The Geometry of Physics

This book is intended to provide a working knowledge of those parts of exterior differential forms, differential geometry, algebraic and differential topology, Lie groups, vector bundles, and Chern forms that are essential for a deeper understanding of both classical and modern physics and engineering. Included are discussions of analytical and fluid dynamics, electromagnetism (in flat and curved space), thermodynamics, elasticity theory, the geometry and topology of Kirchhoff's electric circuit laws, soap films, special and general relativity, the Dirac operator and spinors, and gauge fields, including Yang–Mills, the Aharonov–Bohm effect, Berry phase, and instanton winding numbers, quarks, and the quark model for mesons. Before a discussion of abstract notions of differential geometry, geometric intuition is developed through a rather extensive introduction to the study of surfaces in ordinary space; consequently, the book should be of interest also to mathematics students.

This book will be useful to graduate and advance undergraduate students of physics, engineering, and mathematics. It can be used as a course text or for self-study.

This Third Edition includes a new overview of Cartan's exterior differential forms. It previews many of the geometric concepts developed in the text and illustrates their applications to a single extended problem in engineering; namely, the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

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The Geometry of Physics

An Introduction

Third Edition

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For
Thom-kat, Mont, Dave
and
Jonnie

and

In fond memory of
Raoul Bott
1923–2005

Photograph of Raoul by Montgomery Frankel

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I Manifolds, Tensors, and Exterior Forms

Preface to the Third Edition

A main addition introduced in this third edition is the inclusion of an Overview

An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor

which can be read before starting the text. This appears at the beginning of the text, before Chapter 1. The only prerequisites for reading this overview are sophomore courses in calculus and basic linear algebra. Many of the geometric concepts developed in the text are previewed here and these are illustrated by their applications to a single extended problem in engineering, namely the study of the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

The new shortened version of Appendix A, dealing with elasticity, requires the discussion of Cauchy stresses dealt with in the Overview. The author believes that the use of Cartan's vector valued exterior forms in elasticity is more suitable (both in principle and in computations) than the classical tensor analysis usually employed in engineering (which is also developed in the text.)

The new version of Appendix A also contains contributions by my engineering colleague Professor Hidenori Murakami, including his treatment of the Truesdell stress rate. I am also very grateful to Professor Murakami for many very helpful conversations.

Preface to the Second Edition

This second edition differs mainly in the addition of three new appendices: C, D, and E. Appendices C and D are applications of the elements of representation theory of compact Lie groups.

Appendix C deals with applications to the flavored quark model that revolutionized particle physics. We illustrate how certain observed mesons (pions, kaons, and etas) are described in terms of quarks and how one can "derive" the mass formula of Gell-Mann/Okubo of 1962. This can be read after Section 20.3b.

Appendix D is concerned with isotropic hyperelastic bodies. Here the main result has been used by engineers since the 1850s. My purpose for presenting proofs is that the hypotheses of the Frobenius–Schur theorems of group representations are exactly met here, and so this affords a compelling excuse for developing representation theory, which had not been addressed in the earlier edition. An added bonus is that the group theoretical material is applied to the three-dimensional rotation group $SO(3)$, where these generalities can be pictured explicitly. This material can essentially be read after Appendix A, but some brief excursion into Appendix C might be helpful.

Appendix E delves deeper into the geometry and topology of compact Lie groups. Bott's extension of the presentation of Morse theory that was given in Section 14.3c is sketched and the example of the topology of the Lie group $U(3)$ is worked out in some detail.

Preface to the Revised Printing

In this reprinting I have introduced a new appendix, Appendix B, Harmonic Chains and Kirchhoff's Circuit Laws. This appendix deals with a finite-dimensional version of Hodge's theory, the subject of Chapter 14, and can be read at any time after Chapter 13. It includes a more geometrical view of cohomology, dealt with entirely by matrices and elementary linear algebra. A bonus of this viewpoint is a systematic "geometrical" description of the Kirchhoff laws and their applications to direct current circuits, first considered from roughly this viewpoint by Hermann Weyl in 1923.

I have corrected a number of errors and misprints, many of which were kindly brought to my attention by Professor Friedrich Heyl.

Finally, I would like to take this opportunity to express my great appreciation to my editor, Dr. Alan Harvey of Cambridge University Press.

Preface to the First Edition

The basic ideas at the foundations of point and continuum mechanics, electromagnetism, thermodynamics, special and general relativity, and gauge theories are geometrical, and, I believe, should be approached, by both mathematics and physics students, from this point of view.

This is a textbook that develops some of the geometrical concepts and tools that are helpful in understanding classical and modern physics and engineering. The mathematical subject material is essentially that found in a first-year graduate course in differential geometry. This is not coincidental, for the founders of this part of geometry, among them Euler, Gauss, Jacobi, Riemann and Poincaré, were also profoundly interested in "natural philosophy."

Electromagnetism and fluid flow involve line, surface, and volume integrals. Analytical dynamics brings in multidimensional versions of these objects. In this book these topics are discussed in terms of **exterior differential forms**. One also needs to differentiate such integrals with respect to time, especially when the domains of integration are changing (circulation, vorticity, helicity, Faraday's law, etc.), and this is accomplished most naturally with aid of the **Lie derivative**. Analytical dynamics, thermodynamics, and robotics in engineering deal with **constraints**, including the puzzling nonholonomic ones, and these are dealt with here via the so-called Frobenius theorem on differential forms. All these matters, and more, are considered in Part One of this book.

Einstein created the astonishing principle **field strength** = **curvature** to explain the gravitational field, but if one is not familiar with the classical meaning of surface curvature in ordinary 3-space this is merely a tautology. Consequently I introduce **differential geometry** before discussing general relativity. **Cartan's** version, in terms of exterior differential forms, plays a central role. Differential geometry has applications to more down-to-earth subjects, such as soap bubbles and periodic motions of dynamical systems. Differential geometry occupies the bulk of Part Two.

Einstein's principle has been extended by physicists, and now all the field strengths occurring in elementary particle physics (which are required in order to construct a

Lagrangian) are discussed in terms of curvature and **connections**, but it is the curvature of a **vector bundle**, that is, the *field* space, that arises, not the curvature of space–time. The symmetries of the quantum field play an essential role in these **gauge theories**, as was first emphasized by Hermann Weyl, and these are understood today in terms of **Lie groups**, which are an essential ingredient of the vector bundle. Since many quantum situations (charged particles in an electromagnetic field, Aharonov–Bohm effect, Dirac monopoles, Berry phase, Yang–Mills fields, instantons, etc.) have analogues in elementary differential geometry, we can use the geometric methods and pictures of Part Two as a guide; a picture *is* worth a thousand words! These topics are discussed in Part Three.

Topology is playing an increasing role in physics. A physical problem is "well posed" if there *exists* a solution and it is *unique*, and the topology of the configuration (spherical, toroidal, etc.), in particular the singular **homology groups**, has an essential influence. The **Brouwer degree**, the **Hurewicz homotopy groups**, and **Morse theory** play roles not only in modern gauge theories but also, for example, in the theory of "defects" in materials.

Topological methods are playing an important role in field theory; versions of the **Atiyah–Singer index theorem** are frequently invoked. Although I do not develop this theorem in general, I *do* discuss at length the most famous and elementary example, the **Gauss–Bonnet–Poincaré** theorem, in two dimensions and also the meaning of the **Chern characteristic classes**. These matters are discussed in Parts Two and Three.

The Appendix to this book presents a nontraditional treatment of the **stress tensors** appearing in continuum mechanics, utilizing exterior forms. In this endeavor I am greatly indebted to my engineering colleague Hidenori Murakami. In particular Murakami has supplied, in Section **g** of the Appendix, some typical computations involving stresses and strains, but carried out with the machinery developed in this book. We believe that these computations indicate the efficiency of the use of forms and Lie derivatives in elasticity. The material of this Appendix could be read, except for some minor points, after Section **9.5**.

Mathematical applications to physics occur in at least two aspects. Mathematics is of course the principal tool for solving technical analytical problems, but increasingly it is also a principal guide in our understanding of the basic structure and concepts involved. Analytical computations with elliptic functions *are* important for certain technical problems in rigid body dynamics, but one could not have begun to understand the dynamics before Euler's introducing the moment of inertia tensor. I am very much concerned with the basic concepts in physics. A glance at the Contents will show in detail what mathematical and physical tools are being developed, but frequently physical applications appear also in Exercises. My main philosophy has been to attack physical topics as soon as possible, but only after effective mathematical tools have been introduced. By analogy, one *can* deal with problems of velocity and acceleration after having learned the definition of the derivative as the limit of a quotient (or even before, as in the case of Newton), but we all know how important the *machinery* of calculus (e.g., the power, product, quotient, and chain rules) is for handling specific problems. In the same way, it is a mistake to talk seriously about thermodynamics before understanding that a total differential equation in more than two dimensions need not possess an integrating factor.

In a sense this book is a "final" revision of sets of notes for a year course that I have given in La Jolla over many years. My goal has been to give the reader a *working* knowledge of the tools that are of great value in geometry and physics and (increasingly) engineering. For this it is *absolutely essential* that the reader work (or at least attempt) the Exercises. *Most of the problems are simple and require simple calculations. If you find calculations becoming unmanageable, then in all probability you are not taking advantage of the machinery developed in this book.*

This book is intended primarily for two audiences, first, the physics or engineering student, and second, the mathematics student. My classes in the past have been populated mostly by first-, second-, and third-year graduate students in physics, but there have also been mathematics students and undergraduates. The only real *mathematical* prerequisites are *basic* linear algebra and some familiarity with calculus of several variables. Most students (in the United States) have these by the beginning of the third undergraduate year.

All of the physical subjects, with two exceptions to be noted, are preceded by a brief introduction. The two exceptions are analytical dynamics and the quantum aspects of gauge theories.

Analytical (Hamiltonian) dynamics appears as a problem set in Part One, with very little motivation, for the following reason: the problems form an ideal application of exterior forms and Lie derivatives and involve no knowledge of physics. Only in Part Two, after geodesics have been discussed, do we return for a discussion of analytical dynamics from first principles. (Of course most physics and engineering students will already have seen *some* introduction to analytical mechanics in their course work anyway.) The significance of the Lagrangian (based on special relativity) is discussed in Section **16.4** of Part Three when changes in dynamics are required for discussing the effects of electromagnetism.

An introduction to quantum mechanics would have taken us too far afield. Fortunately (for me) only the simplest quantum ideas are needed for most of our discussions. I would refer the reader to Rabin's article [R] and Sudbery's book [Su] for excellent introductions to the quantum aspects involved.

Physics and engineering readers would profit *greatly* if they would form the habit of translating the vectorial and tensorial statements found in their customary reading of physics articles and books into the language developed in this book, and using the newer methods developed here in their own thinking. (By "newer" I mean methods developed over the last one hundred years!)

As for the mathematics student, I feel that this book gives an overview of a large portion of differential geometry and topology that should be helpful to the mathematics graduate student in this age of very specialized texts and absolute rigor. The student preparing to specialize, say, in differential geometry *will* need to augment this reading with a more rigorous treatment of some of the subjects than that given here (e.g., in Warner's book [Wa] or the five-volume series by Spivak [Sp]). The mathematics student should also have exercises devoted to showing what can go wrong if hypotheses are weakened. I make no pretense of worrying, for example, about the differentiability classes of mappings needed in proofs. (Such matters are studied more carefully in the book [A, M, R] and in the encyclopedia article [T, T]. This latter article (and the accompanying one by Eriksen) are also excellent for questions of historical priorities.) I hope that mathematics students will enjoy the discussions of the physical subjects even if they know very little physics; after all, physics is *the* source of interesting vector fields. Many of the "physical" applications are useful even if they are thought of as simply giving explicit examples of rather abstract concepts. For example, Dirac's equation in *curved space* can be considered as a nontrivial application of the method of connections in associated bundles!

This *is* an introduction and there is much important mathematics that is not developed here. Analytical questions involving existence theorems in partial differential equations, Sobolev spaces, and so on, are missing. Although complex manifolds are defined, there is no discussion of Kaehler manifolds nor the algebraic–geometric notions used in string theory. Infinite dimensional manifolds are not considered. On the physical side, topics are introduced usually only if I felt that geometrical ideas would be a great help in their understanding or in computations.

I have included a small list of references. Most of the articles and books listed have been referred to in this book for specific details. The reader will find that there are many good books on the subject of "geometrical physics" that are not referred to here, primarily because I felt that the development, or sophistication, or notation used was sufficiently different to lead to, perhaps, more confusion than help in the first stages of their struggle. A book that I feel is in very much the same spirit as my own is that by Nash and Sen [N, S]. The standard reference for differential geometry is the two-volume work [K, N] of Kobayashi and Nomizu.

Almost every section of this book begins with a question or a quotation which may concern anything from the main thrust of the section to some small remark that should not be overlooked.

A term being defined will usually appear in **bold type**.

I wish to express my gratitude to Harley Flanders, who introduced me long ago to exterior forms and de Rham's theorem, whose superb book [Fl] was perhaps the first to awaken scientists to the use of exterior forms in their work. I am indebted to my chemical colleague John Wheeler for conversations on thermodynamics and to Donald Fredkin for helpful criticisms of earlier versions of my lecture notes. I have already expressed my deep gratitude to Hidenori Murakami. Joel Broida made many comments on earlier versions, and also prevented my Macintosh from taking me over. I've had many helpful conversations with Bruce Driver, Jay Fillmore, and Michael Freedman. Poul Hjorth made many helpful comments on various drafts and also served as "beater," herding physics students into my course. Above all, my colleague Jeff Rabin used my notes as the text in a one-year graduate course and made many suggestions and corrections. I have also included corrections to the 1997 printing, following helpful remarks from Professor Meinhard Mayer.

Finally I am grateful to the many students in my classes on geometrical physics for their encouragement and enthusiasm in my endeavor. Of course none of the above is responsible for whatever inaccuracies undoubtedly remain.

OVERVIEW

An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor

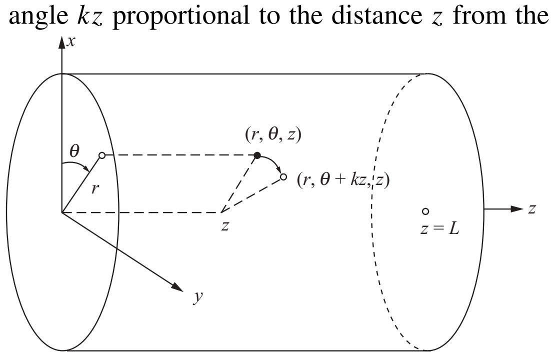
Introduction

0.a. Introduction

My goal in this overview is to introduce **exterior calculus** in a *brief* and *informal* way that leads directly to their use in engineering and physics, both in basic physical concepts and in specific engineering calculations. The presentation will be very informal. Many times a proof will be omitted so that we can get quickly to a calculation. In some "proofs" we shall look only at a typical term.

The chief mathematical prerequisites for this overview are sophomore courses dealing with basic linear algebra, partial derivatives, multiple integrals, and tangent vectors to parameterized curves, but not necessarily "vector calculus," i.e., curls, divergences, line and surface integrals, Stokes' theorem, ... These last topics will be sketched here using Cartan's "exterior calculus."

We shall take advantage of the fact that most engineers live in euclidean 3-space $\mathbb{R}^3$ with its everyday metric structure, but we shall try to use methods that make sense in much more general situations. Instead of including exercises we shall consider, in the section **Elasticity and Stresses**, one main example and illustrate *everything* in terms of this example but hopefully the general principles will be clear. This engineering example will be the following. Take an elastic circular cylindrical rod of radius a and length L, described in cylindrical coordinates r, θ, z, with the ends of the cylinder at $z = 0$ and $z = L$. Look at this same cylinder except that it has been axially twisted through an angle kz proportional to the distance z from the fixed end $z = 0$.



We shall *neglect gravity* and investigate the **stresses** in the cylinder in its final twisted state, in the first approximation, i.e., where we put $k^2 = 0$. Since "stress" and "strain" are "tensors" (as Cauchy and I will show) this is classically treated via "tensor analysis." The final equilibrium state involves surface integrals and the tensor divergence of the Cauchy stress tensor. Our main tool will *not* be the usual *classical* tensor analysis (Christoffel symbols Γ^i_{jk} . . . , etc.) but rather **exterior differential forms** (first used in the nineteenth century by Grassmann, Poincaré, Volterra, . . . , and developed especially by **Elie Cartan**), which, I believe, is a far more appropriate tool.

We are very much at home with cartesian coordinates but curvilinear coordinates play a very important role in physical applications, and the fact that there are *two distinct types of vectors* that arise in curvilinear coordinates (and, even more so, in *curved spaces*) that appear identical in cartesian coordinates *must* be understood, not only when making calculations but also in our understanding of the basic ingredients of the physical world. We shall let x^i, and u^i, $i = 1, 2, 3$, be **general** (curvilinear) coordinates, in euclidean 3 dimensional space $\mathbb{R}^3$. *If cartesian coordinates are wanted, I will say so explicitly*.

Vectors, 1-Forms, and Tensors

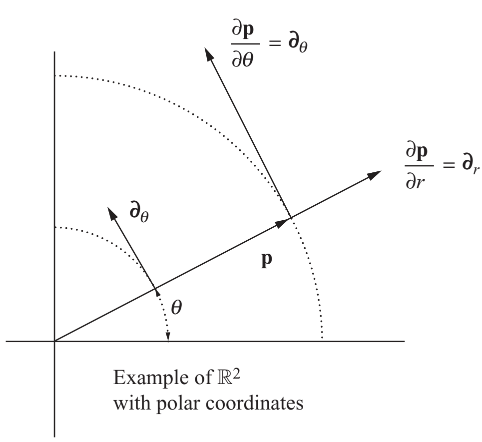
0.b. Two Kinds of Vectors

There are two kinds of vectors that appear in physical applications and it is important that we distinguish between them. First there is the familiar "arrow" version.

Consider n dimensional euclidean space $\mathbb{R}^n$ with cartesian coordinates $x^1, \ldots, x^n$ and local (perhaps curvilinear) coordinates $u^1, \ldots, u^n$.

Example: $\mathbb{R}^2$ with cartesian coordinates $x^1 = x$, $x^2 = y$, and with polar coordinates $u^1 = r$, $u^2 = \theta$.

Example: $\mathbb{R}^3$ with cartesian coordinates x, y, z and with cylindrical coordinates R, Θ, Z.



Let $\mathbf{p}$ be the position vector from the origin of $\mathbb{R}^n$ to the point p. In the curvilinear coordinate system u, the coordinate curve C_i through the point p is the curve where all