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Excerpt
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Counting planar maps, coloured or uncoloured

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Abstract

We present recent results on the enumeration of q-coloured planar maps, where each monochromatic edge carries a weight ν. This is equivalent to weighting each map by its Tutte polynomial, or to solving the q-state Potts model on random planar maps. The associated generating function, obtained by Olivier Bernardi and the author, is differentially algebraic. That is, it satisfies a (non-linear) differential equation. The starting point of this result is a functional equation written by Tutte in 1971, which translates into enumerative terms a simple recursive description of planar maps. The proof follows and adapts Tutte's solution of *properly* q-coloured triangulations (1973-1984).

We put this work in perspective with the much better understood enumeration of families of *uncoloured* planar maps, for which the recursive approach almost systematically yields algebraic generating functions. In the past 15 years, these algebraicity properties have been explained combinatorially by illuminating bijections between maps and families of plane trees. We survey both approaches, recursive and bijective.

Comparing the coloured and uncoloured results raises the question of designing bijections for coloured maps. No complete bijective solution exists at the moment, but we present bijections for certain specialisations of the general problem. We also show that for these specialisations, Tutte's functional equation is much easier to solve that in the general case.

We conclude with some open questions.

1 Introduction

A planar map is a proper embedding in the sphere of a finite connected graph, defined up to continuous deformation. The enumeration of these objects has been a topic of constant interest for 50 years, starting with a series of papers by Tutte in the early 1960s; these papers were mostly based on recursive descriptions of maps (e.g. [103]). The last 15 years have witnessed a new burst of activity in this field, with the development of rich bijective approaches [98, 39], and their applications to the study of random maps of large size [78, 85]. In such enumerative problems, maps are usually *rooted* by orienting one edge. Figure 1 sets a first exercise in map enumeration.

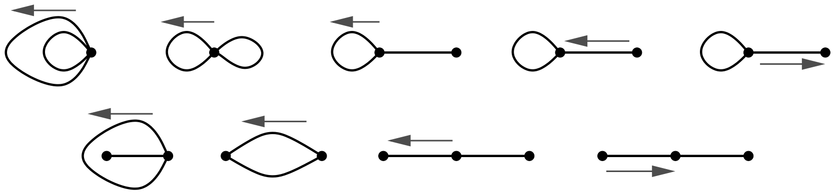

Figure 1: There are 9 rooted planar maps with two edges.

Planar maps are not only studied in combinatorics and probability, but also in theoretical physics. In this context, maps are considered as random surfaces, and constitute a model of 2-dimensional *quantum gravity*. For many years, maps were studied independently in combinatorics and in physics, and another approach for counting them, based on the evaluation of certain matrix integrals, was introduced in the 1970s in physics [42, 18], and much developed since then [55, 88]. More recently, a fruitful exchange started between the two communities. Some physicists have become masters in combinatorial methods [35, 37], while the matrix integral approach has been taken over by some probabilists [71].

From the physics point of view, it is natural to equip maps with additional structures, like particles, trees, spins, and more generally classical models of statistical physics. In combinatorics however, a huge majority of papers deal with the enumeration of bare maps. There has been some exceptions to this rule in the past few years, with combinatorial solutions of the Ising and hard-particle models on planar maps [34, 38, 39]. But there is also an earlier, and major, exception to this rule: Tutte's study of properly q-coloured triangulations (Figure 2).

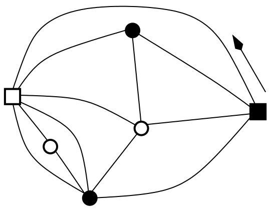

Figure 2: A (rooted) triangulation of the sphere, properly coloured with 4 colours.

This ten years long study (1973-1984) plays a central role in this paper. For a very long time, it remained an isolated *tour de force* with no counterpart for other families of planar maps or for more general colourings, probably because the corresponding series of papers [110, 108, 107, 109, 111, 112, 113, 114, 115, 116] looks quite formidable. Our main point here is to report on recent advances in the enumeration of (non-necessarily properly) q-coloured maps, in the steps of Tutte. In the associated generating function, every monochromatic edge is assigned a weight ν: the case $\nu = 0$ thus captures proper colourings. In physics terms, we are studying the q-state Potts model on planar maps. A third equivalent formulation is that we count planar maps weighted by their Tutte polynomial — a bivariate generalisation of the chromatic polynomial, introduced by Tutte, who called it the dichromatic polynomial. Since the Tutte polynomial has numerous interesting specialisations, giving for example the number of trees, forests, acyclic orientations, proper colourings of course, or the partition function of the Ising model, or the reliability and flow polynomials, we are covering several models at the same time.

We shall put this work in perspective with the (much better understood) enumeration of uncoloured maps, to which we devote Sections 3 and 4. We first present in Section 3 the robust recursive approach found in the early work of Tutte. It applies in a rather uniform way to many families of maps, and yields for their generating functions functional equations that we call *polynomial equations with one catalytic* *variable.* A typical example is (3.1). It is now understood that the solutions of these equations are always algebraic, that is, satisfy a polynomial equation. For instance, there are $2 \cdot 3^n \binom{2n}{n}/((n+1)(n+2))$ rooted planar maps with n edges, and their generating function, that is, the series

$$M(t) := \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} t^n,$$

satisfies

$$M(t) = 1 - 16t + 18tM(t) - 27t^2 M(t)^2.$$

Thus algebraicity is intimately connected with (uncoloured) planar maps. In Section 4, we present two more recent bijective approaches that relate maps to plane trees, which are algebraic objects *par excellence.* Not only do these bijections give a better understanding of algebraicity properties, but they also explain why many families of maps are counted by simple formulas.

In Section 5, we discuss the recursive approach for q-coloured maps. The corresponding functional equation (5.3) was written in 1971 by Tutte —who else?—, but was left untouched since then. It involves two "catalytic" variables, and it has been known for a long time that its solution is not algebraic. The key point of this section, due to Olivier Bernardi and the author, is the solution of this equation, in the form of a system of differential equations that defines the generating function of q-coloured maps. This series is thus *differentially algebraic*, like Tutte's solution of properly coloured triangulations. Halfway on the long path that leads to the solution stands an interesting intermediate result: when $q \neq 4$ is of the form $2 + 2\cos(j\pi/m)$, for integers j and m, the generating function of q-coloured planar maps is algebraic. This includes the values $q = 2$ and $q = 3$, for which we give explicit results. We also discuss certain specialisations for which the equation becomes easier to solve, like the enumeration of maps equipped with a bipolar orientation, or with a spanning tree.

Since we are still in the early days of the enumeration of coloured maps, it is not surprising that bijective approaches are at the moment one step behind. Still, a few bijections are available for some of the simpler specialisations mentioned above. They are presented in Section 6. We conclude with open questions, dealing with both uncoloured and coloured enumeration.

This survey is sometimes written in an informal style, especially when we describe bijections. Proofs are only given when they are new, or especially simple and illuminating. The reference list, although long, is certainly not exhaustive. In particular, the papers cited in this introduction are just examples illustrating our topic, and should be considered as pointers to the relevant literature. More references are given further in the paper. Two approaches that have been used to count maps are utterly absent from this paper: methods based on characters of the symmetric group and symmetric functions [68, 69], which do not exactly address the same range of problems, and the matrix integral approach, which is powerful [55], but is not always fully rigorous. The Potts model has been addressed via matrix integrals [51, 56, 123]. We refer to [15] for a description our current understanding of this work.

2 Definitions and notation

2.1 Planar maps

A *planar map* is a proper embedding of a connected planar graph in the oriented sphere, considered up to orientation preserving homeomorphism. Loops and multiple edges are allowed. The *faces* of a map are the connected components of its complement. The numbers of vertices, edges and faces of a planar map M, denoted by $\mathrm{v}(M)$, $\mathrm{e}(M)$ and $\mathrm{f}(M)$, are related by Euler's relation $\mathrm{v}(M) + \mathrm{f}(M) = \mathrm{e}(M) + 2$. The *degree* of a vertex or face is the number of edges incident to it, counted with multiplicity. A map is *m-valent* if all its vertices have degree m. A *corner* is a sector delimited by two consecutive edges around a vertex; hence a vertex or face of degree k defines k corners. The *dual* of a map M, denoted M^*, is the map obtained by placing a vertex of M^* in each face of M and an edge of M^* across each edge of M; see Figure 3.

For counting purposes it is convenient to consider *rooted* maps. A map is rooted by orienting an edge, called the *root-edge*. The origin of this edge is the *root-vertex*. The face that lies to the right of the root-edge is the *root-face*. In figures, we take the root-face as the infinite face (Figure 3). This explains why we often call the root-face the *outer* (or: *infinite*) face, and its degree the *outer degree*. The other faces are said to be *finite*. From now on, every map is *planar* and *rooted*. By convention, we include among rooted planar maps the *atomic* map m_0 having one vertex and no edge. The set of rooted planar maps is denoted $\mathcal{M}$.

A map is *separable* if it is atomic or can be obtained by gluing two non-atomic maps at a vertex. Observe that both maps with one edge are non-separable.

Figure 3: A rooted planar map and its dual (rooted at the dual edge).

2.2 Power series

Let A be a commutative ring and x an indeterminate. We denote by $A[x]$ (resp. $A[[x]]$) the ring of polynomials (resp. formal power series) in x with coefficients in A. If A is a field, then $A(x)$ denotes the field of rational functions in x, and $A((x))$ the field of Laurent series[1] in x. These notations are generalised to polynomials, fractions and series in several indeterminates. We denote by bars the reciprocals of variables: that is, $\bar{x} = 1/x$, so that $A[x, \bar{x}]$ is the ring of Laurent polynomials in x with coefficients in A. The coefficient of x^n in a Laurent series $F(x)$ is denoted

[1] A *Laurent series* is a series of the form $\sum_{n \geq n_0} a(n)x^n$, for some $n_0 \in \mathbf{Z}$.

by $[x^n]F(x)$. The *valuation* of a Laurent series $F(x)$ is the smallest d such that x^d occurs in $F(x)$ with a non-zero coefficient. If $F(x) = 0$, then the valuation is $+\infty$. If $F(x;t)$ is a power series in t with coefficients in $A((x))$, that is, a series of the form

$$F(x;t) = \sum_{n \geq 0, i \in \mathbf{Z}} f(i;n) x^i t^n,$$

where for all n, almost all coefficients $f(i;n)$ such that $i < 0$ are zero, then the *positive part* of $F(x;t)$ in x is the following series, which has coefficients in $xA[[x]]$:

$$[x^>]F(x;t) := \sum_{n \geq 0, i > 0} f(i;n) x^i t^n.$$

We define similarly the non-negative part of $F(x;t)$ in x.

A power series $F(x_1, \ldots, x_k) \in \mathbb{K}[[x_1, \ldots, x_k]]$, where $\mathbb{K}$ is a field, is *algebraic* (over $\mathbb{K}(x_1, \ldots, x_k)$) if it satisfies a polynomial equation $P(x_1, \ldots, x_k, F(x_1, \ldots, x_k)) = 0$. The series $F(x_1, \ldots, x_k)$ is *D-finite* if for all $i \leq k$, it satisfies a (non-trivial) linear differential equation in x_i with coefficients in $\mathbb{K}[x_1, \ldots, x_k]$. We refer to [81, 82] for a study of these series. All algebraic series are D-finite. A series $F(x)$ is *differentially algebraic* if it satisfies a (non-necessarily linear) differential equation with coefficients in $\mathbb{K}[x]$.

2.3 The Potts model and the Tutte polynomial

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let ν be an indeterminate, and take $q \in \mathbf{N}$. A *colouring* of the vertices of G in q colours is a map $c : V(G) \to \{1, \ldots, q\}$. An edge of G is *monochromatic* if its endpoints share the same colour. Every loop is thus monochromatic. The number of monochromatic edges is denoted by $m(c)$. The *partition function of the Potts model* on G counts colourings by the number of monochromatic edges:

$$\mathrm{P}_G(q,\nu) = \sum_{c: V(G) \to \{1,\ldots,q\}} \nu^{m(c)}.$$

The Potts model is a classical magnetism model in statistical physics, which includes (for $q = 2$) the famous Ising model (with no magnetic field) [120]. Of course, $\mathrm{P}_G(q, 0)$ is the chromatic polynomial of G.

If G_1 and G_2 are disjoint graphs and $G = G_1 \cup G_2$, then clearly

$$\mathrm{P}_G(q,\nu) = \mathrm{P}_{G_1}(q,\nu)\,\mathrm{P}_{G_2}(q,\nu). \tag{2.1}$$

If G is obtained by attaching G_1 and G_2 at one vertex, then

$$\mathrm{P}_G(q,\nu) = \frac{1}{q}\,\mathrm{P}_{G_1}(q,\nu)\,\mathrm{P}_{G_2}(q,\nu). \tag{2.2}$$

The Potts partition function can be computed by induction on the number of edges. If G has no edge, then $\mathrm{P}_G(q,\nu) = q^{|V(G)|}$. Otherwise, let e be an edge of G. Denote by $G\backslash e$ the graph obtained by deleting e, and by G/e the graph obtained by contracting e (if e is a loop, then it is simply deleted). Then

$$\mathrm{P}_G(q,\nu) = \mathrm{P}_{G\backslash e}(q,\nu) + (\nu - 1)\,\mathrm{P}_{G/e}(q,\nu). \tag{2.3}$$

Indeed, it is not hard to see that $\nu\,\mathrm{P}_{G/e}(q,\nu)$ counts colourings for which e is monochromatic, while $\mathrm{P}_{G\backslash e}(q,\nu) - \mathrm{P}_{G/e}(q,\nu)$ counts those for which e is bichromatic. One important consequence of this induction is that $\mathrm{P}_G(q,\nu)$ is always a *polynomial* in q and ν. We call it the *Potts polynomial* of G. Since it is a polynomial, we will no longer consider q as an integer, but as an indeterminate, and sometimes evaluate $\mathrm{P}_G(q,\nu)$ at real values q. We also observe that $\mathrm{P}_G(q,\nu)$ is a multiple of q: this explains why we will weight maps by $\mathrm{P}_G(q,\nu)/q$.

Up to a change of variables, the Potts polynomial is equivalent to another, maybe better known, invariant of graphs, namely the *Tutte polynomial* $\mathrm{T}_G(\mu,\nu)$ (see e.g. [19]):

$$\mathrm{T}_G(\mu,\nu) := \sum_{S\subseteq E(G)} (\mu-1)^{\mathrm{c}(S)-\mathrm{c}(G)}(\nu-1)^{\mathrm{e}(S)+\mathrm{c}(S)-\mathrm{v}(G)},$$

where the sum is over all spanning subgraphs of G (equivalently, over all subsets of edges) and v(.), e(.) and c(.) denote respectively the number of vertices, edges and connected components. For instance, the Tutte polynomial of a graph with no edge is 1. The equivalence with the Potts polynomial was established by Fortuin and Kasteleyn [62]:

$$\mathrm{P}_G(q,\nu) = \sum_{S\subseteq E(G)} q^{\mathrm{c}(S)}(\nu-1)^{\mathrm{e}(S)} = (\mu-1)^{\mathrm{c}(G)}(\nu-1)^{\mathrm{v}(G)}\,\mathrm{T}_G(\mu,\nu), \tag{2.4}$$

for $q = (\mu-1)(\nu-1)$. In this paper, we work with P_G rather than T_G because we wish to assign real values to q (this is more natural than assigning real values to $(\mu-1)(\nu-1)$). However, one property looks more natural in terms of T_G: if G and G^* are dual connected planar graphs (that is, if G and G^* can be embedded as dual planar maps) then

$$\mathrm{T}_{G^*}(\mu,\nu) = \mathrm{T}_G(\nu,\mu).$$

Translating this identity in terms of Potts polynomials thanks to (2.4) gives:

$$\begin{aligned}
\mathrm{P}_{G^*}(q,\nu) &= q(\nu-1)^{\mathrm{v}(G^*)-1}\,\mathrm{T}_{G^*}(\mu,\nu)\\
&= q(\nu-1)^{\mathrm{v}(G^*)-1}\,\mathrm{T}_G(\nu,\mu)\\
&= \frac{(\nu-1)^{\mathrm{e}(G)}}{q^{\mathrm{v}(G)-1}}\,\mathrm{P}_G(q,\mu),
\end{aligned} \tag{2.5}$$

where $\mu = 1+q/(\nu-1)$ and the last equality uses Euler's relation: $\mathrm{v}(G)+\mathrm{v}(G^*)-2 = \mathrm{e}(G)$.

3 Uncoloured planar maps: the recursive approach

In this section, we describe the first approach that was used to count maps: the recursive method. It is based on very simple combinatorial operations (like the deletion or contraction of an edge), which translate into non-trivial functional equations defining the generating functions. A recent theorem, generalising the so-called *quadratic method*, states that the solutions of all equations of this type are algebraic. Since the recursive method applies to many families of maps, numerous algebraicity results follow.

3.1 A functional equation for planar maps

Consider a rooted planar map, distinct from the atomic map. Delete the root-edge. If this edge is an isthmus, one obtains two connected components M_1 and M_2, and otherwise a single component M, which we can root in a canonical way (Figure 4). Conversely, starting from an ordered pair (M_1, M_2) of maps, there is a unique way to connect them by a new (root) edge. If one starts instead from a single map M, there are $d+1$ ways to add a root edge, where $d = \mathrm{df}(M)$ is the degree of the root-face of M (Figure 5).

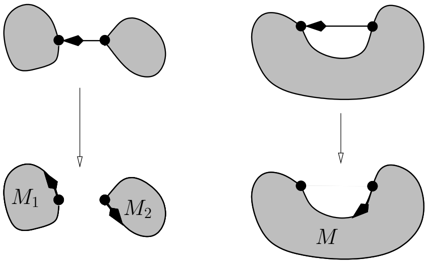

Figure 4: Deletion of the root-edge in a planar map.

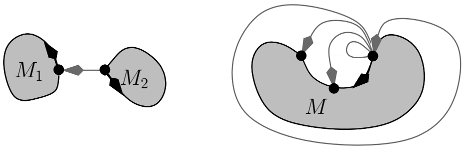

Figure 5: Reconstruction of a planar map.

Hence, to derive from this recursive description of planar maps a functional equation for their generating function, we need to take into account the degree of the root-face, by an additional variable y. Hence, let

$$M(t;y) = \sum_{M \in \mathcal{M}} t^{\mathrm{e}(M)} y^{\mathrm{df}(M)} = \sum_{d \geq 0} y^d M_d(t)$$

be the generating function of planar maps, counted by edges and outer-degree. The series $M_d(t)$ counts by edges maps with outer degree d. The recursive description of maps translates as follows:

$$\begin{aligned} M(t;y) &= 1 + y^2 t M(t;y)^2 + t \sum_{d \geq 0} M_d(t)(y + y^2 + \cdots + y^{d+1}) \\ &= 1 + y^2 t M(t;y)^2 + ty \frac{yM(t;y) - M(t;1)}{y-1}. \end{aligned} \tag{3.1}$$

Indeed, connecting two maps M_1 and M_2 by an edge produces a map of outer-degree $\mathrm{df}(M_1) + \mathrm{df}(M_2) + 2$, while the $d+1$ ways to add an edge to a map M such that $\mathrm{df}(M) = d$ produce $d+1$ maps of respective outer degree $1, 2, \ldots, d+1$, as can be seen on Figure 5. The term 1 records the atomic map.

The above equation was first written by Tutte in 1968 [105]. It is typical of the type of equation obtained in (recursive) map enumeration. More examples will be given in Section 3.2. One important feature in this equation is the divided difference

$$\frac{yM(t;y) - M(t;1)}{y-1},$$

which prevents us from simply setting $y = 1$ to solve for $M(t;1)$ first, and then for $M(t;y)$. The parameter $\mathrm{df}(M)$, and the corresponding variable y, are said to be *catalytic* for this equation — a terminology borrowed to Zeilberger [122].

Such equations do not only occur in connection with maps: they also arise in the enumeration of polyominoes [24, 59, 101], lattice walks [31, 3, 52, 76, 96], permutations [25, 28, 121]... The solution of these equations has naturally attracted some interest. The "guess and check" approach used in the early 1960s is now replaced by a general method, which we present below in Section 3.3. This method implies in particular that *the solution of any (well-founded) polynomial equation with one catalytic variable is algebraic*. It generalises the *quadratic method* developed by Brown [46] for equations of degree 2 that involve a single additional unknown series (like $M(t;1)$ in the equation above) and also the *kernel method* that applies to linear equations, and seems to have first appeared in Knuth's *Art of Computer Programming* [76, Section 2.2.1, Ex. 4] (see also [2, 31, 96]).

Contraction vs. deletion. Before we move to more examples, let us make a simple observation. Another natural way to decrease the edge number of a map is to contract the root-edge, rather than delete it (if this edge is a loop, one just erases it). When one tries to use this to count planar maps, one is lead to introduce the degree of the root-vertex as a catalytic parameter, and a corresponding variable x in the generating function. This yields the same equation as above:

$$M(t;x) = 1 + x^2 t M(t;x)^2 + t \sum_{d \geq 0} M_d(t)(x + x^2 + \cdots + x^{d+1}).$$

As illustrated by Figure 6, the term 1 records the atomic map, the second term corresponds to maps in which the root-edge is a loop, and the third term to the remaining cases. In particular, the sum $(x + x^2 + \cdots + x^{d+1})$ now describes how to

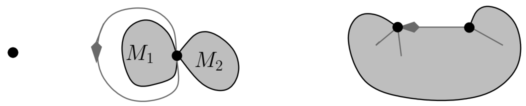

Figure 6: Contraction of the root-edge in a planar map.

distribute the adjacent edges when a new edge is inserted. Given that the contraction operation is the dual of the deletion operation, it is perfectly natural to obtain the same equation as before. The reason why we mention this alternative construction is that, when we establish below a functional equation for maps weighted by their Potts (or Tutte) polynomial, we will have to use simultaneously these two operations, as suggested by the recursive description (2.3) of the Potts polynomial. This will naturally result in equations with *two* catalytic variables x and y.

3.2 More functional equations

The recursive method is extremely robust. We illustrate this by a few examples. Two of them — maps with prescribed face degrees, and Eulerian maps with prescribed face degrees — actually cover infinitely many families of maps. Some of these examples also have a colouring flavour.

Maps with prescribed face degrees. Consider for instance the enumeration of *triangulations*, that is, maps in which all faces have degree 3. The recursive deletion of the root-edge gives maps in which all finite faces have degree 3, but the outer face may have any degree: these maps are called *near-triangulations*. We denote by $\mathcal{T}$ the set of near-triangulations. The deletion of the root-edge in a near triangulation gives either two near-triangulations, or a single one, *the outer degree of which is at least two* (Figure 7). In both cases, there is unique way to reconstruct the map we started from. Let $T(t;y) \equiv T(y)$ be the generating function of near-triangulations, counted by edges and by the outer degree:

$$T(t;y) = \sum_{M \in \mathcal{T}} t^{e(M)} y^{df(M)} = \sum_{d \geq 0} y^d T_d(t).$$

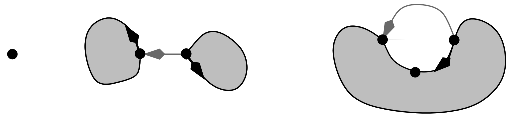

Figure 7: Deletion of the root-edge in a near-triangulation.

The above recursive description translates into

$$T(y) = 1 + ty^2 T(y)^2 + t \, \frac{T(y) - T_0 - yT_1}{y}, \tag{3.2}$$

where $T_0 = 1$ counts the atomic map. We have again a divided difference, this time at $y = 0$. Its combinatorial interpretation (“it is forbidden to add an edge to a map of outer degree 0 or 1”) differs from the interpretation of the divided difference occurring in (3.1) (“there are multiple ways to add an edge”). Still, both equations are of the same type and will be solved by the same method. Note that we have omitted the variable t in the notation $T(y)$, which we will do quite often in this paper, to avoid heavy notation and enhance the catalytic parameter(s).

Consider now *bipartite* planar maps, that is, maps that admit a proper 2-colouring (and then a unique one, if the root-vertex is coloured white). For planar maps, this is equivalent to saying that all faces have an even degree. Let $B(t; y) = \sum_{d\geq 0} B_d(t)y^d$ be the generating function of bipartite maps, counted by edges (variable t) and by half the outer degree (variable y). Then the deletion of the root-edge translates as follows (Figure 8):

$$\begin{aligned} B(y) &= 1 + tyB(y)^2 + t\sum_{d\geq 0} B_d(y + y^2 + \cdots + y^d) \\ &= 1 + tyB(y)^2 + ty\frac{B(y) - B(1)}{y - 1}. \end{aligned} \quad (3.3)$$

This is again a quadratic equation with one catalytic variable, y.

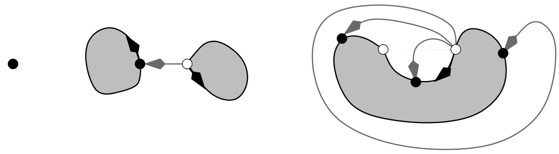

Figure 8: Deletion of the root-edge in a bipartite map.

More generally, it was shown by Bender and Canfield [6] that the recursive approach applies to any family of maps for which the face degrees belong to a given set D, provided D differs from a finite union of arithmetic progressions by a finite set. In all cases, the equation is quadratic, but may involve more than a single additional unknown function. For instance, when counting near-quadrangulations rather than near-triangulations, Eq. (3.2) is replaced by

$$Q(y) = 1 + ty^2Q(y)^2 + t\,\frac{Q(y) - Q_0 - yQ_1 - y^2Q_2}{y^2},$$

where Q_i counts near-quadrangulations of outer degree i. Bender and Canfield solved these equations using a theorem of Brown from which the quadratic method is derived, proving in particular that the resulting generating function is always algebraic. Their result only involves the edge number, but, when D is finite, it can be refined by keeping track of the vertex degree distribution [29].

Eulerian maps with prescribed face degrees. A planar map is Eulerian if all vertices have an even degree. Equivalently, its faces admit a proper 2-colouring (and a unique one, if the root-face is coloured white). Of course, Eulerian maps are the duals of bipartite maps, so that their generating function (by edges, and half-degree of the root-vertex) satisfies (3.3). But we wish to impose conditions on the face degrees of Eulerian maps (dually, on the vertex degrees of bipartite maps). This includes as a special case the enumeration of (non-necessarily Eulerian) maps with prescribed