.	Compact Matrix Groups Are Subgroups of Unitary Groups	542
20.4d.	$Ad$ Invariant Scalar Products in the Lie Algebra of a Compact Group	543
20.4e.	The Yang–Mills Action	544
20.5.	The Yang–Mills Equation	545
20.5a.	The Exterior Covariant Divergence $\nabla^*$	545
20.5b.	The Yang–Mills Analogy with Electromagnetism	547
20.5c.	Further Remarks on the Yang–Mills Equations	548

CONTENTS	xvii
20.6. Yang–Mills Instantons	550
20.6a. Instantons	550
20.6b. Chern’s Proof Revisited	553
20.6c. Instantons and the Vacuum	557
<b>21 Betti Numbers and Covering Spaces</b>	<b>561</b>
21.1. Bi-invariant Forms on Compact Groups	561
21.1a. Bi-invariant $p$ -Forms	561
21.1b. The Cartan $p$ -Forms	562
21.1c. Bi-invariant Riemannian Metrics	563
21.1d. Harmonic Forms in the Bi-invariant Metric	564
21.1e. Weyl and Cartan on the Betti Numbers of $G$	565
21.2. The Fundamental Group and Covering Spaces	567
21.2a. Poincaré’s Fundamental Group $\pi_1(M)$	567
21.2b. The Concept of a Covering Space	569
21.2c. The Universal Covering	570
21.2d. The Orientable Covering	573
21.2e. Lifting Paths	574
21.2f. Subgroups of $\pi_1(M)$	575
21.2g. The Universal Covering Group	575
21.3. The Theorem of S. B. Myers: A Problem Set	576
21.4. The Geometry of a Lie Group	580
21.4a. The Connection of a Bi-invariant Metric	580
21.4b. The Flat Connections	581
<b>22 Chern Forms and Homotopy Groups</b>	<b>583</b>
22.1. Chern Forms and Winding Numbers	583
22.1a. The Yang–Mills “Winding Number”	583
22.1b. Winding Number in Terms of Field Strength	585
22.1c. The Chern Forms for a $U(n)$ Bundle	587
22.2. Homotopies and Extensions	591
22.2a. Homotopy	591
22.2b. Covering Homotopy	592
22.2c. Some Topology of $SU(n)$	594
22.3. The Higher Homotopy Groups $\pi_k(M)$	596
22.3a. $\pi_k(M)$	596
22.3b. Homotopy Groups of Spheres	597
22.3c. Exact Sequences of Groups	598
22.3d. The Homotopy Sequence of a Bundle	600
22.3e. The Relation Between Homotopy and Homology Groups	603
22.4. Some Computations of Homotopy Groups	605
22.4a. Lifting Spheres from $M$ into the Bundle $P$	605
22.4b. $SU(n)$ Again	606
22.4c. The Hopf Map and Fiberings	606

<b>22.5.</b>	Chern Forms as Obstructions	608
<b>22.5a.</b>	The Chern Forms $c_r$ for an $SU(n)$ Bundle Revisited	608
<b>22.5b.</b>	$c_2$ as an “Obstruction Cocycle”	609
<b>22.5c.</b>	The Meaning of the Integer $j(\Delta_4)$	612
<b>22.5d.</b>	Chern’s Integral	612
<b>22.5e.</b>	Concluding Remarks	615
<b>Appendix A.</b>	<b>Forms in Continuum Mechanics</b>	<b>617</b>
<b>A.a.</b>	The Equations of Motion of a Stressed Body	617
<b>A.b.</b>	Stresses are Vector Valued $(n - 1)$ <i>Pseudo</i> -Forms	618
<b>A.c.</b>	The Piola–Kirchhoff Stress Tensors $S$ and $P$	619
<b>A.d.</b>	Strain Energy Rate	620
<b>A.e.</b>	Some Typical Computations Using Forms	622
<b>A.f.</b>	Concluding Remarks	627
<b>Appendix B.</b>	<b>Harmonic Chains and Kirchhoff’s Circuit Laws</b>	<b>628</b>
<b>B.a.</b>	Chain Complexes	628
<b>B.b.</b>	Cochains and Cohomology	630
<b>B.c.</b>	Transpose and Adjoint	631
<b>B.d.</b>	Laplacians and Harmonic Cochains	633
<b>B.e.</b>	Kirchhoff’s Circuit Laws	635
<b>Appendix C.</b>	<b>Symmetries, Quarks, and Meson Masses</b>	<b>640</b>
<b>C.a.</b>	Flavored Quarks	640
<b>C.b.</b>	Interactions of Quarks and Antiquarks	642
<b>C.c.</b>	The Lie Algebra of $SU(3)$	644
<b>C.d.</b>	Pions, Kaons, and Etas	645
<b>C.e.</b>	A Reduced Symmetry Group	648
<b>C.f.</b>	Meson Masses	650
<b>Appendix D.</b>	<b>Representations and Hyperelastic Bodies</b>	<b>652</b>
<b>D.a.</b>	Hyperelastic Bodies	652
<b>D.b.</b>	Isotropic Bodies	653
<b>D.c.</b>	Application of Schur’s Lemma	654
<b>D.d.</b>	Frobenius–Schur Relations	656
<b>D.e.</b>	The Symmetric Traceless $3 \times 3$ Matrices Are Irreducible	658
<b>Appendix E.</b>	<b>Orbits and Morse–Bott Theory in Compact Lie Groups</b>	<b>662</b>
<b>E.a.</b>	The Topology of Conjugacy Orbits	662
<b>E.b.</b>	Application of Bott’s Extension of Morse Theory	665
	<i>References</i>	671
	<i>Index</i>	675