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Introduction

Consider a linear system $y = \Phi x$ where Φ can be taken as an $m \times n$ matrix acting on Euclidean space or more generally, a linear operator on a Hilbert space. We call the vector x a signal or input, Φ the transform–sample matrix–filter and the vector y the sample or output. The problem is to reconstruct x from y, or more generally, to reconstruct an altered version of x from an altered y. For example, we might analyze the signal x in terms of frequency components and various combinations of time and frequency components y. Once we have analyzed the signal we may alter some of the component parts to eliminate undesirable features or to compress the signal for more efficient transmission and storage. Finally, we reconstitute the signal from its component parts.

The three typical steps in this process are:

- **Analysis.** Decompose the signal into basic components. This is called analysis. We will think of the signal space as a vector space and break it up into a sum of subspaces, each of which captures a special feature of a signal.
- **Processing.** Modify some of the basic components of the signal that were obtained through the analysis. This is called processing.
- **Synthesis.** Reconstitute the signal from its (altered) component parts. This is called synthesis. Sometimes, we will want perfect reconstruction. Sometimes only perfect reconstruction with high probability. If we don't alter the component parts, we usually want the synthesized signal to agree exactly with the original signal. We will also be interested in the convergence properties of an altered signal with respect to the original signal, e.g., how well a reconstituted signal, from which some information may have been dropped, approximates the original signal. Finally we look at problems where the "signal" lies in some

high dimensional Euclidean space in the form of discrete data and where the "filter" is not necessarily linear.

We will look at several methods for signal analysis. We will cover:

- Fourier series and Fourier integrals, infinite products.
- Windowed Fourier transforms.
- Continuous wavelet transforms.
- Filter banks, bases, frames.
- Discrete transforms, Z transforms, Haar wavelets and Daubechies wavelets, singular value decomposition.
- Compressive sampling/compressive sensing.
- Topics in the the parsimonious representation of data.

We break up our treatment into several cases, both theoretical and applied: (1) The system is invertible (Fourier series, Fourier integrals, finite Fourier transform, Z transform, Riesz basis, discrete wavelets, etc.). (2) The system is underdetermined, so that a unique solution can be obtained only if x is restricted (compressive sensing). (3) The system is overdetermined (bandlimited functions, windowed Fourier transform, continuous wavelet transform, frames). In the last case one can throw away some information from y and still recover x. This is the motivation of frame theory, discrete wavelets from continuous wavelets, Shannon sampling, filterbanks, etc. (4) The signal space is a collection of data in some containing Euclidean space.

Each of these cases has its own mathematical peculiarities and opportunity for application. Taken together, they form a logically coherent whole.

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1

Normed vector spaces

The purpose of this chapter is to introduce key structural concepts that are needed for theoretical transform analysis and are part of the common language of modern signal processing and computer vision. One of the great insights of this approach is the recognition that natural abstractions which occur in analysis, algebra and geometry help to unify the study of the principal objects which occur in modern signal processing. Everything in this book takes place in a *vector space*, a linear space of objects closed under associative, distributive and commutative laws. The vector spaces we study include vectors in Euclidean and complex space and spaces of functions such as polynomials, integrable functions, approximation spaces such as wavelets and images, spaces of bounded linear operators and compression operators (infinite dimensional). We also need geometrical concepts such as distance and shortest (perpendicular) distance, and sparsity. This chapter first introduces important concepts of vector space and subspace which allow for general ideas of linear independence, span and basis to be defined. Span tells us for example, that a linear space may be generated from a smaller collection of its members by linear combinations. Thereafter, we discuss Riemann integrals and introduce the notion of a normed linear space and metric space. Metric spaces are spaces, nonlinear in general, where a notion of distance and hence limit makes sense. Normed spaces are generalizations of "absolute value" spaces. All normed spaces are metric spaces. The geometry of Euclidean space is founded on the familiar properties of length and angle. In Euclidean geometry, the angle between two vectors is specified in terms of the dot product, which itself is formalized by the notion of inner product. In this chapter, we introduce inner product space, completeness and Hilbert space with important examples. An inner product space is a generalization of a dot product space which preserves the concept of "perpendicular/orthonormal" or "shortest distance." It is a normed linear space satisfying a parallelogram law. Completeness means, roughly, closed under limiting processes and the most general function space admitting an inner product structure is a Hilbert Space. Hilbert spaces lie at the foundation of much of modern analysis, function theory and Fourier analysis, and provide the theoretical setting for modern signal processing. A Hilbert space is a complete inner product space. A basis is a spanning set which is linearly independent. We introduce orthonormal bases in finite- and infinite-dimensional Hilbert spaces and study bounded linear operators on Hilbert spaces. The characterizations of inner products on Euclidean space allows us to study least square and minimization approximations, singular values of matrices and ℓ_1 optimization. An important idea developed is of natural examples motivating an abstract theory which in turn leads to the ability to understand more complex objects but with the same underlying features.

1.1 Definitions

The most basic object in this text is a vector space V over a field $\mathbb{F}$, the latter being the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. The elements of $\mathbb{F}$ are called scalars. Vector spaces V or (linear spaces) over fields $\mathbb{F}$ capture the essential properties of $n \geq 1$ Euclidean space $\mathbb{V}^n$ which is the space of all real ($\mathbb{R}^n$) or complex ($\mathbb{C}^n$) column vectors with n entries closed under addition and scalar multiplication.

Definition 1.1 A vector space V over $\mathbb{F}$ is a collection of elements (vectors) with the following properties:[1]

- For every pair $u, v \in V$ there is defined a unique vector $w = u + v \in V$ (the sum of u and v)
- For every $\alpha \in \mathbb{F}$, $u \in V$ there is defined a unique vector $z = \alpha u \in V$ (product of α and u)
- Commutative, Associative and Distributive laws

  1. $u + v = v + u$
  2. $(u + v) + w = u + (v + w)$
  3. There exists a vector $\Theta \in V$ such that $u + \Theta = u$ for all $u \in V$

[1] We should strictly write $(V, \mathbb{F})$ since V depends on the field over which is defined. As this will be clear always, we suppress this notation.

4. For every $u \in V$ there is a $-u \in V$ such that $u + (-u) = \Theta$
5. $1u = u$ for all $u \in V$
6. $\alpha(\beta u) = (\alpha\beta)u$ for all $\alpha, \beta \in \mathbb{F}$
7. $(\alpha + \beta)u = \alpha u + \beta u$
8. $\alpha(u + v) = \alpha u + \alpha v$

Vector spaces are often called "linear" spaces. Given any two elements $u, v \in V$, by a linear combination of u, v we mean the sum $\alpha u + \beta v$ for any scalars α, β. Since V is a vector space, $\alpha u + \beta v \in V$ and is well defined. Θ is called the zero vector.

Examples 1.2 (a) As we have noted above, for $n \geq 1$, the space V_n is a vector space if given $u := (u_1, ..., u_n)$ and $v := (v_1, ..., v_n)$ in V_n we define $u + v := (u_1 + v_1, ..., u_n + v_n)$ and $cu := (cu_1, ..., cu_n)$ for any $c \in \mathbb{F}$.

(b) Let $n, m \geq 1$ and let $\mathcal{M}_{m\times n}$ denote the space of all real matrices of size $m \times n$. Then $\mathcal{M}_{m\times n}$ forms a real vector space under the laws of matrix addition and scalar multiplication. The Θ element is the matrix with all zero entries.

(c) Consider the space Π_n, $n \geq 1$ of real polynomials with degree $\leq n$. Then, with addition and scalar multiplication defined pointwise, Π_n becomes a real vector space. Note that the space of polynomials of degree equal to n is not closed under addition and so is not a vector space.[2]

(d) Let J be an arbitrary set and consider the space of functions $\mathcal{F}_V(J)$ as the space of all $f : J \to V$. Then defining addition and scalar multiplication by $(f + g)(x) := f(x) + g(x)$ and $(cf)(x) := cf(x)$ for $f, g \in \mathcal{F}_V$ and $x \in J$, $\mathcal{F}_V(J)$ is a vector space over the same field as V.

Sometimes, we are given a vector space V and a nonempty subset W of V that we need to study. It may happen that W is not closed under linear combinations. A nonempty subset of V will be called a subspace of V if it is closed under linear combinations of its elements with respect to the same field as V. More precisely we have

Definition 1.3 A nonempty set W in V is a subspace of V if $\alpha u + \beta v \in W$ for all $\alpha, \beta \in \mathbb{F}$ and $u, v \in W$.

[2] Indeed, consider $(x^3 + 100) + (-x^3 - x) = 100 - x$.

It's easy to prove that W is itself a vector space over $\mathbb{F}$ and contains in particular the zero element of of W. Here, V and the set $\{\Theta\}$ are called the trivial subspaces of V.

We now list some examples of subspaces and non subspaces, some of which are important in what follows, in order to remind the reader of this idea.

Examples 1.4 (a) Let $V_n = \mathbb{R}^n$ and scalars c_i, $1 \le i \le n$ be given. Then the half space consisting of all n-tuples $(u_1, \ldots, u_{n-1}, 0)$ with $u_i \in V_n$, $1 \le i \le n-1$ and the set of solutions $(v_1, \ldots, v_n) \in V_n$ to the homogeneous linear equation

$$c_1 x_1 + \cdots + c_n x_n = 0$$

are each nontrivial subspaces.

(b) $C^{(n)}[a, b]$: The space of all complex-valued functions with continuous derivatives of orders $0, 1, 2, \ldots n$ on the closed, bounded interval $[a, b]$ of the real line. Let $t \in [a, b]$, i.e., $a \le t \le b$ with $a < b$. Vector addition and scalar multiplication of functions $u, v \in C^{(n)}[a, b]$ are defined by

$$[u + v](t) = u(t) + v(t) \qquad [\alpha u](t) = \alpha u(t).$$

The zero vector is the function $\Theta(t) \equiv 0$.

(c) $S(I)$: The space of all complex-valued step functions on the (bounded or unbounded) interval I on the real line.[3] s is a step function on J if there are a finite number of non-intersecting bounded intervals $I_1, \ldots, I_m$ and complex numbers $c_1, \ldots, c_m$ such that $s(t) = c_k$ for $t \in I_k$, $k = 1, \ldots, m$ and $s(t) = 0$ for $t \in I - \cup_{k=1}^m I_k$. Vector addition and scalar multiplication of step functions $s_1, s_2 \in S(I)$ are defined by

$$[s_1 + s_2](t) = s_1(t) + s_2(t) \qquad [\alpha s_1](t) = \alpha s_1(t).$$

(One needs to check that $s_1 + s_2$ and αs_1 are step functions.) The zero vector is the function $\Theta(t) \equiv 0$.

[3] Intervals I are the only connected subsets of $\mathbb{R}$ of the form:

- closed, meaning $[a, b] := \{x \in \mathbb{R} : a \le x \le b\}$
- open, meaning $(a, b) := \{x \in \mathbb{R} : a < x < b\}$
- half open, meaning $[a, b)$ or $(a, b]$ where

$$[a, b) := \{x \in \mathbb{R} : a \le x < b\}$$

and $(a, b]$ is similarly defined. If either a or b is $\pm\infty$, then J is open at a or b and J is unbounded. Otherwise, it is bounded.

(d) $A(I)$: The space of all analytic functions on an open interval I. Here, we recall that f is analytic on I if all its $n \geq 1$ orders derivatives exist and are finite on I and given any fixed $a \in I$, the series $\sum_{n=0}^{\infty} f^{(n)}(a)$ $(x-a)^n/n!$ converges to $f(x)$ for all x close enough to a.

(e) For clarity, we add a few examples of sets which are not subspaces. We consider V_3 for simplicity as our underlying vector space.

- The set of all vectors of the form $(u_1, u_2, 1) \in \mathbb{R}^3$. Note that $(0, 0, 0)$ is not in this set.
- The positive octant $\{(u_1, u_2, u_3) : u_i \geq 0, 1 \leq i \leq 3\}$. Note that this set is not closed under multiplication by negative scalars. [4]

We now show that given any finite collection of vectors say $u^{(1)}, u^{(2)}, \ldots, u^{(m)}$ in V for some $m \geq 1$, it is always possible to construct a subspace of V containing $u^{(1)}, u^{(2)}, \ldots, u^{(m)}$ and, moreover, this is the smallest (nontrivial) subspace of V containing all these vectors. Indeed, we have

Lemma 1.5 *Let $u^{(1)}, u^{(2)}, \ldots, u^{(m)}$ be a set of vectors in the vector space V. Denote by $[u^{(1)}, u^{(2)}, \ldots, u^{(m)}]$ the set of all vectors of the form $\alpha_1 u^{(1)} + \alpha_2 u^{(2)} + \cdots + \alpha_m u^{(m)}$ for $\alpha_i \in \mathbb{F}$. The set $[u^{(1)}, u^{(2)}, \ldots, u^{(m)}]$, called the span of the set $\{u^{(1)}, ..., u^{(m)}\}$, is the smallest subspace of V containing $u^{(1)}, u^{(2)}, \ldots, u^{(m)}$.*

Proof Let $u, v \in [u^{(1)}, u^{(2)}, \ldots, u^{(m)}]$. Thus,

$$u = \sum_{i=1}^{m} \alpha_i u^{(i)}, \qquad v = \sum_{i=1}^{m} \beta_i u^{(i)}$$

so

$$\alpha u + \beta v = \sum_{i=1}^{m} (\alpha \alpha_i + \beta \beta_i) u^{(i)} \in [u^{(1)}, u^{(2)}, \ldots, u^{(m)}].$$

Clearly any subspace of V, containing $u^{(1)}, u^{(2)}, \ldots, u^{(m)}$ will contain $[u^{(1)}, u^{(2)}, \ldots, u^{(m)}]$. □

Note that spanning sets in vector spaces generalize the geometric notion of two vectors spanning a plane in $\mathbb{R}^3$. We now present three definitions of linear independence, dimensionality and basis and a useful characterization of a basis. We begin with the idea of linear independence. Often, all the vectors used to form a spanning set are essential.

[4] In fact, see Exercise 1.7, it is instructive to show that there are only two nontrivial subspaces of $\mathbb{R}^3$. (1) A plane passing through the origin and (2) a line passing through the origin.

For example, if we wish to span a plane in $\mathbb{R}^3$, we cannot use fewer than two vectors since the span of one vector is a line and thus not a plane. In some problems, however, some elements of a spanning set may be redundant. For example, we only need one vector to describe a line in $\mathbb{R}^3$ but if we are given two vectors which are parallel then their span is a line and so we really only need one of these to prescribe the line. The idea of removing superfluous elements from a spanning set leads to the idea of *linear dependence* which is given below.

Definition 1.6 The elements $u^{(1)}, u^{(2)}, \ldots, u^{(p)}$ of V are *linearly independent* if the relation $\alpha_1 u^{(1)} + \alpha_2 u^{(2)} + \cdots + \alpha_p u^{(p)} = \Theta$ for $\alpha_i \in \mathbb{F}$ holds *only* for $\alpha_1 = \alpha_2 = \cdots = \alpha_p = 0$. Otherwise $u^{(1)}, \ldots, u^{(p)}$ are *linearly dependent.*

Examples 1.7 (a) Any collection of vectors which includes the zero vector is linearly dependent.

(b) Two vectors are linearly dependent iff they are parallel. Indeed, if $u^{(1)} = cu^{(2)}$ for some $c \in \mathbb{F}$ and vectors $u^{(1)}, u^{(2)} \in V$, then $u^{(1)} - cu^{(2)} = \Theta$ is a nontrivial linear combination of $u^{(1)}$ and $u^{(2)}$ summing to Θ. Conversely, if $c_1 u^{(1)} + c_2 u^{(2)} = \Theta$ and $c_1 \neq 0$, then $u^{(1)} = -(c_2/c_1)u^{(2)}$ and if $c_1 = 0$ and $c_2 \neq 0$, then $u^{(2)} = \Theta$.

(c) The basic monomials $\{1, x, x^2, x^3, ..., x^n\}$ are linearly independent. See Exercise 1.9.

(d) The set of quadratic trigonometric functions

$$\{1, \cos x, \sin x, \cos^2 x, \cos x \sin x, \sin^2 x\}$$

is linearly dependent. Hint: use the fact that $\cos^2 x + \sin^2 x = 1$.

Next, we have

Definition 1.8 V is n-dimensional if there exist n linearly independent vectors in V and any $n + 1$ vectors in V are linearly dependent.

Definition 1.9 V is finite dimensional if V is n-dimensional for some integer n. Otherwise V is infinite dimensional.

For example, V_n is finite dimensional and all of the spaces $C^{(n)}[a, b]$, $S(I)$ and $A(I)$ are infinite dimensional. Next, we define the concept of a basis. In order to span a vector space or subspace, we know that we must use a sufficient number of distinct elements. On the other hand, we also know that including too many elements in a spanning set causes problems with linear independence. Optimal spanning sets are called bases. *Bases* are fundamental in signal processing, linear algebra, data compression, imaging, control and many other areas of research. We have

Definition 1.10 If there exist vectors $u^{(1)}, \ldots, u^{(n)}$, linearly independent in V and such that every vector $u \in V$ can be written in the form

$$u = \alpha_1 u^{(1)} + \alpha_2 u^{(2)} + \cdots + \alpha_n u^{(n)}, \qquad \alpha_i \in \mathbb{F},$$

($\{u^{(1)}, \ldots, u^{(n)}\}$ *spans* V), then V is n-dimensional. Such a set $\{u^{(1)}, \ldots, u^{(n)}\}$ is called a basis for V.

The following theorem gives a useful characterization of a basis.

Theorem 1.11 *Let V be an n-dimensional vector space and $u^{(1)}, \ldots, u^{(n)}$ a linearly independent set in V. Then $u^{(1)}, \ldots, u^{(n)}$ is a basis for V and every $u \in V$ can be written uniquely in the form*

$$u = \beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_n u^{(n)}.$$

Proof Let $u \in V$. Then the set $u^{(1)}, \ldots, u^{(n)}, u$ is linearly dependent. Thus there exist $\alpha_1, \cdots, \alpha_{n+1} \in \mathbb{F}$, not all zero, such that

$$\alpha_1 u^{(1)} + \alpha_2 u^{(2)} + \cdots + \alpha_n u^{(n)} + \alpha_{n+1} u = \Theta.$$

If $\alpha_{n+1} = 0$ then $\alpha_1 = \cdots = \alpha_n = 0$. But this cannot happen, so $\alpha_{n+1} \neq 0$ and hence

$$u = \beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_n u^{(n)}, \qquad \beta_i = -\frac{\alpha_i}{\alpha_{n+1}}.$$

Now suppose

$$u = \beta_1 u^{(1)} + \beta_2 u^{(2)} + \cdots + \beta_n u^{(n)} = \gamma_1 u^{(1)} + \gamma_2 u^{(2)} + \cdots + \gamma_n u^{(n)}.$$

Then

$$(\beta_1 - \gamma_1) u^{(1)} + \cdots + (\beta_n - \gamma_n) u^{(n)} = \Theta.$$

But the u_i form a linearly independent set, so $\beta_1 - \gamma_1 = 0, \ldots, \beta_n - \gamma_n = 0$. □

Examples 1.12 • V_n: A standard basis is:

$$e^{(1)} = (1, 0, \ldots, 0),\ e^{(2)} = (0, 1, 0, \ldots, 0),\ \ldots,\ e^{(n)} = (0, 0, \ldots, 1).$$

Proof

$$(\alpha_1, \ldots, \alpha_n) = \alpha_1 e^{(1)} + \cdots + \alpha_n e^{(n)},$$

so the vectors span. They are linearly independent because

$$(\beta_1, \cdots, \beta_n) = \beta_1 e^{(1)} + \cdots + \beta_n e^{(n)} = \Theta = (0, \cdots, 0)$$

if and only if $\beta_1 = \cdots = \beta_n = 0$. □

- V_∞, the space of all (real or complex) infinity-tuples

$$(\alpha_1, \alpha_2, \ldots, \alpha_n, \cdots).$$

1.2 Inner products and norms

The geometry of Euclidean space is built on properties of length and angle. The abstract concept of a norm on a vector space formalizes the geometrical notion of the length of a vector. In Euclidean geometry, the angle between two vectors is specified by their dot product which is itself formalized by the concept of inner product. As we shall see, norms and inner products are basic in signal processing. As a warm up, in this section, we will prove one of the most important inequalities in the theory, namely the Cauchy–Schwarz inequality, valid in any inner product space. The more familiar triangle inequality, follows from the definition of a norm. Complete inner product spaces, Hilbert spaces, are fundamental in what follows.

Definition 1.13 A vector space $\mathcal{N}$ over $\mathbb{F}$ is a normed linear space (pre Banach space) if to every $u \in \mathcal{N}$ there corresponds a real scalar $||u||$ (called the norm of u) such that

1. $||u|| \geq 0$ and $||u|| = 0$ if and only if $u = 0$.
2. $||\alpha u|| = |\alpha|\ ||u||$ for all $\alpha \in \mathbb{F}$.
3. Triangle inequality. $||u + v|| \leq ||u|| + ||v||$ for all $u, v \in \mathcal{N}$.

Assumption 3 is a generalization of the familiar triangle inequality in $\mathbb{R}^2$ that the length of any side of a triangle is bounded by the sum of the lengths of the other sides. This fact is actually an immediate consequence of the *Cauchy–Schwarz* inequality which we will state and prove later in this chapter.

Examples 1.14 • $C^{(n)}[a, b]$: Set of all complex-valued functions with continuous derivatives of orders $0, 1, 2, \ldots, n$ on the closed interval $[a, b]$ of the real line. Let $t \in [a, b]$, i.e., $a \leq t \leq b$ with $a < b$. Vector addition and scalar multiplication of functions $u, v \in C^{(n)}[a, b]$ are defined by

$$[u + v](t) = u(t) + v(t), \qquad [\alpha u](t) = \alpha u(t).$$

The zero vector is the function $\Theta(t) \equiv 0$. We defined this space earlier, but now we provide it with a norm defined by $||u|| = \int_a^b |u(t)|\, dt$.