PART ONE

Manifolds, Tensors, and Exterior Forms

CHAPTER 1

Manifolds and Vector Fields

Better is the end of a thing than the beginning thereof.

Ecclesiastes 7:8

As students we learn differential and integral calculus in the context of euclidean space $\mathbb{R}^n$, but it is necessary to apply calculus to problems involving "curved" spaces. Geodesy and cartography, for example, are devoted to the study of the most familiar curved surface of all, the surface of planet Earth. In discussing maps of the Earth, latitude and longitude serve as "coordinates," allowing us to use calculus by considering functions on the Earth's surface (temperature, height above sea level, etc.) as being functions of latitude and longitude. The familiar Mercator's projection, with its stretching of the polar regions, vividly informs us that these coordinates are badly behaved at the poles: that is, that they are not defined everywhere; they are not "global." (We shall refer to such coordinates as being "local," even though they might cover a huge portion of the surface. Precise definitions will be given in Section 1.2.) Of course we may use two sets of "polar" projections to study the Arctic and Antarctic regions. With these three maps we can study the entire surface, provided we know how to relate the Mercator to the polar maps.

We shall soon define a "manifold" to be a space that, like the surface of the Earth, can be covered by a family of local coordinate systems. *A manifold will turn out to be the most general space in which one can use differential and integral calculus with roughly the same facility as in euclidean space*. It should be recalled, though, that calculus in $\mathbb{R}^3$ demands special care when curvilinear coordinates are required.

The most familiar manifold is N-dimensional euclidean space $\mathbb{R}^N$, that is, the space of ordered N tuples $(x^1, \dots, x^N)$ of real numbers. Before discussing manifolds in general we shall talk about the more familiar (and less abstract) concept of a submanifold of $\mathbb{R}^N$, generalizing the notions of curve and surface in $\mathbb{R}^3$.

1.1. Submanifolds of Euclidean Space

What is the configuration space of a rigid body fixed at one point of $\mathbb{R}^n$?

1.1a. Submanifolds of $\mathbb{R}^N$

Euclidean space, $\mathbb{R}^N$, is endowed with a global coordinate system $(x^1, \dots, x^N)$ and is the most important example of a manifold.

In our familiar $\mathbb{R}^3$, with coordinates (x, y, z), a locus $z = F(x, y)$ describes a (2-dimensional) surface, whereas a locus of the form $y = G(x)$, $z = H(x)$, describes a (1-dimensional) curve. We shall need to consider higher-dimensional versions of these important notions.

A subset $M = M^n \subset \mathbb{R}^{n+r}$ is said to be an n-dimensional **submanifold** of $\mathbb{R}^{n+r}$, if *locally* M can be described by giving r of the coordinates differentiably in terms of the n remaining ones. This means that given $p \in M$, a neighborhood of p on M can be described in *some* coordinate system $(x, y) = (x^1, \dots, x^n, y^1, \dots, y^r)$ of $\mathbb{R}^{n+r}$ by r differentiable functions

$$y^\alpha = f^\alpha(x^1, \dots, x^n), \qquad \alpha = 1, \dots r$$

We abbreviate this by $y = f(x)$, or even $y = y(x)$. We say that $x^1, \dots, x^n$ are **local (curvilinear) coordinates** for M near p.

Examples:

(i) $y^1 = f(x^1, \dots, x^n)$ describes an n-dimensional submanifold of $\mathbb{R}^{n+1}$.

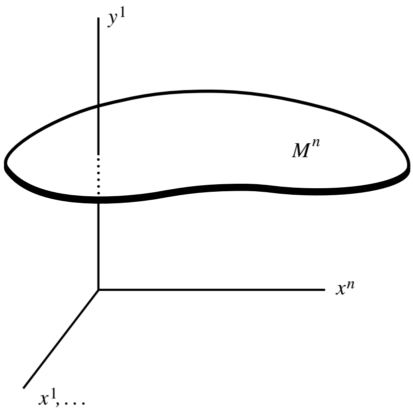

Figure 1.1

In Figure 1.1 we have drawn a portion of the submanifold M. This M is the **graph** of a function $f : \mathbb{R}^n \to \mathbb{R}$, that is, $M = \{(\mathbf{x}, y) \in \mathbb{R}^{n+1} \mid y = f(\mathbf{x})\}$. When $n = 1$, M is a curve; while if $n = 2$, it is a surface.

(ii) The *unit sphere* $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$. Points in the northern hemisphere can be described by $z = F(x, y) = (1 - x^2 - y^2)^{1/2}$ and this function is differentiable everywhere except at the equator $x^2 + y^2 = 1$. Thus x and y are local coordinates for the northern hemisphere except at the equator. For points on the equator one can solve for x or y in terms of the others. If we have solved for x then y and z are the two local coordinates. For points in the southern hemisphere one can use the negative square root for z. The unit sphere in $\mathbb{R}^3$ is a 2-dimensional submanifold of $\mathbb{R}^3$. We note that we have *not* been able to describe the *entire* sphere by expressing one of the coordinates, say z, in terms of the two remaining ones, $z = F(x, y)$. We settle for local coordinates.

More generally, given r functions $F^\alpha(x_1, \ldots, x_n, y_1, \ldots, y_r)$ of $n + r$ variables, we may consider the locus $M^n \subset \mathbb{R}^{n+r}$ defined by the equations

$$F^\alpha(x, y) = c^\alpha, \qquad (c^1, \ldots, c^r) \text{ constants}$$

If the **Jacobian** determinant

$$\left[\frac{\partial(F^1, \ldots, F^r)}{\partial(y^1, \ldots, y^r)}\right](x_0, y_0)$$

at $(x_0, y_0) \in M$ of the locus is not 0, the **implicit function theorem** assures us that locally, near (x_0, y_0), we may solve $F^\alpha(x, y) = c^\alpha$, $\alpha = 1, \ldots, r$, for the y's in terms of the x's

$$y^\alpha = f^\alpha(x^1, \ldots, x^n)$$

We may say that "a portion of M^n near (x_0, y_0) is a submanifold of $\mathbb{R}^{n+r}$." If the Jacobian $\neq 0$ *at all points of the locus*, then the entire M^n is a submanifold.

Recall that the Jacobian condition arises as follows. If $F^\alpha(x, y) = c^\alpha$ can be solved for the y's differentiably in terms of the x's, $y^\beta = y^\beta(x)$, then if, for fixed i, we differentiate the identity $F^\alpha(x, y(x)) = c^\alpha$ with respect to x^i, we get

$$\frac{\partial F^\alpha}{\partial x^i} + \sum_\beta \left[\frac{\partial F^\alpha}{\partial y^\beta}\right]\frac{\partial y^\beta}{\partial x^i} = 0$$

and

$$\frac{\partial y^\beta}{\partial x^i} = -\sum_\alpha \left(\left[\frac{\partial F}{\partial y}\right]^{-1}\right)^\beta{}_\alpha \left[\frac{\partial F^\alpha}{\partial x^i}\right]$$

provided the subdeterminant $\partial(F^1, \ldots, F^r)/\partial(y^1, \ldots, y^r)$ is not zero. (Here $([\partial F/\partial y]^{-1})^\beta{}_\alpha$ is the $\beta\alpha$ entry of the inverse to the matrix $\partial F/\partial y$; we shall use the convention that for matrix indices, the index to the *left* always is the *row* index, whether it is up or down.) This *suggests* that if the indicated Jacobian is nonzero then we might indeed be able to solve for the y's in terms of the x's, and the implicit function theorem confirms this. The (nontrivial) *proof* of the implicit function theorem can be found in most books on real analysis.

Still more generally, suppose that we have r functions of $n+r$ variables, $F^\alpha(x^1, \ldots, x^{n+r})$. Consider the locus $F^\alpha(x) = c^\alpha$. Suppose that at each point x_0 of the locus the Jacobian *matrix*

$$\left(\frac{\partial F^\alpha}{\partial x^i}\right) \qquad \alpha = 1, \ldots, r \qquad i = 1, \ldots, n + r$$

has rank r. Then the equations $F^\alpha = c^\alpha$ define an n-dimensional submanifold of $\mathbb{R}^{n+r}$, since we may locally solve for r of the coordinates in terms of the remaining n.

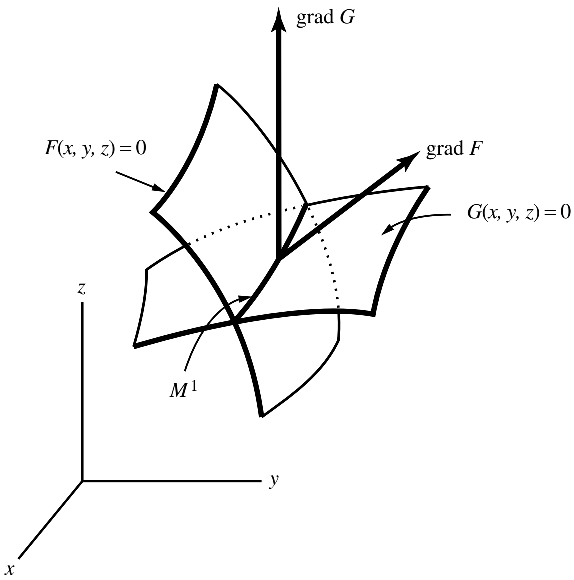

Figure 1.2

In Figure 1.2, two surfaces $F = 0$ and $G = 0$ in $\mathbb{R}^3$ intersect to yield a curve M.

The simplest case is *one* function F of N variables $(x^1, \dots, x^N)$. If *at each point of the locus* $F = c$ there is always *at least one partial derivative that does not vanish*, then the Jacobian (row) matrix $[\partial F/\partial x^1, \partial F/\partial x^2, \dots, \partial F/\partial x^N]$ has rank 1 and we may conclude that *this locus is indeed an* $(N-1)$*-dimensional submanifold of* $\mathbb{R}^N$. This criterion is easily verified, for example, in the case of the 2-sphere $F(x, y, z) = x^2 + y^2 + z^2 = 1$ of Example (ii). The column version of this row matrix is called in calculus the gradient vector of F. In $\mathbb{R}^3$ this vector

$$\begin{bmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \end{bmatrix}$$

is orthogonal to the locus $F = 0$, and we may conclude, for example, that if this gradient vector has a nontrivial component in the z direction at a point of $F = 0$, then locally we can solve for $z = z(x, y)$.

A submanifold of dimension $(N-1)$ in $\mathbb{R}^N$, that is, of "**codimension**" 1, is called a **hypersurface**.

(iii) The x axis of the xy plane $\mathbb{R}^2$ can be described (perversely) as the locus of the quadratic $F(x, y) := y^2 = 0$. Both partial derivatives vanish on the locus, the x axis, and our criteria would not allow us to say that the x axis is a 1-dimensional submanifold of $\mathbb{R}^2$. Of course the x axis *is* a submanifold; we should have used the usual description $G(x, y) := y = 0$. Our Jacobian criteria are *sufficient* conditions, not necessary ones.

(iv) The locus $F(x, y) := xy = 0$ in $\mathbb{R}^2$, consisting of the union of the x and y axes, is not a 1-dimensional submanifold of $\mathbb{R}^2$. It seems "clear" (and can be proved) that in a neighborhood of the intersection of the two lines we are not going to be able to describe the locus in the form of $y = f(x)$ or $x = g(y)$, where f, g, are differentiable functions. The best we can say is that this locus *with the origin removed* is a 1-dimensional submanifold.

1.1b. The Geometry of Jacobian Matrices: The "Differential"

The **tangent space** to $\mathbb{R}^n$ at the point x, written here as $\mathbb{R}^n_x$, is by definition the vector space of all vectors in $\mathbb{R}^n$ based at x (i.e., it is a copy of $\mathbb{R}^n$ with origin shifted to x).

Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^r$ be coordinates for $\mathbb{R}^n$ and $\mathbb{R}^r$ respectively. Let $F : \mathbb{R}^n \to \mathbb{R}^r$ be a **smooth** map. ("Smooth" ordinarily means infinitely differentiable. For our purposes, however, it will mean differentiable at least as many times as is necessary in the present context. For example, if F is once continuously differentiable, we may use the chain rule in the argument to follow.) In coordinates, F is described by giving r functions of n variables

$$y^\alpha = F^\alpha(x) \qquad \alpha = 1, \ldots, r$$

or simply $y = F(x)$. We will frequently use the more dangerous notation $y = y(x)$.

Let $y_0 = F(x_0)$; the Jacobian *matrix* $(\partial y^\alpha / \partial x^i)(x_0)$ has the following significance.

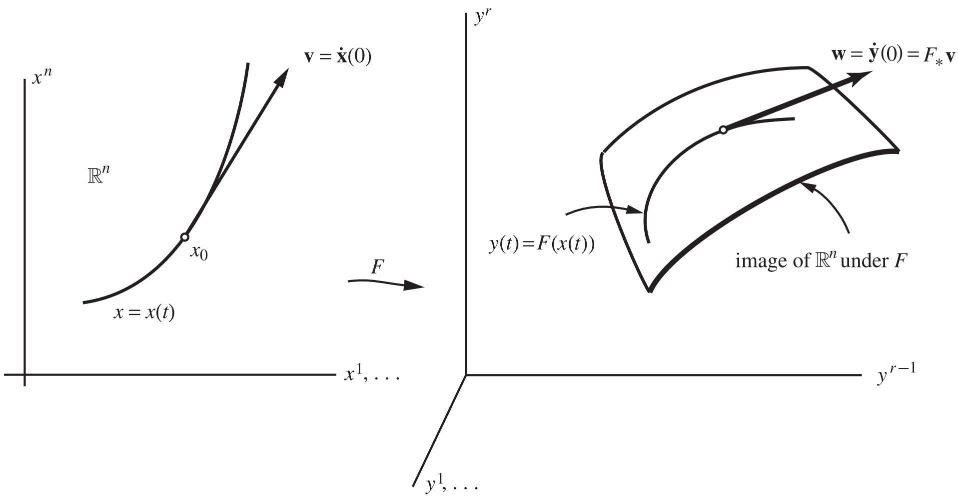

Figure 1.3

Let $\mathbf{v}$ be a tangent vector to $\mathbb{R}^n$ at x_0. Take *any* smooth curve $x(t)$ such that $x(0) = x_0$ and $\dot{x}(0) := (dx/dt)(0) = \mathbf{v}$, for example, the straight line $x(t) = x_0 + t\mathbf{v}$. The image of this curve

$$y(t) = F(x(t))$$

has a tangent vector $\mathbf{w}$ at y_0 given by the chain rule

$$w^\alpha = \dot{y}^\alpha(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)\dot{x}^i(0) = \sum_{i=1}^{n} \left(\frac{\partial y^\alpha}{\partial x^i}\right)(x_0)v^i$$

The assignment $\mathbf{v} \mapsto \mathbf{w}$ is, from this expression, independent of the curve $x(t)$ chosen, and defines a *linear transformation*, the **differential** of F at x_0

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0} \qquad F_*(\mathbf{v}) = \mathbf{w} \tag{1.1}$$

whose matrix is simply the Jacobian matrix $(\partial y^\alpha/\partial x^i)(x_0)$. *This interpretation of the Jacobian matrix, as a linear transformation sending tangents to curves into tangents to the image curves under F, can sometimes be used to replace the direct computation of matrices*. This philosophy will be illustrated in Section 1.1d.

1.1c. The Main Theorem on Submanifolds of $\mathbb{R}^N$

The main theorem is a geometric interpretation of what we have discussed. Note that the statement "F has rank r at x_0," that is, $[\partial y^\alpha/\partial x^i](x_0)$ has rank r, is geometrically the statement that the differential

$$F_* : \mathbb{R}^n_{x_0} \to \mathbb{R}^r_{y_0=F(x_0)}$$

is **onto** or "surjective"; that is, given any vector $\mathbf{w}$ at y_0 there is at least one vector $\mathbf{v}$ at x_0 such that $F_*(\mathbf{v}) = \mathbf{w}$. We then have

Theorem (1.2): *Let $F : \mathbb{R}^{r+n} \to \mathbb{R}^r$ and suppose that the locus*

$$F^{-1}(y_0) := \{x \in \mathbb{R}^{r+n} \mid F(x) = y_0\}$$

is not empty. Suppose further that for all $x_0 \in F^{-1}(y_0)$

$$F_* : \mathbb{R}^{n+r}_{x_0} \to \mathbb{R}^r_{y_0}$$

is onto. Then $F^{-1}(y_0)$ is an n-dimensional submanifold of $\mathbb{R}^{n+r}$.

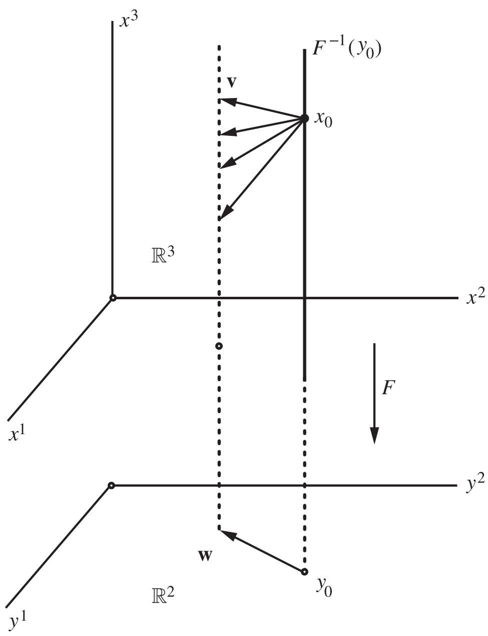

Figure 1.4

The best example to keep in mind is the linear "projection" $F : \mathbb{R}^3 \to \mathbb{R}^2$, $F(x^1, x^2, x^3) = (x^1, x^2)$, that is, $y^1 = x^1$ and $y^2 = x^2$. In this case, x^3 serves as global coordinate for the submanifold $x^1 = y_0^1$, $x^2 = y_0^2$, that is, the vertical line.

1.1d. A Nontrivial Example: The Configuration Space of a Rigid Body

Assume a rigid body has one point, the origin of $\mathbb{R}^3$, fixed. By comparing a cartesian right-handed system fixed in the body with that of $\mathbb{R}^3$ we see that the configuration of the body at any time is described by the rotation matrix taking us from the basis of $\mathbb{R}^3$ to the basis fixed in the body. The configuration space of the body is then the **rotation group** SO(3), that is, the 3×3 real matrices $x = (x_{ij})$ such that

$$x^T = x^{-1} \quad \text{and} \quad \det x = 1$$

where T denotes transpose. (If we omit the determinant condition, the group is the full **orthogonal** group, O(3).) By assigning (in some fixed order) the nine coordinates $x_{11}, x_{12}, \ldots, x_{33}$ to *any* matrix x, we see that the space of all 3×3 real matrices, $M(3 \times 3)$, is the euclidean space $\mathbb{R}^9$. The group O(3) is then the locus in this $\mathbb{R}^9$ defined by the equations $x^T x = I$, that is, by the system of nine quadratic equations (i, k)

$$(i, k) \qquad \sum_{j=1}^{3} x_{ji} x_{jk} = \delta_{ik}$$

We then have the following situation. The configuration of the body at time t can be represented by a point $x(t)$ in $\mathbb{R}^9$, but in fact the point $x(t)$ lies on the locus O(3) in $\mathbb{R}^9$. We shall see shortly that *this locus is in fact a 3-dimensional submanifold* of $\mathbb{R}^9$. As time t evolves, the point $x(t)$ traces out a curve on this 3-dimensional locus. Since O(3) *is* a submanifold, we shall see, in Section 10.2c from the principle of least action, that this path is a very special one, a "geodesic" on the submanifold O(3), and this in turn will yield important information on the existence of periodic motions of the body even when the body is subject to an unusual potential field. All this depends on the fact that O(3) is a submanifold, and we turn now to the proof of this crucial result.

Note first that since $x^T x$ is a symmetric matrix, equation (i, k) is the same as equation (k, i); there are, then, only 6 independent equations. This suggests the following. Let

$$\text{Sym}^6 := \{x \in M(3 \times 3) \mid x^T = x\}$$

be the space of all *symmetric* 3×3 matrices. Since this is defined by the three *linear* equations $x_{ik} - x_{ki} = 0$, $i \neq k$, we see that Sym^6 is a 6-dimensional linear subspace of $\mathbb{R}^9$; that is, it can be considered as a copy of $\mathbb{R}^6$. To exhibit O(3) as a locus in $\mathbb{R}^9$, we consider the map

$$F : \mathbb{R}^9 \to \mathbb{R}^6 = \text{Sym}^6 \quad \text{defined by } F(x) = x^T x - I$$

O(3) is then the locus $F^{-1}(0)$. Let $x_0 \in F^{-1}(0) = \text{O}(3)$. We shall show that $F_* : \mathbb{R}^9_{x_0} \to \text{Sym}^6$ is onto.

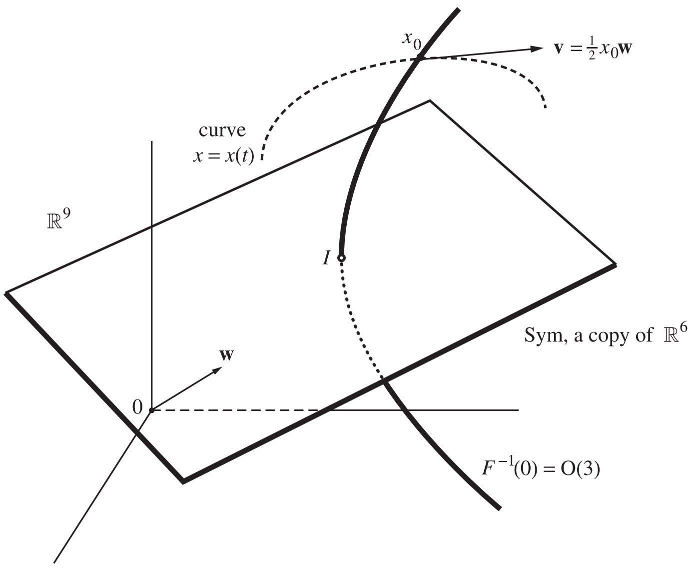

Figure 1.5

Let $\mathbf{w}$ be tangent to Sym^6 at the zero matrix. As usual, we identify a vector at the origin of $\mathbb{R}^n$ with its endpoint. *Then* $\mathbf{w}$ *is itself a symmetric matrix*. We must find $\mathbf{v}$, a tangent vector to $\mathbb{R}^9$ at x_0, such that $F_*\mathbf{v} = \mathbf{w}$. Consider a general curve $x = x(t)$ of matrices such that $x(0) = x_0$; its tangent vector at x_0 is $\dot{x}(0)$. The image curve

$$F(x(t)) = x(t)^T x(t) - I$$

has tangent at $t = 0$ given by

$$\frac{d}{dt}[F(x(t))]_{t=0} = \dot{x}(0)^T x_0 + x_0^T \dot{x}(0)$$

We wish this quantity to be $\mathbf{w}$. You should verify that it is sufficient to satisfy the matrix equation $x_0^T \dot{x}(0) = \mathbf{w}/2$. Since $x_0 \in \mathrm{O}(3)$, $x_0^T = x_0^{-1}$ and we have as solution the matrix product $\mathbf{v} = \dot{x} = x_0\mathbf{w}/2$. Thus F_* *is* onto at x_0 and by our main theorem O(3)= $F^{-1}(0)$ is a $(9 - 6) = 3$-dimensional submanifold of $\mathbb{R}^9$.

What about the subset SO(3) of O(3)? Recall that each orthogonal matrix has determinant ± 1, whereas SO(3) consists of those orthogonal matrices with determinant $+1$. The mapping

$$\det : \mathbb{R}^9 \to \mathbb{R}$$

that sends each matrix x into its determinant is continuous (it is a cubic polynomial function of the coordinates x_{ik}) and consequently the two subsets of O(3) where det is $+1$ and where det is -1 must be separated. This means that SO(3) itself must have the property that it is locally described by giving 6 of the coordinates in terms of the remaining 3, that is, SO(3) is a 3-dimensional submanifold of $\mathbb{R}^9$.

Thus the configuration space of a rigid body with one point fixed is the group SO(3). *This is a 3-dimensional submanifold of* $\mathbb{R}^9$. *Each point of this configuration space lies in some local curvilinear coordinate system.*