

The Geometry of Physics

This book is intended to provide a working knowledge of those parts of exterior differential forms, differential geometry, algebraic and differential topology, Lie groups, vector bundles, and Chern forms that are essential for a deeper understanding of both classical and modern physics and engineering. Included are discussions of analytical and fluid dynamics, electromagnetism (in flat and curved space), thermodynamics, elasticity theory, the geometry and topology of Kirchhoff's electric circuit laws, soap films, special and general relativity, the Dirac operator and spinors, and gauge fields, including Yang–Mills, the Aharonov–Bohm effect, Berry phase, and instanton winding numbers, quarks, and the quark model for mesons. Before a discussion of abstract notions of differential geometry, geometric intuition is developed through a rather extensive introduction to the study of surfaces in ordinary space; consequently, the book should be of interest also to mathematics students.

This book will be useful to graduate and advance undergraduate students of physics, engineering, and mathematics. It can be used as a course text or for self-study.

This Third Edition includes a new overview of Cartan's exterior differential forms. It previews many of the geometric concepts developed in the text and illustrates their applications to a single extended problem in engineering; namely, the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

THEODORE FRANKEL received his Ph.D. from the University of California, Berkeley. He is currently Emeritus Professor of Mathematics at the University of California, San Diego.

The Geometry of Physics
An Introduction
Third Edition

Theodore Frankel

University of California, San Diego



CAMBRIDGE
UNIVERSITY PRESS



Shaftesbury Road, Cambridge CB2 8EA, United Kingdom
One Liberty Plaza, 20th Floor, New York, NY 10006, USA
477 Williamstown Road, Port Melbourne, VIC 3207, Australia
314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India
103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

We share the University's mission to contribute to society through the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org

Information on this title: www.cambridge.org/9781107602601

© Cambridge University Press & Assessment 1997, 2004, 2012

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press & Assessment.

First published 1997

Revised paperback edition 1999

Second edition 2004

Reprinted 2006, 2007 (twice), 2009

Third edition 2012

Reprinted 2017

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data

Frankel, Theodore, 1929–

The geometry of physics : an introduction / Theodore Frankel. – 3rd ed.

p. cm.

Includes bibliographical references and index.

ISBN 978-1-107-60260-1 (pbk.)

1. Geometry, Differential. 2. Mathematical physics. I. Title.

QC20.7.D52F73 2011

530.15'636 – dc23 2011027890

ISBN 978-1-107-60260-1 Paperback

Cambridge University Press & Assessment has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

*For
Thom-kat, Mont, Dave
and
Jonnie*

and

*In fond memory of
Raoul Bott
1923–2005*



Photograph of Raoul by Montgomery Frankel

Contents

<i>Preface to the Third Edition</i>	page xix
<i>Preface to the Second Edition</i>	xxi
<i>Preface to the Revised Printing</i>	xxiii
<i>Preface to the First Edition</i>	xxv
Overview. An Informal Overview of Cartan’s Exterior Differential Forms, Illustrated with an Application to Cauchy’s Stress Tensor	xxix
Introduction	xxix
0.a. Introduction	xxix
Vectors, 1-Forms, and Tensors	xxx
0.b. Two Kinds of Vectors	xxx
0.c. Superscripts, Subscripts, Summation Convention	xxxiii
0.d. Riemannian Metrics	xxxiv
0.e. Tensors	xxxvii
Integrals and Exterior Forms	xxxvii
0.f. Line Integrals	xxxvii
0.g. Exterior 2-Forms	xxxix
0.h. Exterior p -Forms and Algebra in \mathbb{R}^n	xl
0.i. The Exterior Differential d	xli
0.j. The Push-Forward of a Vector and the Pull-Back of a Form	xlii
0.k. Surface Integrals and “Stokes’ Theorem”	xliv
0.l. Electromagnetism, or, Is it a Vector or a Form?	xlvi
0.m. Interior Products	xlvii
0.n. Volume Forms and Cartan’s Vector Valued Exterior Forms	xlviii
0.o. Magnetic Field for Current in a Straight Wire	1
Elasticity and Stresses	li
0.p. Cauchy Stress, Floating Bodies, Twisted Cylinders, and Strain Energy	li
0.q. Sketch of Cauchy’s “First Theorem”	lvii
0.r. Sketch of Cauchy’s “Second Theorem,” Moments as Generators of Rotations	lix
0.s. A Remarkable Formula for Differentiating Line, Surface, and . . . , Integrals	lxi

I Manifolds, Tensors, and Exterior Forms	
1 Manifolds and Vector Fields	3
1.1. Submanifolds of Euclidean Space	3
1.1a. Submanifolds of \mathbb{R}^N	4
1.1b. The Geometry of Jacobian Matrices: The “Differential”	7
1.1c. The Main Theorem on Submanifolds of \mathbb{R}^N	8
1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body	9
1.2. Manifolds	11
1.2a. Some Notions from Point Set Topology	11
1.2b. The Idea of a Manifold	13
1.2c. A Rigorous Definition of a Manifold	19
1.2d. Complex Manifolds: The Riemann Sphere	21
1.3. Tangent Vectors and Mappings	22
1.3a. Tangent or “Contravariant” Vectors	23
1.3b. Vectors as Differential Operators	24
1.3c. The Tangent Space to M^n at a Point	25
1.3d. Mappings and Submanifolds of Manifolds	26
1.3e. Change of Coordinates	29
1.4. Vector Fields and Flows	30
1.4a. Vector Fields and Flows on \mathbb{R}^n	30
1.4b. Vector Fields on Manifolds	33
1.4c. Straightening Flows	34
2 Tensors and Exterior Forms	37
2.1. Covectors and Riemannian Metrics	37
2.1a. Linear Functionals and the Dual Space	37
2.1b. The Differential of a Function	40
2.1c. Scalar Products in Linear Algebra	42
2.1d. Riemannian Manifolds and the Gradient Vector	45
2.1e. Curves of Steepest Ascent	46
2.2. The Tangent Bundle	48
2.2a. The Tangent Bundle	48
2.2b. The Unit Tangent Bundle	50
2.3. The Cotangent Bundle and Phase Space	52
2.3a. The Cotangent Bundle	52
2.3b. The Pull-Back of a Covector	52
2.3c. The Phase Space in Mechanics	54
2.3d. The Poincaré 1-Form	56
2.4. Tensors	58
2.4a. Covariant Tensors	58
2.4b. Contravariant Tensors	59
2.4c. Mixed Tensors	60
2.4d. Transformation Properties of Tensors	62
2.4e. Tensor Fields on Manifolds	63

	CONTENTS	ix
2.5.	The Grassmann or Exterior Algebra	66
2.5a.	The Tensor Product of Covariant Tensors	66
2.5b.	The Grassmann or Exterior Algebra	66
2.5c.	The Geometric Meaning of Forms in \mathbb{R}^n	70
2.5d.	Special Cases of the Exterior Product	70
2.5e.	Computations and Vector Analysis	71
2.6.	Exterior Differentiation	73
2.6a.	The Exterior Differential	73
2.6b.	Examples in \mathbb{R}^3	75
2.6c.	A Coordinate Expression for d	76
2.7.	Pull-Backs	77
2.7a.	The Pull-Back of a Covariant Tensor	77
2.7b.	The Pull-Back in Elasticity	80
2.8.	Orientation and Pseudoforms	82
2.8a.	Orientation of a Vector Space	82
2.8b.	Orientation of a Manifold	83
2.8c.	Orientability and 2-Sided Hypersurfaces	84
2.8d.	Projective Spaces	85
2.8e.	Pseudoforms and the Volume Form	85
2.8f.	The Volume Form in a Riemannian Manifold	87
2.9.	Interior Products and Vector Analysis	89
2.9a.	Interior Products and Contractions	89
2.9b.	Interior Product in \mathbb{R}^3	90
2.9c.	Vector Analysis in \mathbb{R}^3	92
2.10.	Dictionary	94
3	Integration of Differential Forms	95
3.1.	Integration over a Parameterized Subset	95
3.1a.	Integration of a p -Form in \mathbb{R}^p	95
3.1b.	Integration over Parameterized Subsets	96
3.1c.	Line Integrals	97
3.1d.	Surface Integrals	99
3.1e.	Independence of Parameterization	101
3.1f.	Integrals and Pull-Backs	102
3.1g.	Concluding Remarks	102
3.2.	Integration over Manifolds with Boundary	104
3.2a.	Manifolds with Boundary	105
3.2b.	Partitions of Unity	106
3.2c.	Integration over a Compact Oriented Submanifold	108
3.2d.	Partitions and Riemannian Metrics	109
3.3.	Stokes's Theorem	110
3.3a.	Orienting the Boundary	110
3.3b.	Stokes's Theorem	111
3.4.	Integration of Pseudoforms	114
3.4a.	Integrating Pseudo- n -Forms on an n -Manifold	115
3.4b.	Submanifolds with Transverse Orientation	115

3.4c.	Integration over a Submanifold with Transverse Orientation	116
3.4d.	Stokes's Theorem for Pseudoforms	117
3.5.	Maxwell's Equations	118
3.5a.	Charge and Current in Classical Electromagnetism	118
3.5b.	The Electric and Magnetic Fields	119
3.5c.	Maxwell's Equations	120
3.5d.	Forms and Pseudoforms	122
4	The Lie Derivative	125
4.1.	The Lie Derivative of a Vector Field	125
4.1a.	The Lie Bracket	125
4.1b.	Jacobi's Variational Equation	127
4.1c.	The Flow Generated by $[X, Y]$	129
4.2.	The Lie Derivative of a Form	132
4.2a.	Lie Derivatives of Forms	132
4.2b.	Formulas Involving the Lie Derivative	134
4.2c.	Vector Analysis Again	136
4.3.	Differentiation of Integrals	138
4.3a.	The Autonomous (Time-Independent) Case	138
4.3b.	Time-Dependent Fields	140
4.3c.	Differentiating Integrals	142
4.4.	A Problem Set on Hamiltonian Mechanics	145
4.4a.	Time-Independent Hamiltonians	147
4.4b.	Time-Dependent Hamiltonians and Hamilton's Principle	151
4.4c.	Poisson brackets	154
5	The Poincaré Lemma and Potentials	155
5.1.	A More General Stokes's Theorem	155
5.2.	Closed Forms and Exact Forms	156
5.3.	Complex Analysis	158
5.4.	The Converse to the Poincaré Lemma	160
5.5.	Finding Potentials	162
6	Holonomic and Nonholonomic Constraints	165
6.1.	The Frobenius Integrability Condition	165
6.1a.	Planes in \mathbb{R}^3	165
6.1b.	Distributions and Vector Fields	167
6.1c.	Distributions and 1-Forms	167
6.1d.	The Frobenius Theorem	169
6.2.	Integrability and Constraints	172
6.2a.	Foliations and Maximal Leaves	172
6.2b.	Systems of Mayer–Lie	174
6.2c.	Holonomic and Nonholonomic Constraints	175

CONTENTS

xi

6.3.	Heuristic Thermodynamics via Caratheodory	178
6.3a.	Introduction	178
6.3b.	The First Law of Thermodynamics	179
6.3c.	Some Elementary Changes of State	180
6.3d.	The Second Law of Thermodynamics	181
6.3e.	Entropy	183
6.3f.	Increasing Entropy	185
6.3g.	Chow's Theorem on Accessibility	187

II Geometry and Topology

7	\mathbb{R}^3 and Minkowski Space	191
7.1.	Curvature and Special Relativity	191
7.1a.	Curvature of a Space Curve in \mathbb{R}^3	191
7.1b.	Minkowski Space and Special Relativity	192
7.1c.	Hamiltonian Formulation	196
7.2.	Electromagnetism in Minkowski Space	196
7.2a.	Minkowski's Electromagnetic Field Tensor	196
7.2b.	Maxwell's Equations	198
8	The Geometry of Surfaces in \mathbb{R}^3	201
8.1.	The First and Second Fundamental Forms	201
8.1a.	The First Fundamental Form, or Metric Tensor	201
8.1b.	The Second Fundamental Form	203
8.2.	Gaussian and Mean Curvatures	205
8.2a.	Symmetry and Self-Adjointness	205
8.2b.	Principal Normal Curvatures	206
8.2c.	Gauss and Mean Curvatures: The Gauss Normal Map	207
8.3.	The Brouwer Degree of a Map: A Problem Set	210
8.3a.	The Brouwer Degree	210
8.3b.	Complex Analytic (Holomorphic) Maps	214
8.3c.	The Gauss Normal Map Revisited: The Gauss–Bonnet Theorem	215
8.3d.	The Kronecker Index of a Vector Field	215
8.3e.	The Gauss Looping Integral	218
8.4.	Area, Mean Curvature, and Soap Bubbles	221
8.4a.	The First Variation of Area	221
8.4b.	Soap Bubbles and Minimal Surfaces	226
8.5.	Gauss's <i>Theorema Egregium</i>	228
8.5a.	The Equations of Gauss and Codazzi	228
8.5b.	The <i>Theorema Egregium</i>	230
8.6.	Geodesics	232
8.6a.	The First Variation of Arc Length	232
8.6b.	The Intrinsic Derivative and the Geodesic Equation	234
8.7.	The Parallel Displacement of Levi-Civita	236

9 Covariant Differentiation and Curvature	241
9.1. Covariant Differentiation	241
9.1a. Covariant Derivative	241
9.1b. Curvature of an Affine Connection	244
9.1c. Torsion and Symmetry	245
9.2. The Riemannian Connection	246
9.3. Cartan's Exterior Covariant Differential	247
9.3a. Vector-Valued Forms	247
9.3b. The Covariant Differential of a Vector Field	248
9.3c. Cartan's Structural Equations	249
9.3d. The Exterior Covariant Differential of a Vector-Valued Form	250
9.3e. The Curvature 2-Forms	251
9.4. Change of Basis and Gauge Transformations	253
9.4a. Symmetric Connections Only	253
9.4b. Change of Frame	253
9.5. The Curvature Forms in a Riemannian Manifold	255
9.5a. The Riemannian Connection	255
9.5b. Riemannian Surfaces M^2	257
9.5c. An Example	257
9.6. Parallel Displacement and Curvature on a Surface	259
9.7. Riemann's Theorem and the Horizontal Distribution	263
9.7a. Flat metrics	263
9.7b. The Horizontal Distribution of an Affine Connection	263
9.7c. Riemann's Theorem	266
10 Geodesics	269
10.1. Geodesics and Jacobi Fields	269
10.1a. Vector Fields Along a Surface in M^n	269
10.1b. Geodesics	271
10.1c. Jacobi Fields	272
10.1d. Energy	274
10.2. Variational Principles in Mechanics	275
10.2a. Hamilton's Principle in the Tangent Bundle	275
10.2b. Hamilton's Principle in Phase Space	277
10.2c. Jacobi's Principle of "Least" Action	278
10.2d. Closed Geodesics and Periodic Motions	281
10.3. Geodesics, Spiders, and the Universe	284
10.3a. Gaussian Coordinates	284
10.3b. Normal Coordinates on a Surface	287
10.3c. Spiders and the Universe	288
11 Relativity, Tensors, and Curvature	291
11.1. Heuristics of Einstein's Theory	291
11.1a. The Metric Potentials	291
11.1b. Einstein's Field Equations	293
11.1c. Remarks on Static Metrics	296

CONTENTS		xiii
11.2.	Tensor Analysis	298
11.2a.	Covariant Differentiation of Tensors	298
11.2b.	Riemannian Connections and the Bianchi Identities	299
11.2c.	Second Covariant Derivatives: The Ricci Identities	301
11.3.	Hilbert's Action Principle	303
11.3a.	Geodesics in a Pseudo-Riemannian Manifold	303
11.3b.	Normal Coordinates, the Divergence and Laplacian	303
11.3c.	Hilbert's Variational Approach to General Relativity	305
11.4.	The Second Fundamental Form in the Riemannian Case	309
11.4a.	The Induced Connection and the Second Fundamental Form	309
11.4b.	The Equations of Gauss and Codazzi	311
11.4c.	The Interpretation of the Sectional Curvature	313
11.4d.	Fixed Points of Isometries	314
11.5.	The Geometry of Einstein's Equations	315
11.5a.	The Einstein Tensor in a (Pseudo-)Riemannian Space-Time	315
11.5b.	The Relativistic Meaning of Gauss's Equation	316
11.5c.	The Second Fundamental Form of a Spatial Slice	318
11.5d.	The Codazzi Equations	319
11.5e.	Some Remarks on the Schwarzschild Solution	320
12	Curvature and Topology: Synge's Theorem	323
12.1.	Synge's Formula for Second Variation	324
12.1a.	The Second Variation of Arc Length	324
12.1b.	Jacobi Fields	326
12.2.	Curvature and Simple Connectivity	329
12.2a.	Synge's Theorem	329
12.2b.	Orientability Revisited	331
13	Betti Numbers and De Rham's Theorem	333
13.1.	Singular Chains and Their Boundaries	333
13.1a.	Singular Chains	333
13.1b.	Some 2-Dimensional Examples	338
13.2.	The Singular Homology Groups	342
13.2a.	Coefficient Fields	342
13.2b.	Finite Simplicial Complexes	343
13.2c.	Cycles, Boundaries, Homology and Betti Numbers	344
13.3.	Homology Groups of Familiar Manifolds	347
13.3a.	Some Computational Tools	347
13.3b.	Familiar Examples	350
13.4.	De Rham's Theorem	355
13.4a.	The Statement of de Rham's Theorem	355
13.4b.	Two Examples	357

14 Harmonic Forms	361
14.1. The Hodge Operators	361
14.1a. The $*$ Operator	361
14.1b. The Codifferential Operator $\delta = d^*$	364
14.1c. Maxwell's Equations in Curved Space–Time M^4	366
14.1d. The Hilbert Lagrangian	367
14.2. Harmonic Forms	368
14.2a. The Laplace Operator on Forms	368
14.2b. The Laplacian of a 1-Form	369
14.2c. Harmonic Forms on Closed Manifolds	370
14.2d. Harmonic Forms and de Rham's Theorem	372
14.2e. Bochner's Theorem	374
14.3. Boundary Values, Relative Homology, and Morse Theory	375
14.3a. Tangential and Normal Differential Forms	376
14.3b. Hodge's Theorem for Tangential Forms	377
14.3c. Relative Homology Groups	379
14.3d. Hodge's Theorem for Normal Forms	381
14.3e. Morse's Theory of Critical Points	382
III Lie Groups, Bundles, and Chern Forms	
15 Lie Groups	391
15.1. Lie Groups, Invariant Vector Fields and Forms	391
15.1a. Lie Groups	391
15.1b. Invariant Vector Fields and Forms	395
15.2. One Parameter Subgroups	398
15.3. The Lie Algebra of a Lie Group	402
15.3a. The Lie Algebra	402
15.3b. The Exponential Map	403
15.3c. Examples of Lie Algebras	404
15.3d. Do the 1-Parameter Subgroups Cover G ?	405
15.4. Subgroups and Subalgebras	407
15.4a. Left Invariant Fields Generate Right Translations	407
15.4b. Commutators of Matrices	408
15.4c. Right Invariant Fields	409
15.4d. Subgroups and Subalgebras	410
16 Vector Bundles in Geometry and Physics	413
16.1. Vector Bundles	413
16.1a. Motivation by Two Examples	413
16.1b. Vector Bundles	415
16.1c. Local Trivializations	417
16.1d. The Normal Bundle to a Submanifold	419
16.2. Poincaré's Theorem and the Euler Characteristic	421
16.2a. Poincaré's Theorem	422
16.2b. The Stiefel Vector Field and Euler's Theorem	426

CONTENTS

xv

16.3.	Connections in a Vector Bundle	428
16.3a.	Connection in a Vector Bundle	428
16.3b.	Complex Vector Spaces	431
16.3c.	The Structure Group of a Bundle	433
16.3d.	Complex Line Bundles	433
16.4.	The Electromagnetic Connection	435
16.4a.	Lagrange's Equations Without Electromagnetism	435
16.4b.	The Modified Lagrangian and Hamiltonian	436
16.4c.	Schrödinger's Equation in an Electromagnetic Field	439
16.4d.	Global Potentials	443
16.4e.	The Dirac Monopole	444
16.4f.	The Aharonov–Bohm Effect	446
17	Fiber Bundles, Gauss–Bonnet, and Topological Quantization	451
17.1.	Fiber Bundles and Principal Bundles	451
17.1a.	Fiber Bundles	451
17.1b.	Principal Bundles and Frame Bundles	453
17.1c.	Action of the Structure Group on a Principal Bundle	454
17.2.	Coset Spaces	456
17.2a.	Cosets	456
17.2b.	Grassmann Manifolds	459
17.3.	Chern's Proof of the Gauss–Bonnet–Poincaré Theorem	460
17.3a.	A Connection in the Frame Bundle of a Surface	460
17.3b.	The Gauss–Bonnet–Poincaré Theorem	462
17.3c.	Gauss–Bonnet as an Index Theorem	465
17.4.	Line Bundles, Topological Quantization, and Berry Phase	465
17.4a.	A Generalization of Gauss–Bonnet	465
17.4b.	Berry Phase	468
17.4c.	Monopoles and the Hopf Bundle	473
18	Connections and Associated Bundles	475
18.1.	Forms with Values in a Lie Algebra	475
18.1a.	The Maurer–Cartan Form	475
18.1b.	\mathfrak{g} -Valued p -Forms on a Manifold	477
18.1c.	Connections in a Principal Bundle	479
18.2.	Associated Bundles and Connections	481
18.2a.	Associated Bundles	481
18.2b.	Connections in Associated Bundles	483
18.2c.	The Associated Ad Bundle	485
18.3.	r -Form Sections of a Vector Bundle: Curvature	488
18.3a.	r -Form sections of E	488
18.3b.	Curvature and the Ad Bundle	489
19	The Dirac Equation	491
19.1.	The Groups $SO(3)$ and $SU(2)$	491
19.1a.	The Rotation Group $SO(3)$ of \mathbb{R}^3	492
19.1b.	$SU(2)$: The Lie algebra $\mathfrak{su}(2)$	493

19.1c.	$SU(2)$ is Topologically the 3-Sphere	495
19.1d.	$Ad : SU(2) \rightarrow SO(3)$ in More Detail	496
19.2.	Hamilton, Clifford, and Dirac	497
19.2a.	Spinors and Rotations of \mathbb{R}^3	497
19.2b.	Hamilton on Composing Two Rotations	499
19.2c.	Clifford Algebras	500
19.2d.	The Dirac Program: The Square Root of the d'Alembertian	502
19.3.	The Dirac Algebra	504
19.3a.	The Lorentz Group	504
19.3b.	The Dirac Algebra	509
19.4.	The Dirac Operator $\not{\partial}$ in Minkowski Space	511
19.4a.	Dirac Spinors	511
19.4b.	The Dirac Operator	513
19.5.	The Dirac Operator in Curved Space–Time	515
19.5a.	The Spinor Bundle	515
19.5b.	The Spin Connection in \mathcal{SM}	518
20	Yang–Mills Fields	523
20.1.	Noether's Theorem for Internal Symmetries	523
20.1a.	The Tensorial Nature of Lagrange's Equations	523
20.1b.	Boundary Conditions	526
20.1c.	Noether's Theorem for Internal Symmetries	527
20.1d.	Noether's Principle	528
20.2.	Weyl's Gauge Invariance Revisited	531
20.2a.	The Dirac Lagrangian	531
20.2b.	Weyl's Gauge Invariance Revisited	533
20.2c.	The Electromagnetic Lagrangian	534
20.2d.	Quantization of the A Field: Photons	536
20.3.	The Yang–Mills Nucleon	537
20.3a.	The Heisenberg Nucleon	537
20.3b.	The Yang–Mills Nucleon	538
20.3c.	A Remark on Terminology	540
20.4.	Compact Groups and Yang–Mills Action	541
20.4a.	The Unitary Group Is Compact	541
20.4b.	Averaging over a Compact Group	541
20.4c.	Compact Matrix Groups Are Subgroups of Unitary Groups	542
20.4d.	Ad Invariant Scalar Products in the Lie Algebra of a Compact Group	543
20.4e.	The Yang–Mills Action	544
20.5.	The Yang–Mills Equation	545
20.5a.	The Exterior Covariant Divergence ∇^*	545
20.5b.	The Yang–Mills Analogy with Electromagnetism	547
20.5c.	Further Remarks on the Yang–Mills Equations	548

CONTENTS	xvii
20.6. Yang–Mills Instantons	550
20.6a. Instantons	550
20.6b. Chern’s Proof Revisited	553
20.6c. Instantons and the Vacuum	557
21 Betti Numbers and Covering Spaces	561
21.1. Bi-invariant Forms on Compact Groups	561
21.1a. Bi-invariant p -Forms	561
21.1b. The Cartan p -Forms	562
21.1c. Bi-invariant Riemannian Metrics	563
21.1d. Harmonic Forms in the Bi-invariant Metric	564
21.1e. Weyl and Cartan on the Betti Numbers of G	565
21.2. The Fundamental Group and Covering Spaces	567
21.2a. Poincaré’s Fundamental Group $\pi_1(M)$	567
21.2b. The Concept of a Covering Space	569
21.2c. The Universal Covering	570
21.2d. The Orientable Covering	573
21.2e. Lifting Paths	574
21.2f. Subgroups of $\pi_1(M)$	575
21.2g. The Universal Covering Group	575
21.3. The Theorem of S. B. Myers: A Problem Set	576
21.4. The Geometry of a Lie Group	580
21.4a. The Connection of a Bi-invariant Metric	580
21.4b. The Flat Connections	581
22 Chern Forms and Homotopy Groups	583
22.1. Chern Forms and Winding Numbers	583
22.1a. The Yang–Mills “Winding Number”	583
22.1b. Winding Number in Terms of Field Strength	585
22.1c. The Chern Forms for a $U(n)$ Bundle	587
22.2. Homotopies and Extensions	591
22.2a. Homotopy	591
22.2b. Covering Homotopy	592
22.2c. Some Topology of $SU(n)$	594
22.3. The Higher Homotopy Groups $\pi_k(M)$	596
22.3a. $\pi_k(M)$	596
22.3b. Homotopy Groups of Spheres	597
22.3c. Exact Sequences of Groups	598
22.3d. The Homotopy Sequence of a Bundle	600
22.3e. The Relation Between Homotopy and Homology Groups	603
22.4. Some Computations of Homotopy Groups	605
22.4a. Lifting Spheres from M into the Bundle P	605
22.4b. $SU(n)$ Again	606
22.4c. The Hopf Map and Fiberings	606

22.5.	Chern Forms as Obstructions	608
22.5a.	The Chern Forms c_r for an $SU(n)$ Bundle Revisited	608
22.5b.	c_2 as an “Obstruction Cocycle”	609
22.5c.	The Meaning of the Integer $j(\Delta_4)$	612
22.5d.	Chern’s Integral	612
22.5e.	Concluding Remarks	615
Appendix A.	Forms in Continuum Mechanics	617
A.a.	The Equations of Motion of a Stressed Body	617
A.b.	Stresses are Vector Valued $(n - 1)$ <i>Pseudo</i> -Forms	618
A.c.	The Piola–Kirchhoff Stress Tensors S and P	619
A.d.	Strain Energy Rate	620
A.e.	Some Typical Computations Using Forms	622
A.f.	Concluding Remarks	627
Appendix B.	Harmonic Chains and Kirchhoff’s Circuit Laws	628
B.a.	Chain Complexes	628
B.b.	Cochains and Cohomology	630
B.c.	Transpose and Adjoint	631
B.d.	Laplacians and Harmonic Cochains	633
B.e.	Kirchhoff’s Circuit Laws	635
Appendix C.	Symmetries, Quarks, and Meson Masses	640
C.a.	Flavored Quarks	640
C.b.	Interactions of Quarks and Antiquarks	642
C.c.	The Lie Algebra of $SU(3)$	644
C.d.	Pions, Kaons, and Etas	645
C.e.	A Reduced Symmetry Group	648
C.f.	Meson Masses	650
Appendix D.	Representations and Hyperelastic Bodies	652
D.a.	Hyperelastic Bodies	652
D.b.	Isotropic Bodies	653
D.c.	Application of Schur’s Lemma	654
D.d.	Frobenius–Schur Relations	656
D.e.	The Symmetric Traceless 3×3 Matrices Are Irreducible	658
Appendix E.	Orbits and Morse–Bott Theory in Compact Lie Groups	662
E.a.	The Topology of Conjugacy Orbits	662
E.b.	Application of Bott’s Extension of Morse Theory	665
	<i>References</i>	671
	<i>Index</i>	675

Preface to the Third Edition

A main addition introduced in this third edition is the inclusion of an Overview

An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor

which can be read before starting the text. This appears at the beginning of the text, before Chapter 1. The only prerequisites for reading this overview are sophomore courses in calculus and basic linear algebra. Many of the geometric concepts developed in the text are previewed here and these are illustrated by their applications to a single extended problem in engineering, namely the study of the Cauchy stresses created by a small twist of an elastic cylindrical rod about its axis.

The new shortened version of Appendix A, dealing with elasticity, requires the discussion of Cauchy stresses dealt with in the Overview. The author believes that the use of Cartan's vector valued exterior forms in elasticity is more suitable (both in principle and in computations) than the classical tensor analysis usually employed in engineering (which is also developed in the text.)

The new version of Appendix A also contains contributions by my engineering colleague Professor Hidenori Murakami, including his treatment of the Truesdell stress rate. I am also very grateful to Professor Murakami for many very helpful conversations.

Preface to the Second Edition

This second edition differs mainly in the addition of three new appendices: C, D, and E. Appendices C and D are applications of the elements of representation theory of compact Lie groups.

Appendix C deals with applications to the flavored quark model that revolutionized particle physics. We illustrate how certain observed mesons (pions, kaons, and etas) are described in terms of quarks and how one can “derive” the mass formula of Gell-Mann/Okubo of 1962. This can be read after Section 20.3b.

Appendix D is concerned with isotropic hyperelastic bodies. Here the main result has been used by engineers since the 1850s. My purpose for presenting proofs is that the hypotheses of the Frobenius–Schur theorems of group representations are exactly met here, and so this affords a compelling excuse for developing representation theory, which had not been addressed in the earlier edition. An added bonus is that the group theoretical material is applied to the three-dimensional rotation group $SO(3)$, where these generalities can be pictured explicitly. This material can essentially be read after Appendix A, but some brief excursion into Appendix C might be helpful.

Appendix E delves deeper into the geometry and topology of compact Lie groups. Bott’s extension of the presentation of Morse theory that was given in Section 14.3c is sketched and the example of the topology of the Lie group $U(3)$ is worked out in some detail.

Preface to the Revised Printing

In this reprinting I have introduced a new appendix, Appendix B, Harmonic Chains and Kirchhoff's Circuit Laws. This appendix deals with a finite-dimensional version of Hodge's theory, the subject of Chapter 14, and can be read at any time after Chapter 13. It includes a more geometrical view of cohomology, dealt with entirely by matrices and elementary linear algebra. A bonus of this viewpoint is a systematic "geometrical" description of the Kirchhoff laws and their applications to direct current circuits, first considered from roughly this viewpoint by Hermann Weyl in 1923.

I have corrected a number of errors and misprints, many of which were kindly brought to my attention by Professor Friedrich Heyl.

Finally, I would like to take this opportunity to express my great appreciation to my editor, Dr. Alan Harvey of Cambridge University Press.

Preface to the First Edition

The basic ideas at the foundations of point and continuum mechanics, electromagnetism, thermodynamics, special and general relativity, and gauge theories are geometrical, and, I believe, should be approached, by both mathematics and physics students, from this point of view.

This is a textbook that develops some of the geometrical concepts and tools that are helpful in understanding classical and modern physics and engineering. The mathematical subject material is essentially that found in a first-year graduate course in differential geometry. This is not coincidental, for the founders of this part of geometry, among them Euler, Gauss, Jacobi, Riemann and Poincaré, were also profoundly interested in “natural philosophy.”

Electromagnetism and fluid flow involve line, surface, and volume integrals. Analytical dynamics brings in multidimensional versions of these objects. In this book these topics are discussed in terms of **exterior differential forms**. One also needs to differentiate such integrals with respect to time, especially when the domains of integration are changing (circulation, vorticity, helicity, Faraday’s law, etc.), and this is accomplished most naturally with aid of the **Lie derivative**. Analytical dynamics, thermodynamics, and robotics in engineering deal with **constraints**, including the puzzling nonholonomic ones, and these are dealt with here via the so-called Frobenius theorem on differential forms. All these matters, and more, are considered in Part One of this book.

Einstein created the astonishing principle **field strength = curvature** to explain the gravitational field, but if one is not familiar with the classical meaning of surface curvature in ordinary 3-space this is merely a tautology. Consequently I introduce **differential geometry** before discussing general relativity. **Cartan’s** version, in terms of exterior differential forms, plays a central role. Differential geometry has applications to more down-to-earth subjects, such as soap bubbles and periodic motions of dynamical systems. Differential geometry occupies the bulk of Part Two.

Einstein’s principle has been extended by physicists, and now all the field strengths occurring in elementary particle physics (which are required in order to construct a

Lagrangian) are discussed in terms of curvature and **connections**, but it is the curvature of a **vector bundle**, that is, the *field* space, that arises, not the curvature of space–time. The symmetries of the quantum field play an essential role in these **gauge theories**, as was first emphasized by Hermann Weyl, and these are understood today in terms of **Lie groups**, which are an essential ingredient of the vector bundle. Since many quantum situations (charged particles in an electromagnetic field, Aharonov–Bohm effect, Dirac monopoles, Berry phase, Yang–Mills fields, instantons, etc.) have analogues in elementary differential geometry, we can use the geometric methods and pictures of Part Two as a guide; a picture *is* worth a thousand words! These topics are discussed in Part Three.

Topology is playing an increasing role in physics. A physical problem is “well posed” if there *exists* a solution and it is *unique*, and the topology of the configuration (spherical, toroidal, etc.), in particular the singular **homology groups**, has an essential influence. The **Brouwer degree**, the **Hurewicz homotopy groups**, and **Morse theory** play roles not only in modern gauge theories but also, for example, in the theory of “defects” in materials.

Topological methods are playing an important role in field theory; versions of the **Atiyah–Singer index theorem** are frequently invoked. Although I do not develop this theorem in general, I *do* discuss at length the most famous and elementary example, the **Gauss–Bonnet–Poincaré** theorem, in two dimensions and also the meaning of the **Chern characteristic classes**. These matters are discussed in Parts Two and Three.

The Appendix to this book presents a nontraditional treatment of the **stress tensors** appearing in continuum mechanics, utilizing exterior forms. In this endeavor I am greatly indebted to my engineering colleague Hidenori Murakami. In particular Murakami has supplied, in Section **g** of the Appendix, some typical computations involving stresses and strains, but carried out with the machinery developed in this book. We believe that these computations indicate the efficiency of the use of forms and Lie derivatives in elasticity. The material of this Appendix could be read, except for some minor points, after Section **9.5**.

Mathematical applications to physics occur in at least two aspects. Mathematics is of course the principal tool for solving technical analytical problems, but increasingly it is also a principal guide in our understanding of the basic structure and concepts involved. Analytical computations with elliptic functions *are* important for certain technical problems in rigid body dynamics, but one could not have begun to understand the dynamics before Euler’s introducing the moment of inertia tensor. I am very much concerned with the basic concepts in physics. A glance at the Contents will show in detail what mathematical and physical tools are being developed, but frequently physical applications appear also in Exercises. My main philosophy has been to attack physical topics as soon as possible, but only after effective mathematical tools have been introduced. By analogy, one *can* deal with problems of velocity and acceleration after having learned the definition of the derivative as the limit of a quotient (or even before, as in the case of Newton), but we all know how important the *machinery* of calculus (e.g., the power, product, quotient, and chain rules) is for handling specific problems. In the same way, it is a mistake to talk seriously about thermodynamics

before understanding that a total differential equation in more than two dimensions need not possess an integrating factor.

In a sense this book is a “final” revision of sets of notes for a year course that I have given in La Jolla over many years. My goal has been to give the reader a *working* knowledge of the tools that are of great value in geometry and physics and (increasingly) engineering. For this it is *absolutely essential* that the reader work (or at least attempt) the Exercises. *Most of the problems are simple and require simple calculations. If you find calculations becoming unmanageable, then in all probability you are not taking advantage of the machinery developed in this book.*

This book is intended primarily for two audiences, first, the physics or engineering student, and second, the mathematics student. My classes in the past have been populated mostly by first-, second-, and third-year graduate students in physics, but there have also been mathematics students and undergraduates. The only real *mathematical* prerequisites are *basic* linear algebra and some familiarity with calculus of several variables. Most students (in the United States) have these by the beginning of the third undergraduate year.

All of the physical subjects, with two exceptions to be noted, are preceded by a brief introduction. The two exceptions are analytical dynamics and the quantum aspects of gauge theories.

Analytical (Hamiltonian) dynamics appears as a problem set in Part One, with very little motivation, for the following reason: the problems form an ideal application of exterior forms and Lie derivatives and involve no knowledge of physics. Only in Part Two, after geodesics have been discussed, do we return for a discussion of analytical dynamics from first principles. (Of course most physics and engineering students will already have seen *some* introduction to analytical mechanics in their course work anyway.) The significance of the Lagrangian (based on special relativity) is discussed in Section 16.4 of Part Three when changes in dynamics are required for discussing the effects of electromagnetism.

An introduction to quantum mechanics would have taken us too far afield. Fortunately (for me) only the simplest quantum ideas are needed for most of our discussions. I would refer the reader to Rabin’s article [R] and Sudbery’s book [Su] for excellent introductions to the quantum aspects involved.

Physics and engineering readers would profit *greatly* if they would form the habit of translating the vectorial and tensorial statements found in their customary reading of physics articles and books into the language developed in this book, and using the newer methods developed here in their own thinking. (By “newer” I mean methods developed over the last one hundred years!)

As for the mathematics student, I feel that this book gives an overview of a large portion of differential geometry and topology that should be helpful to the mathematics graduate student in this age of very specialized texts and absolute rigor. The student preparing to specialize, say, in differential geometry *will* need to augment this reading with a more rigorous treatment of some of the subjects than that given here (e.g., in Warner’s book [Wa] or the five-volume series by Spivak [Sp]). The mathematics student should also have exercises devoted to showing what can go wrong if hypotheses are weakened. I make no pretense of worrying, for example, about the differentiability

classes of mappings needed in proofs. (Such matters are studied more carefully in the book [A, M, R] and in the encyclopedia article [T, T]. This latter article (and the accompanying one by Eriksen) are also excellent for questions of historical priorities.) I hope that mathematics students will enjoy the discussions of the physical subjects even if they know very little physics; after all, physics is *the* source of interesting vector fields. Many of the “physical” applications are useful even if they are thought of as simply giving explicit examples of rather abstract concepts. For example, Dirac’s equation in *curved space* can be considered as a nontrivial application of the method of connections in associated bundles!

This *is* an introduction and there is much important mathematics that is not developed here. Analytical questions involving existence theorems in partial differential equations, Sobolev spaces, and so on, are missing. Although complex manifolds are defined, there is no discussion of Kaehler manifolds nor the algebraic–geometric notions used in string theory. Infinite dimensional manifolds are not considered. On the physical side, topics are introduced usually only if I felt that geometrical ideas would be a great help in their understanding or in computations.

I have included a small list of references. Most of the articles and books listed have been referred to in this book for specific details. The reader will find that there are many good books on the subject of “geometrical physics” that are not referred to here, primarily because I felt that the development, or sophistication, or notation used was sufficiently different to lead to, perhaps, more confusion than help in the first stages of their struggle. A book that I feel is in very much the same spirit as my own is that by Nash and Sen [N, S]. The standard reference for differential geometry is the two-volume work [K, N] of Kobayashi and Nomizu.

Almost every section of this book begins with a question or a quotation which may concern anything from the main thrust of the section to some small remark that should not be overlooked.

A term being defined will usually appear in **bold type**.

I wish to express my gratitude to Harley Flanders, who introduced me long ago to exterior forms and de Rham’s theorem, whose superb book [FI] was perhaps the first to awaken scientists to the use of exterior forms in their work. I am indebted to my chemical colleague John Wheeler for conversations on thermodynamics and to Donald Fredkin for helpful criticisms of earlier versions of my lecture notes. I have already expressed my deep gratitude to Hidenori Murakami. Joel Broida made many comments on earlier versions, and also prevented my Macintosh from taking me over. I’ve had many helpful conversations with Bruce Driver, Jay Fillmore, and Michael Freedman. Poul Hjorth made many helpful comments on various drafts and also served as “beater,” herding physics students into my course. Above all, my colleague Jeff Rabin used my notes as the text in a one-year graduate course and made many suggestions and corrections. I have also included corrections to the 1997 printing, following helpful remarks from Professor Meinhard Mayer.

Finally I am grateful to the many students in my classes on geometrical physics for their encouragement and enthusiasm in my endeavor. Of course none of the above is responsible for whatever inaccuracies undoubtedly remain.

OVERVIEW

An Informal Overview of Cartan's Exterior Differential Forms, Illustrated with an Application to Cauchy's Stress Tensor

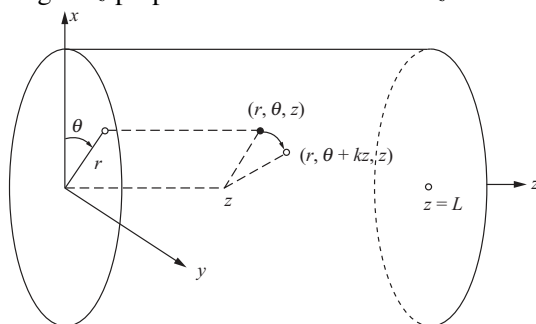
Introduction

0.a. Introduction

My goal in this overview is to introduce **exterior calculus** in a *brief* and *informal* way that leads directly to their use in engineering and physics, both in basic physical concepts and in specific engineering calculations. The presentation will be very informal. Many times a proof will be omitted so that we can get quickly to a calculation. In some “proofs” we shall look only at a typical term.

The chief mathematical prerequisites for this overview are sophomore courses dealing with basic linear algebra, partial derivatives, multiple integrals, and tangent vectors to parameterized curves, but not necessarily “vector calculus,” i.e., curls, divergences, line and surface integrals, Stokes’ theorem, These last topics will be sketched here using Cartan’s “exterior calculus.”

We shall take advantage of the fact that most engineers live in euclidean 3-space \mathbb{R}^3 with its everyday metric structure, but we shall try to use methods that make sense in much more general situations. Instead of including exercises we shall consider, in the section **Elasticity and Stresses**, one main example and illustrate *everything* in terms of this example but hopefully the general principles will be clear. This engineering example will be the following. Take an elastic circular cylindrical rod of radius a and length L , described in cylindrical coordinates r, θ, z , with the ends of the cylinder at $z = 0$ and $z = L$. Look at this same cylinder except that it has been axially twisted through an angle kz proportional to the distance z from the fixed end $z = 0$.



We shall *neglect gravity* and investigate the **stresses** in the cylinder in its final twisted state, in the first approximation, i.e., where we put $k^2 = 0$. Since “stress” and “strain” are “tensors” (as Cauchy and I will show) this is classically treated via “tensor analysis.” The final equilibrium state involves surface integrals and the tensor divergence of the Cauchy stress tensor. Our main tool will *not* be the usual *classical* tensor analysis (Christoffel symbols $\Gamma_{jk}^i \dots$, etc.) but rather **exterior differential forms** (first used in the nineteenth century by Grassmann, Poincaré, Volterra, . . . , and developed especially by **Elie Cartan**), which, I believe, is a far more appropriate tool.

We are very much at home with cartesian coordinates but curvilinear coordinates play a very important role in physical applications, and the fact that there are *two distinct types of vectors* that arise in curvilinear coordinates (and, even more so, in *curved spaces*) that appear identical in cartesian coordinates *must* be understood, not only when making calculations but also in our understanding of the basic ingredients of the physical world. We shall let x^i , and u^i , $i = 1, 2, 3$, be **general** (curvilinear) coordinates, in euclidean 3 dimensional space \mathbb{R}^3 . *If cartesian coordinates are wanted, I will say so explicitly.*

Vectors, 1-Forms, and Tensors

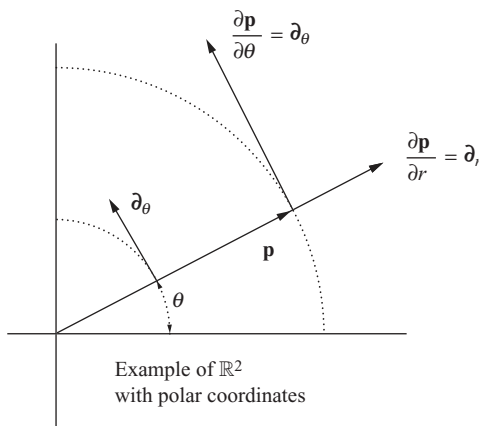
0.b. Two Kinds of Vectors

There are two kinds of vectors that appear in physical applications and it is important that we distinguish between them. First there is the familiar “arrow” version.

Consider n dimensional euclidean space \mathbb{R}^n with cartesian coordinates x^1, \dots, x^n and local (perhaps curvilinear) coordinates u^1, \dots, u^n .

Example: \mathbb{R}^2 with cartesian coordinates $x^1 = x$, $x^2 = y$, and with polar coordinates $u^1 = r$, $u^2 = \theta$.

Example: \mathbb{R}^3 with cartesian coordinates x, y, z and with cylindrical coordinates R, Θ, Z .



Let \mathbf{p} be the position vector from the origin of \mathbb{R}^n to the point p . In the curvilinear coordinate system u , the coordinate curve C_i through the point p is the curve where all