

Contents

| | |
|--|-----------------|
| <i>Preface to the Third Edition</i> | <i>page</i> xix |
| <i>Preface to the Second Edition</i> | xxi |
| <i>Preface to the Revised Printing</i> | xxiii |
| <i>Preface to the First Edition</i> | xxv |
| | |
| Overview. An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor | xxix |
| Introduction | xxix |
| O.a. Introduction | xxix |
| Vectors, 1-Forms, and Tensors | xxx |
| O.b. Two Kinds of Vectors | xxx |
| O.c. Superscripts, Subscripts, Summation Convention | xxxiii |
| O.d. Riemannian Metrics | xxxiv |
| O.e. Tensors | xxxvii |
| Integrals and Exterior Forms | xxxvii |
| O.f. Line Integrals | xxxvii |
| O.g. Exterior 2-Forms | xxxix |
| O.h. Exterior p -Forms and Algebra in \mathbb{R}^n | xl |
| O.i. The Exterior Differential d | xli |
| O.j. The Push-Forward of a Vector and the Pull-Back of a Form | xlii |
| O.k. Surface Integrals and "Stokes' Theorem" | xliv |
| O.l. Electromagnetism, or, Is it a Vector or a Form? | xlv |
| O.m. Interior Products | xlvii |
| O.n. Volume Forms and Cartan's Vector Valued Exterior Forms | xlviii |
| O.o. Magnetic Field for Current in a Straight Wire | l |
| Elasticity and Stresses | li |
| O.p. Cauchy Stress, Floating Bodies, Twisted Cylinders, and Strain Energy | li |
| O.q. Sketch of Cauchy's "First Theorem" | lvii |
| O.r. Sketch of Cauchy's "Second Theorem," Moments as Generators of Rotations | lix |
| O.s. A Remarkable Formula for Differentiating Line, Surface, and . . . , Integrals | lxii |

I Manifolds, Tensors, and Exterior Forms

| | |
|---|-----------|
| 1 Manifolds and Vector Fields | 3 |
| 1.1. Submanifolds of Euclidean Space | 3 |
| 1.1a. Submanifolds of \mathbb{R}^N | 4 |
| 1.1b. The Geometry of Jacobian Matrices: The “Differential” | 7 |
| 1.1c. The Main Theorem on Submanifolds of \mathbb{R}^N | 8 |
| 1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body | 9 |
| 1.2. Manifolds | 11 |
| 1.2a. Some Notions from Point Set Topology | 11 |
| 1.2b. The Idea of a Manifold | 13 |
| 1.2c. A Rigorous Definition of a Manifold | 19 |
| 1.2d. Complex Manifolds: The Riemann Sphere | 21 |
| 1.3. Tangent Vectors and Mappings | 22 |
| 1.3a. Tangent or “Contravariant” Vectors | 23 |
| 1.3b. Vectors as Differential Operators | 24 |
| 1.3c. The Tangent Space to M^n at a Point | 25 |
| 1.3d. Mappings and Submanifolds of Manifolds | 26 |
| 1.3e. Change of Coordinates | 29 |
| 1.4. Vector Fields and Flows | 30 |
| 1.4a. Vector Fields and Flows on \mathbb{R}^n | 30 |
| 1.4b. Vector Fields on Manifolds | 33 |
| 1.4c. Straightening Flows | 34 |
| 2 Tensors and Exterior Forms | 37 |
| 2.1. Covectors and Riemannian Metrics | 37 |
| 2.1a. Linear Functionals and the Dual Space | 37 |
| 2.1b. The Differential of a Function | 40 |
| 2.1c. Scalar Products in Linear Algebra | 42 |
| 2.1d. Riemannian Manifolds and the Gradient Vector | 45 |
| 2.1e. Curves of Steepest Ascent | 46 |
| 2.2. The Tangent Bundle | 48 |
| 2.2a. The Tangent Bundle | 48 |
| 2.2b. The Unit Tangent Bundle | 50 |
| 2.3. The Cotangent Bundle and Phase Space | 52 |
| 2.3a. The Cotangent Bundle | 52 |
| 2.3b. The Pull-Back of a Covector | 52 |
| 2.3c. The Phase Space in Mechanics | 54 |
| 2.3d. The Poincaré 1-Form | 56 |
| 2.4. Tensors | 58 |
| 2.4a. Covariant Tensors | 58 |
| 2.4b. Contravariant Tensors | 59 |
| 2.4c. Mixed Tensors | 60 |
| 2.4d. Transformation Properties of Tensors | 62 |
| 2.4e. Tensor Fields on Manifolds | 63 |

CONTENTS

ix

| | | |
|--------------|--|-----|
| 2.5. | The Grassmann or Exterior Algebra | 66 |
| 2.5a. | The Tensor Product of Covariant Tensors | 66 |
| 2.5b. | The Grassmann or Exterior Algebra | 66 |
| 2.5c. | The Geometric Meaning of Forms in \mathbb{R}^n | 70 |
| 2.5d. | Special Cases of the Exterior Product | 70 |
| 2.5e. | Computations and Vector Analysis | 71 |
| 2.6. | Exterior Differentiation | 73 |
| 2.6a. | The Exterior Differential | 73 |
| 2.6b. | Examples in \mathbb{R}^3 | 75 |
| 2.6c. | A Coordinate Expression for d | 76 |
| 2.7. | Pull-Backs | 77 |
| 2.7a. | The Pull-Back of a Covariant Tensor | 77 |
| 2.7b. | The Pull-Back in Elasticity | 80 |
| 2.8. | Orientation and Pseudoforms | 82 |
| 2.8a. | Orientation of a Vector Space | 82 |
| 2.8b. | Orientation of a Manifold | 83 |
| 2.8c. | Orientability and 2-Sided Hypersurfaces | 84 |
| 2.8d. | Projective Spaces | 85 |
| 2.8e. | Pseudoforms and the Volume Form | 85 |
| 2.8f. | The Volume Form in a Riemannian Manifold | 87 |
| 2.9. | Interior Products and Vector Analysis | 89 |
| 2.9a. | Interior Products and Contractions | 89 |
| 2.9b. | Interior Product in \mathbb{R}^3 | 90 |
| 2.9c. | Vector Analysis in \mathbb{R}^3 | 92 |
| 2.10. | Dictionary | 94 |
| 3 | Integration of Differential Forms | 95 |
| 3.1. | Integration over a Parameterized Subset | 95 |
| 3.1a. | Integration of a p -Form in \mathbb{R}^p | 95 |
| 3.1b. | Integration over Parameterized Subsets | 96 |
| 3.1c. | Line Integrals | 97 |
| 3.1d. | Surface Integrals | 99 |
| 3.1e. | Independence of Parameterization | 101 |
| 3.1f. | Integrals and Pull-Backs | 102 |
| 3.1g. | Concluding Remarks | 102 |
| 3.2. | Integration over Manifolds with Boundary | 104 |
| 3.2a. | Manifolds with Boundary | 105 |
| 3.2b. | Partitions of Unity | 106 |
| 3.2c. | Integration over a Compact Oriented Submanifold | 108 |
| 3.2d. | Partitions and Riemannian Metrics | 109 |
| 3.3. | Stokes's Theorem | 110 |
| 3.3a. | Orienting the Boundary | 110 |
| 3.3b. | Stokes's Theorem | 111 |
| 3.4. | Integration of Pseudoforms | 114 |
| 3.4a. | Integrating Pseudo- n -Forms on an n -Manifold | 115 |
| 3.4b. | Submanifolds with Transverse Orientation | 115 |

| | |
|---|------------|
| 3.4c. Integration over a Submanifold with Transverse Orientation | 116 |
| 3.4d. Stokes's Theorem for Pseudoforms | 117 |
| 3.5. Maxwell's Equations | 118 |
| 3.5a. Charge and Current in Classical Electromagnetism | 118 |
| 3.5b. The Electric and Magnetic Fields | 119 |
| 3.5c. Maxwell's Equations | 120 |
| 3.5d. Forms and Pseudoforms | 122 |
| 4 The Lie Derivative | 125 |
| 4.1. The Lie Derivative of a Vector Field | 125 |
| 4.1a. The Lie Bracket | 125 |
| 4.1b. Jacobi's Variational Equation | 127 |
| 4.1c. The Flow Generated by $[X, Y]$ | 129 |
| 4.2. The Lie Derivative of a Form | 132 |
| 4.2a. Lie Derivatives of Forms | 132 |
| 4.2b. Formulas Involving the Lie Derivative | 134 |
| 4.2c. Vector Analysis Again | 136 |
| 4.3. Differentiation of Integrals | 138 |
| 4.3a. The Autonomous (Time-Independent) Case | 138 |
| 4.3b. Time-Dependent Fields | 140 |
| 4.3c. Differentiating Integrals | 142 |
| 4.4. A Problem Set on Hamiltonian Mechanics | 145 |
| 4.4a. Time-Independent Hamiltonians | 147 |
| 4.4b. Time-Dependent Hamiltonians and Hamilton's Principle | 151 |
| 4.4c. Poisson brackets | 154 |
| 5 The Poincaré Lemma and Potentials | 155 |
| 5.1. A More General Stokes's Theorem | 155 |
| 5.2. Closed Forms and Exact Forms | 156 |
| 5.3. Complex Analysis | 158 |
| 5.4. The Converse to the Poincaré Lemma | 160 |
| 5.5. Finding Potentials | 162 |
| 6 Holonomic and Nonholonomic Constraints | 165 |
| 6.1. The Frobenius Integrability Condition | 165 |
| 6.1a. Planes in \mathbb{R}^3 | 165 |
| 6.1b. Distributions and Vector Fields | 167 |
| 6.1c. Distributions and 1-Forms | 167 |
| 6.1d. The Frobenius Theorem | 169 |
| 6.2. Integrability and Constraints | 172 |
| 6.2a. Foliations and Maximal Leaves | 172 |
| 6.2b. Systems of Mayer–Lie | 174 |
| 6.2c. Holonomic and Nonholonomic Constraints | 175 |

CONTENTS

xi

| | |
|---|------------|
| 6.3. Heuristic Thermodynamics via Caratheodory | 178 |
| 6.3a. Introduction | 178 |
| 6.3b. The First Law of Thermodynamics | 179 |
| 6.3c. Some Elementary Changes of State | 180 |
| 6.3d. The Second Law of Thermodynamics | 181 |
| 6.3e. Entropy | 183 |
| 6.3f. Increasing Entropy | 185 |
| 6.3g. Chow's Theorem on Accessibility | 187 |
| | |
| II Geometry and Topology | |
| 7 \mathbb{R}^3 and Minkowski Space | 191 |
| 7.1. Curvature and Special Relativity | 191 |
| 7.1a. Curvature of a Space Curve in \mathbb{R}^3 | 191 |
| 7.1b. Minkowski Space and Special Relativity | 192 |
| 7.1c. Hamiltonian Formulation | 196 |
| 7.2. Electromagnetism in Minkowski Space | 196 |
| 7.2a. Minkowski's Electromagnetic Field Tensor | 196 |
| 7.2b. Maxwell's Equations | 198 |
| 8 The Geometry of Surfaces in \mathbb{R}^3 | 201 |
| 8.1. The First and Second Fundamental Forms | 201 |
| 8.1a. The First Fundamental Form, or Metric Tensor | 201 |
| 8.1b. The Second Fundamental Form | 203 |
| 8.2. Gaussian and Mean Curvatures | 205 |
| 8.2a. Symmetry and Self-Adjointness | 205 |
| 8.2b. Principal Normal Curvatures | 206 |
| 8.2c. Gauss and Mean Curvatures: The Gauss Normal Map | 207 |
| 8.3. The Brouwer Degree of a Map: A Problem Set | 210 |
| 8.3a. The Brouwer Degree | 210 |
| 8.3b. Complex Analytic (Holomorphic) Maps | 214 |
| 8.3c. The Gauss Normal Map Revisited: The Gauss–Bonnet Theorem | 215 |
| 8.3d. The Kronecker Index of a Vector Field | 215 |
| 8.3e. The Gauss Looping Integral | 218 |
| 8.4. Area, Mean Curvature, and Soap Bubbles | 221 |
| 8.4a. The First Variation of Area | 221 |
| 8.4b. Soap Bubbles and Minimal Surfaces | 226 |
| 8.5. Gauss's <i>Theorema Egregium</i> | 228 |
| 8.5a. The Equations of Gauss and Codazzi | 228 |
| 8.5b. The <i>Theorema Egregium</i> | 230 |
| 8.6. Geodesics | 232 |
| 8.6a. The First Variation of Arc Length | 232 |
| 8.6b. The Intrinsic Derivative and the Geodesic Equation | 234 |
| 8.7. The Parallel Displacement of Levi-Civita | 236 |

| | |
|--|------------|
| 9 Covariant Differentiation and Curvature | 241 |
| 9.1. Covariant Differentiation | 241 |
| 9.1a. Covariant Derivative | 241 |
| 9.1b. Curvature of an Affine Connection | 244 |
| 9.1c. Torsion and Symmetry | 245 |
| 9.2. The Riemannian Connection | 246 |
| 9.3. Cartan's Exterior Covariant Differential | 247 |
| 9.3a. Vector-Valued Forms | 247 |
| 9.3b. The Covariant Differential of a Vector Field | 248 |
| 9.3c. Cartan's Structural Equations | 249 |
| 9.3d. The Exterior Covariant Differential of a Vector-Valued Form | 250 |
| 9.3e. The Curvature 2-Forms | 251 |
| 9.4. Change of Basis and Gauge Transformations | 253 |
| 9.4a. Symmetric Connections Only | 253 |
| 9.4b. Change of Frame | 253 |
| 9.5. The Curvature Forms in a Riemannian Manifold | 255 |
| 9.5a. The Riemannian Connection | 255 |
| 9.5b. Riemannian Surfaces M^2 | 257 |
| 9.5c. An Example | 257 |
| 9.6. Parallel Displacement and Curvature on a Surface | 259 |
| 9.7. Riemann's Theorem and the Horizontal Distribution | 263 |
| 9.7a. Flat metrics | 263 |
| 9.7b. The Horizontal Distribution of an Affine Connection | 263 |
| 9.7c. Riemann's Theorem | 266 |
| 10 Geodesics | 269 |
| 10.1. Geodesics and Jacobi Fields | 269 |
| 10.1a. Vector Fields Along a Surface in M^n | 269 |
| 10.1b. Geodesics | 271 |
| 10.1c. Jacobi Fields | 272 |
| 10.1d. Energy | 274 |
| 10.2. Variational Principles in Mechanics | 275 |
| 10.2a. Hamilton's Principle in the Tangent Bundle | 275 |
| 10.2b. Hamilton's Principle in Phase Space | 277 |
| 10.2c. Jacobi's Principle of "Least" Action | 278 |
| 10.2d. Closed Geodesics and Periodic Motions | 281 |
| 10.3. Geodesics, Spiders, and the Universe | 284 |
| 10.3a. Gaussian Coordinates | 284 |
| 10.3b. Normal Coordinates on a Surface | 287 |
| 10.3c. Spiders and the Universe | 288 |
| 11 Relativity, Tensors, and Curvature | 291 |
| 11.1. Heuristics of Einstein's Theory | 291 |
| 11.1a. The Metric Potentials | 291 |
| 11.1b. Einstein's Field Equations | 293 |
| 11.1c. Remarks on Static Metrics | 296 |

CONTENTS

xiii

| | | |
|---------------|--|------------|
| 11.2. | Tensor Analysis | 298 |
| 11.2a. | Covariant Differentiation of Tensors | 298 |
| 11.2b. | Riemannian Connections and the Bianchi Identities | 299 |
| 11.2c. | Second Covariant Derivatives: The Ricci Identities | 301 |
| 11.3. | Hilbert's Action Principle | 303 |
| 11.3a. | Geodesics in a Pseudo-Riemannian Manifold | 303 |
| 11.3b. | Normal Coordinates, the Divergence and Laplacian | 303 |
| 11.3c. | Hilbert's Variational Approach to General Relativity | 305 |
| 11.4. | The Second Fundamental Form in the Riemannian Case | 309 |
| 11.4a. | The Induced Connection and the Second Fundamental Form | 309 |
| 11.4b. | The Equations of Gauss and Codazzi | 311 |
| 11.4c. | The Interpretation of the Sectional Curvature | 313 |
| 11.4d. | Fixed Points of Isometries | 314 |
| 11.5. | The Geometry of Einstein's Equations | 315 |
| 11.5a. | The Einstein Tensor in a (Pseudo-)Riemannian Space–Time | 315 |
| 11.5b. | The Relativistic Meaning of Gauss's Equation | 316 |
| 11.5c. | The Second Fundamental Form of a Spatial Slice | 318 |
| 11.5d. | The Codazzi Equations | 319 |
| 11.5e. | Some Remarks on the Schwarzschild Solution | 320 |
| 12 | Curvature and Topology: Synge's Theorem | 323 |
| 12.1. | Synge's Formula for Second Variation | 324 |
| 12.1a. | The Second Variation of Arc Length | 324 |
| 12.1b. | Jacobi Fields | 326 |
| 12.2. | Curvature and Simple Connectivity | 329 |
| 12.2a. | Synge's Theorem | 329 |
| 12.2b. | Orientability Revisited | 331 |
| 13 | Betti Numbers and De Rham's Theorem | 333 |
| 13.1. | Singular Chains and Their Boundaries | 333 |
| 13.1a. | Singular Chains | 333 |
| 13.1b. | Some 2-Dimensional Examples | 338 |
| 13.2. | The Singular Homology Groups | 342 |
| 13.2a. | Coefficient Fields | 342 |
| 13.2b. | Finite Simplicial Complexes | 343 |
| 13.2c. | Cycles, Boundaries, Homology and Betti Numbers | 344 |
| 13.3. | Homology Groups of Familiar Manifolds | 347 |
| 13.3a. | Some Computational Tools | 347 |
| 13.3b. | Familiar Examples | 350 |
| 13.4. | De Rham's Theorem | 355 |
| 13.4a. | The Statement of de Rham's Theorem | 355 |
| 13.4b. | Two Examples | 357 |

| | |
|---|------------|
| 14 Harmonic Forms | 361 |
| 14.1. The Hodge Operators | 361 |
| 14.1a. The $*$ Operator | 361 |
| 14.1b. The Codifferential Operator $\delta = d^*$ | 364 |
| 14.1c. Maxwell's Equations in Curved Space-Time M^4 | 366 |
| 14.1d. The Hilbert Lagrangian | 367 |
| 14.2. Harmonic Forms | 368 |
| 14.2a. The Laplace Operator on Forms | 368 |
| 14.2b. The Laplacian of a 1-Form | 369 |
| 14.2c. Harmonic Forms on Closed Manifolds | 370 |
| 14.2d. Harmonic Forms and de Rham's Theorem | 372 |
| 14.2e. Bochner's Theorem | 374 |
| 14.3. Boundary Values, Relative Homology, and Morse Theory | 375 |
| 14.3a. Tangential and Normal Differential Forms | 376 |
| 14.3b. Hodge's Theorem for Tangential Forms | 377 |
| 14.3c. Relative Homology Groups | 379 |
| 14.3d. Hodge's Theorem for Normal Forms | 381 |
| 14.3e. Morse's Theory of Critical Points | 382 |
| III Lie Groups, Bundles, and Chern Forms | |
| 15 Lie Groups | 391 |
| 15.1. Lie Groups, Invariant Vector Fields and Forms | 391 |
| 15.1a. Lie Groups | 391 |
| 15.1b. Invariant Vector Fields and Forms | 395 |
| 15.2. One Parameter Subgroups | 398 |
| 15.3. The Lie Algebra of a Lie Group | 402 |
| 15.3a. The Lie Algebra | 402 |
| 15.3b. The Exponential Map | 403 |
| 15.3c. Examples of Lie Algebras | 404 |
| 15.3d. Do the 1-Parameter Subgroups Cover G ? | 405 |
| 15.4. Subgroups and Subalgebras | 407 |
| 15.4a. Left Invariant Fields Generate Right Translations | 407 |
| 15.4b. Commutators of Matrices | 408 |
| 15.4c. Right Invariant Fields | 409 |
| 15.4d. Subgroups and Subalgebras | 410 |
| 16 Vector Bundles in Geometry and Physics | 413 |
| 16.1. Vector Bundles | 413 |
| 16.1a. Motivation by Two Examples | 413 |
| 16.1b. Vector Bundles | 415 |
| 16.1c. Local Trivializations | 417 |
| 16.1d. The Normal Bundle to a Submanifold | 419 |
| 16.2. Poincaré's Theorem and the Euler Characteristic | 421 |
| 16.2a. Poincaré's Theorem | 422 |
| 16.2b. The Stiefel Vector Field and Euler's Theorem | 426 |

CONTENTS

xv

| | | |
|---------------|--|------------|
| 16.3. | Connections in a Vector Bundle | 428 |
| 16.3a. | Connection in a Vector Bundle | 428 |
| 16.3b. | Complex Vector Spaces | 431 |
| 16.3c. | The Structure Group of a Bundle | 433 |
| 16.3d. | Complex Line Bundles | 433 |
| 16.4. | The Electromagnetic Connection | 435 |
| 16.4a. | Lagrange's Equations Without Electromagnetism | 435 |
| 16.4b. | The Modified Lagrangian and Hamiltonian | 436 |
| 16.4c. | Schrödinger's Equation in an Electromagnetic Field | 439 |
| 16.4d. | Global Potentials | 443 |
| 16.4e. | The Dirac Monopole | 444 |
| 16.4f. | The Aharonov–Bohm Effect | 446 |
| 17 | Fiber Bundles, Gauss–Bonnet, and Topological Quantization | 451 |
| 17.1. | Fiber Bundles and Principal Bundles | 451 |
| 17.1a. | Fiber Bundles | 451 |
| 17.1b. | Principal Bundles and Frame Bundles | 453 |
| 17.1c. | Action of the Structure Group on a Principal Bundle | 454 |
| 17.2. | Coset Spaces | 456 |
| 17.2a. | Cosets | 456 |
| 17.2b. | Grassmann Manifolds | 459 |
| 17.3. | Chern's Proof of the Gauss–Bonnet–Poincaré Theorem | 460 |
| 17.3a. | A Connection in the Frame Bundle of a Surface | 460 |
| 17.3b. | The Gauss–Bonnet–Poincaré Theorem | 462 |
| 17.3c. | Gauss–Bonnet as an Index Theorem | 465 |
| 17.4. | Line Bundles, Topological Quantization, and Berry Phase | 465 |
| 17.4a. | A Generalization of Gauss–Bonnet | 465 |
| 17.4b. | Berry Phase | 468 |
| 17.4c. | Monopoles and the Hopf Bundle | 473 |
| 18 | Connections and Associated Bundles | 475 |
| 18.1. | Forms with Values in a Lie Algebra | 475 |
| 18.1a. | The Maurer–Cartan Form | 475 |
| 18.1b. | g -Valued p -Forms on a Manifold | 477 |
| 18.1c. | Connections in a Principal Bundle | 479 |
| 18.2. | Associated Bundles and Connections | 481 |
| 18.2a. | Associated Bundles | 481 |
| 18.2b. | Connections in Associated Bundles | 483 |
| 18.2c. | The Associated Ad Bundle | 485 |
| 18.3. | r -Form Sections of a Vector Bundle: Curvature | 488 |
| 18.3a. | r -Form sections of E | 488 |
| 18.3b. | Curvature and the Ad Bundle | 489 |
| 19 | The Dirac Equation | 491 |
| 19.1. | The Groups $SO(3)$ and $SU(2)$ | 491 |
| 19.1a. | The Rotation Group $SO(3)$ of \mathbb{R}^3 | 492 |
| 19.1b. | $SU(2)$: The Lie algebra $\mathfrak{su}(2)$ | 493 |

| | |
|--|------------|
| 19.1c. <i>SU(2) is Topologically the 3-Sphere</i> | 495 |
| 19.1d. <i>Ad : SU(2) → SO(3) in More Detail</i> | 496 |
| 19.2. Hamilton, Clifford, and Dirac | 497 |
| 19.2a. Spinors and Rotations of \mathbb{R}^3 | 497 |
| 19.2b. Hamilton on Composing Two Rotations | 499 |
| 19.2c. Clifford Algebras | 500 |
| 19.2d. The Dirac Program: The Square Root of the d'Alembertian | 502 |
| 19.3. The Dirac Algebra | 504 |
| 19.3a. The Lorentz Group | 504 |
| 19.3b. The Dirac Algebra | 509 |
| 19.4. The Dirac Operator \not{d} in Minkowski Space | 511 |
| 19.4a. Dirac Spinors | 511 |
| 19.4b. The Dirac Operator | 513 |
| 19.5. The Dirac Operator in Curved Space–Time | 515 |
| 19.5a. The Spinor Bundle | 515 |
| 19.5b. The Spin Connection in \mathcal{SM} | 518 |
| 20 Yang–Mills Fields | 523 |
| 20.1. Noether's Theorem for Internal Symmetries | 523 |
| 20.1a. The Tensorial Nature of Lagrange's Equations | 523 |
| 20.1b. Boundary Conditions | 526 |
| 20.1c. Noether's Theorem for Internal Symmetries | 527 |
| 20.1d. Noether's Principle | 528 |
| 20.2. Weyl's Gauge Invariance Revisited | 531 |
| 20.2a. The Dirac Lagrangian | 531 |
| 20.2b. Weyl's Gauge Invariance Revisited | 533 |
| 20.2c. The Electromagnetic Lagrangian | 534 |
| 20.2d. Quantization of the A Field: Photons | 536 |
| 20.3. The Yang–Mills Nucleon | 537 |
| 20.3a. The Heisenberg Nucleon | 537 |
| 20.3b. The Yang–Mills Nucleon | 538 |
| 20.3c. A Remark on Terminology | 540 |
| 20.4. Compact Groups and Yang–Mills Action | 541 |
| 20.4a. The Unitary Group Is Compact | 541 |
| 20.4b. Averaging over a Compact Group | 541 |
| 20.4c. Compact Matrix Groups Are Subgroups of Unitary Groups | 542 |
| 20.4d. Ad Invariant Scalar Products in the Lie Algebra of a Compact Group | 543 |
| 20.4e. The Yang–Mills Action | 544 |
| 20.5. The Yang–Mills Equation | 545 |
| 20.5a. The Exterior Covariant Divergence ∇^* | 545 |
| 20.5b. The Yang–Mills Analogy with Electromagnetism | 547 |
| 20.5c. Further Remarks on the Yang–Mills Equations | 548 |

CONTENTS

xvii

| | |
|---|------------|
| 20.6. Yang–Mills Instantons | 550 |
| 20.6a. Instantons | 550 |
| 20.6b. Chern’s Proof Revisited | 553 |
| 20.6c. Instantons and the Vacuum | 557 |
| 21 Betti Numbers and Covering Spaces | 561 |
| 21.1. Bi-invariant Forms on Compact Groups | 561 |
| 21.1a. Bi-invariant p -Forms | 561 |
| 21.1b. The Cartan p -Forms | 562 |
| 21.1c. Bi-invariant Riemannian Metrics | 563 |
| 21.1d. Harmonic Forms in the Bi-invariant Metric | 564 |
| 21.1e. Weyl and Cartan on the Betti Numbers of G | 565 |
| 21.2. The Fundamental Group and Covering Spaces | 567 |
| 21.2a. Poincaré’s Fundamental Group $\pi_1(M)$ | 567 |
| 21.2b. The Concept of a Covering Space | 569 |
| 21.2c. The Universal Covering | 570 |
| 21.2d. The Orientable Covering | 573 |
| 21.2e. Lifting Paths | 574 |
| 21.2f. Subgroups of $\pi_1(M)$ | 575 |
| 21.2g. The Universal Covering Group | 575 |
| 21.3. The Theorem of S. B. Myers: A Problem Set | 576 |
| 21.4. The Geometry of a Lie Group | 580 |
| 21.4a. The Connection of a Bi-invariant Metric | 580 |
| 21.4b. The Flat Connections | 581 |
| 22 Chern Forms and Homotopy Groups | 583 |
| 22.1. Chern Forms and Winding Numbers | 583 |
| 22.1a. The Yang–Mills “Winding Number” | 583 |
| 22.1b. Winding Number in Terms of Field Strength | 585 |
| 22.1c. The Chern Forms for a $U(n)$ Bundle | 587 |
| 22.2. Homotopies and Extensions | 591 |
| 22.2a. Homotopy | 591 |
| 22.2b. Covering Homotopy | 592 |
| 22.2c. Some Topology of $SU(n)$ | 594 |
| 22.3. The Higher Homotopy Groups $\pi_k(M)$ | 596 |
| 22.3a. $\pi_k(M)$ | 596 |
| 22.3b. Homotopy Groups of Spheres | 597 |
| 22.3c. Exact Sequences of Groups | 598 |
| 22.3d. The Homotopy Sequence of a Bundle | 600 |
| 22.3e. The Relation Between Homotopy and Homology Groups | 603 |
| 22.4. Some Computations of Homotopy Groups | 605 |
| 22.4a. Lifting Spheres from M into the Bundle P | 605 |
| 22.4b. $SU(n)$ Again | 606 |
| 22.4c. The Hopf Map and Fibering | 606 |

| | |
|---|------------|
| 22.5. Chern Forms as Obstructions | 608 |
| 22.5a. The Chern Forms c_r for an $SU(n)$ Bundle Revisited | 608 |
| 22.5b. c_2 as an “Obstruction Cocycle” | 609 |
| 22.5c. The Meaning of the Integer $j(\Delta_4)$ | 612 |
| 22.5d. Chern’s Integral | 612 |
| 22.5e. Concluding Remarks | 615 |
| Appendix A. Forms in Continuum Mechanics | 617 |
| A.a. The Equations of Motion of a Stressed Body | 617 |
| A.b. Stresses are Vector Valued $(n - 1)$ Pseudo-Forms | 618 |
| A.c. The Piola–Kirchhoff Stress Tensors S and P | 619 |
| A.d. Strain Energy Rate | 620 |
| A.e. Some Typical Computations Using Forms | 622 |
| A.f. Concluding Remarks | 627 |
| Appendix B. Harmonic Chains and Kirchhoff’s Circuit Laws | 628 |
| B.a. Chain Complexes | 628 |
| B.b. Cochains and Cohomology | 630 |
| B.c. Transpose and Adjoint | 631 |
| B.d. Laplacians and Harmonic Cochains | 633 |
| B.e. Kirchhoff’s Circuit Laws | 635 |
| Appendix C. Symmetries, Quarks, and Meson Masses | 640 |
| C.a. Flavored Quarks | 640 |
| C.b. Interactions of Quarks and Antiquarks | 642 |
| C.c. The Lie Algebra of $SU(3)$ | 644 |
| C.d. Pions, Kaons, and Etas | 645 |
| C.e. A Reduced Symmetry Group | 648 |
| C.f. Meson Masses | 650 |
| Appendix D. Representations and Hyperelastic Bodies | 652 |
| D.a. Hyperelastic Bodies | 652 |
| D.b. Isotropic Bodies | 653 |
| D.c. Application of Schur’s Lemma | 654 |
| D.d. Frobenius–Schur Relations | 656 |
| D.e. The Symmetric Traceless 3×3 Matrices Are Irreducible | 658 |
| Appendix E. Orbits and Morse–Bott Theory in Compact Lie Groups | 662 |
| E.a. The Topology of Conjugacy Orbits | 662 |
| E.b. Application of Bott’s Extension of Morse Theory | 665 |
| <i>References</i> | 671 |
| <i>Index</i> | 675 |