CONVEX BODIES: THE BRUNN–MINKOWSKI THEORY

At the heart of this monograph is the Brunn–Minkowski theory, which can be used to great effect in studying such ideas as volume and surface area and their generalizations. In particular, the notions of mixed volume and mixed area measure arise naturally and the fundamental inequalities that are satisfied by mixed volumes are considered here in detail.

The author presents a comprehensive introduction to convex bodies, including full proofs for some deeper theorems. The book provides hints and pointers to connections with other fields and an exhaustive reference list.

This second edition has been considerably expanded to reflect the rapid developments of the past two decades. It includes new chapters on valuations on convex bodies, on extensions like the L_p Brunn–Minkowski theory, and on affine constructions and inequalities. There are also many supplements and updates to the original chapters, and a substantial expansion of chapter notes and references.

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Encyclopedia of Mathematics and its Applications

Convex Bodies: The Brunn–Minkowski Theory

Second Expanded Edition

ROLF SCHNEIDER
Albert-Ludwigs-Universität Freiburg, Germany

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Preface to the second edition

Wie machen wirs, daß alles frisch und neu
Und mit Bedeutung auch gefällig sei?

Goethe, *Faust I*

The past 20 years have seen considerable progress and lively activity in various different areas of convex geometry. In order that this book still meet its intended purpose, it had to be updated and expanded. It remains the aim of the book to serve the newcomer to the field who wants an introduction from the very beginning, as well as the experienced reader who is either doing research in the field or is looking for some special result to be used elsewhere. In the introductory parts of the book, no greater changes have been necessary, but already here recent developments are reflected in a number of supplements. The main additions to the book are three new chapters, on valuations, on extensions and analogues of the Brunn–Minkowski theory, and on affine constructions and inequalities in the theory of convex bodies. The contents of Chapter 7 from the first edition are now found in Chapters 8 and 10 of the second edition, considerably extended. The structure of some other chapters has also been changed by, for example, dividing them into subsections, regrouping some material, or adding a new section. A few more technical proofs, which had been carried out in the first edition, have been replaced by hints to the original literature.

While the new topics added to the book all have their origins in the Brunn–Minkowski theory, their natural intrinsic developments may gradually have led them farther away. Proofs in this book are restricted to results which may have been basic for further developments, but are still close to the classical Brunn–Minkowski theory. In the remaining parts, we survey many recent results without giving proofs, but we always provide references to the sources where the proofs can be found. The section notes contain additional information.

It has become clear that this book, even with its restriction to the Brunn–Minkowski theory and its ramifications, could never be exhaustive. Therefore, I felt comfortable with being brief in the treatment of projections and sections of convex bodies and related topics, subsumed under geometric tomography, having the chance to refer instead to the book by Gardner [675], which already exists in its second edition.

The Fourier analytic approach to sections and projections is fully covered by the books of Koldobsky [1136] and of Koldobsky and Yaskin [1142]. For all questions related to projections or sections of convex bodies, these are the three books the reader should consult.

It was also comforting to learn that the important developments in another active branch of convexity, the asymptotic theory, will be the subject of forthcoming books. With an easy conscience I could, therefore, refrain from inadequate attempts to cross my borders in this direction.

This second edition would not have come into life without Erwin Lutwak's friendly persuasion and without his invaluable help. In particular, he formed a team of young collaborators who produced a LaTeX version of the first edition, thus providing a highly appreciated technical basis for my later work. Special thanks for this very useful support go to Varvara Liti and Guangxian Zhu. An advanced version of the second edition was read by Franz Schuster, and selected chapters were read by Richard Gardner, Daniel Hug, Erwin Lutwak and Gaoyong Zhang. They all helped me with many useful comments and suggestions, for which I express my sincere thanks.

Freiburg i. Br., December 2012

Preface to the first edition

The Brunn–Minkowski theory is the classical core of the geometry of convex bodies. It originated with the thesis of Hermann Brunn in 1887 and is in its essential parts the creation of Hermann Minkowski, around the turn of the century. The well-known survey of Bonnesen and Fenchel in 1934 collected what was already an impressive body of results, though important developments were still to come, through the work of A. D. Aleksandrov and others in the thirties. In recent decades, the theory of convex bodies has expanded considerably; new topics have been developed and originally neglected branches of the subject have gained in interest. For instance, the combinatorial aspects, the theory of convex polytopes and the local theory of Banach spaces attract particular attention now. Nevertheless, the Brunn–Minkowski theory has remained of constant interest owing to its various new applications, its connections with other fields, and the challenge of some resistant open problems.

Aiming at a brief characterization of Brunn–Minkowski theory, one might say that it is the result of merging two elementary notions for point sets in Euclidean space: vector addition and volume. The vector addition of convex bodies, usually called Minkowski addition, has many facets of independent geometric interest. Combined with volume, it leads to the fundamental Brunn–Minkowski inequality and the notion of mixed volumes. The latter satisfy a series of inequalities which, due to their flexibility, solve many extremal problems and yield several uniqueness results. Looking at mixed volumes from a local point of view, one is led to mixed area measures. Quermassintegrals, or Minkowski functionals, and their local versions, surface area measures and curvature measures, are a special case of mixed volumes and mixed area measures. They are related to the differential geometry of convex hypersurfaces and to integral geometry.

Chapter 1 of the present book treats the basic properties of convex bodies and thus lays the foundations for subsequent developments. This chapter does not claim much originality; in large parts, it follows the procedures in standard books such as McMullen and Shephard [1398], Roberts and Varberg [1581] and Rockafellar [1583]. Together with Sections 2.1, 2.2, 2.4 and 2.5, it serves as a general introduction to the metric geometry of convex bodies. Chapter 2 is devoted to the boundary structure of convex bodies. Most of its material is needed later, except for Section[1] 2.6, on generic boundary structure, which just rounds off the picture. Minkowski addition is the subject of Chapter 3. Several different aspects are considered here such as decomposability, approximation problems with special regard to addition, additive maps and sums of segments. Quermassintegrals, which constitute a fundamental class of functionals on convex bodies, are studied in Chapter 4, where they are viewed as specializations of curvature measures, their local versions. For these, some integral-geometric formulae are established in Section 4.5. Here I try to follow the tradition set by Blaschke and Hadwiger, of incorporating parts of integral geometry into the theory of convex bodies. Some of this, however, is also a necessary prerequisite for Section 4.6. The remaining part of the book is devoted to mixed volumes and their applications. Chapter 5 develops the basic properties of mixed volumes and mixed area measures and treats special formulae, extensions, and analogues.[2] Chapter 6, the heart of the book, is devoted to the inequalities satisfied by mixed volumes, with special emphasis on improvements, the equality cases (as far as they are known) and stability questions. Chapter 7 presents a small selection of applications. The classical theorems of Minkowski and the Aleksandrov–Fenchel–Jessen theorem are treated here, the latter in refined versions. Section 7.4 serves as an overview of affine extremal problems for convex bodies. In this promising field, Brunn–Minkowski theory is of some use, but it appears that for the solution of some long-standing open problems new methods still have to be invented.

Concerning the choice of topics treated in this book, I wish to point out that it is guided by Minkowski's original work also in the following sense. Some subjects that Minkowski touched only briefly have later expanded considerably, and I pay special attention to these. Examples are projection bodies (zonoids), tangential bodies, the use of spherical harmonics in convexity and strengthenings of Minkowskian inequalities in the form of stability estimates.

The necessary prerequisites for reading this book are modest: the usual geometry of Euclidean space, elementary analysis, and basic measure and integration theory for Chapter 4. Occasionally, use is made of spherical harmonics; relevant information is collected in the Appendix. My intended attitude towards the presentation of proofs cannot be summarized better than by quoting from the preface to the book on Hausdorff measures by C. A. Rogers: 'As the book is largely based on lectures, and as I like my students to follow my lectures, proofs are given in great detail; this may bore the mature mathematician, but it will, I believe, be a great help to anyone trying to learn the subject *ab initio*.' On the other hand, some important results are stated as theorems but not proved, since this would lead us too far from the main theme, and no proofs are given in the survey sections 5.4, 6.8 and 7.4.

The notes at the end of nearly all sections are an essential part of the book. As a rule, this is where I have given references to original literature, considered questions

[1] Section numbers here refer to the first edition; they may differ in the second edition.
[2] This description of the chapters concerns the first edition. Beginning with Chapter 6, the second edition has a different structure.

of priority, made various comments and, in particular, given hints about applications, generalizations and ramifications. As an important purpose of the notes is to demonstrate the connections of convex geometry with other fields, some notes do take us further from the main theme of the book, mentioning, for example, infinite-dimensional results or non-convex sets or giving more detailed information on applications in, for instance, stochastic geometry.

The list of references does not have much overlap with the older bibliographies in the books by Bonnesen and Fenchel and by Hadwiger. Hence, a reader wishing to have a more complete picture should consult these bibliographies also, as well as those in the survey articles listed in part B of the References.

My thanks go to Sabine Linsenbold for her careful typing of the manuscript and to Daniel Hug who read the typescript and made many valuable comments and suggestions.

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Rolf Schneider
Frontmatter
More information

General hints to the literature

This book treats convex bodies with a special view to its classical part known as Brunn–Minkowski theory. To give the reader a wider picture of convex geometry, we collect here a list of textbooks and monographs, roughly ordered by category and year of publication. Each of the books listed under *'General'* has parts which can serve as an introduction to convex geometry, although at different levels and with different choices of topics in the advanced parts.

Classical

1916 Blaschke [241]: *Kreis und Kugel*
1934 Bonnesen and Fenchel [284]: *Theorie der Konvexen Körper*

General

1955 Hadwiger [908]: *Altes und Neues über Konvexe Körper*
1957 Hadwiger [911]: *Vorlesungen über Inhalt, Oberfläche und Isoperimetrie*
1958 Eggleston [532]: *Convexity*
1964 Valentine [1866]: *Convex Sets*
1966 Benson [193]: *Euclidean Geometry and Convexity*
1979 Kelly and Weiss [1070]: *Geometry and Convexity*
1980 Leichtweiß [1184]: *Konvexe Mengen*
1982 Lay [1178]: *Convex Sets and Their Applications*
1993 Schneider [1717]: *Convex Bodies: The Brunn–Minkowski Theory*
1994 Webster [1928]: *Convexity*
2002 Barvinok [169]: *A Course in Convexity*
2006 Berger [200]: *Convexité dans le Plan, dans l'Espace et Au-delà*
2006 Moszyńska [1452]: *Selected Topics in Convex Geometry*
2007 Gruber [834]: *Convex and Discrete Geometry*
2010 Berger [201]: Chapter VII in *Geometry Revealed*

Convex analysis, convex functions, related topics

1951 Fenchel [570]: *Convex Cones, Sets and Functions*
1970 Rockafellar [1583]: *Convex Analysis*
1973 Roberts and Varberg [1581]: *Convex Functions*

1977 Marti [1331]: *Konvexe Analysis*
1994 Hörmander [988]: *Notions of Convexity*
2010 Borwein and Vanderwerff [305]: *Convex Functions: Constructions, Characterizations and Counterexamples*

Convex polytopes

1950 Aleksandrov [25]: *Konvexe Polyeder*
1967 Grünbaum [848]: *Convex Polytopes*
1971 McMullen and Shephard [1398]: *Convex Polytopes and the Upper Bound Conjecture*
1983 Brøndsted [338]: *An Introduction to Convex Polytopes*
1995 Ziegler [2079]: *Lectures on Polytopes*
2003 Grünbaum [849]: *Convex Polytopes,* 2nd edn

Particular aspects

1929 Bonnesen [282]: *Les Problèmes des Isopérimètres et des Isépiphanes*
1948 Aleksandrov [23]: *Die Innere Geometrie der Konvexen Flächen*
1951 Jaglom and Boltjanski [1032]: *Konvexe Figuren*
1956 Lyusternik [1310]: *Convex Figures and Polyhedra*
1958 Busemann [371]: *Convex Surfaces*
1969 Pogorelov [1538]: *Extrinsic Geometry of Convex Surfaces*
1977 Guggenheimer [867]: *Applicable Geometry: Global and Local Convexity*
1995 Gardner [672]: *Geometric Tomography*
1996 Groemer [800]: *Geometric Applications of Fourier Series and Spherical Harmonics*
1996 Thompson [1845]: *Minkowski Geometry*
1996 Zong [2081]: *Strange Phenomena in Convex and Discrete Geometry*
1998 Leichtweiß [1193]: *Affine Geometry of Convex Bodies*
2005 Koldobsky [1136]: *Fourier Analysis in Convex Geometry*
2006 Gardner [675]: *Geometric Tomography,* 2nd edn
2006 Zong [2082]: *The Cube: A Window to Convex and Discrete Geometry*
2008 Koldobsky and Yaskin [1142]: *The Interface between Convex Geometry and Harmonic Analysis*
2009 Zamfirescu [2053]: *The Majority in Convexity*

Collections

1963 Klee (ed.) [1113]: *Convexity*
1967 Fenchel (ed.) [571]: *Proceedings of the Colloquium on Convexity, Copenhagen 1965*
1979 Tölke, Wills (eds) [1848]: *Contributions to Geometry, Part 1: Geometric Convexity*
1983 Gruber, Wills (eds) [845] : *Convexity and Its Applications*

1985 Goodman, Lutwak, Malkevitch, Pollack (eds) [758]: *Discrete Geometry and Convexity*
1993 Gruber, Wills (eds) [846]: *Handbook of Convex Geometry*
1994 Bisztriczky, McMullen, Schneider, Ivić Weiss (eds) [232]: *Polytopes: Abstract, Convex and Computational*
2004 Brandolini, Colzani, Iosevich, Travaglini (eds) [328]: *Fourier Analysis and Convexity*

Conventions and notation

Here we fix our notation and collect some basic definitions. We shall work in n-dimensional real Euclidean vector space, $\mathbb{R}^n$, with origin o, standard scalar product $\langle \cdot, \cdot \rangle$, and induced norm $|\cdot|$. We do not distinguish formally between the vector space $\mathbb{R}^n$ and its corresponding affine space, although our alternating use of the words 'vector' and 'point' is deliberate and should support the reader's intuition. As long as we deal with Euclidean geometry, we also do not distinguish between $\mathbb{R}^n$ and its dual space, but use the standard scalar product to identify both spaces (being well aware of the fact that it is often considered more elegant not to make this identification).

As a rule, elements of $\mathbb{R}^n$ are denoted by lower-case letters, subsets by capitals, and real numbers by small Greek letters.

The vector $x \in \mathbb{R}^n$ is a *linear combination* of the vectors $x_1, \dots, x_k \in \mathbb{R}^n$ if $x = \lambda_1 x_1 + \dots + \lambda_k x_k$ with suitable $\lambda_1, \dots, \lambda_k \in \mathbb{R}$. If such λ_i exist with $\lambda_1 + \dots + \lambda_k = 1$, then x is an *affine combination* of $x_1, \dots, x_k$. For $A \subset \mathbb{R}^n$, $\operatorname{lin} A$ ($\operatorname{aff} A$) denotes the *linear hull* (*affine hull*) of A; this is the set of all linear (affine) combinations of elements of A and at the same time the smallest linear subspace (affine subspace) of $\mathbb{R}^n$ containing A. Points $x_1, \dots, x_k \in \mathbb{R}^n$ are *affinely independent* if none of them is an affine combination of the others, i.e., if

$$\sum_{i=1}^{k} \lambda_i x_i = o \quad \text{with } \lambda_i \in \mathbb{R} \text{ and } \quad \sum_{i=1}^{k} \lambda_i = 0$$

implies that $\lambda_1 = \dots = \lambda_k = 0$. This is equivalent to the linear independence of the vectors $x_2 - x_1, \dots, x_k - x_1$. We may also define a map $\tau : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}$ by $\tau(x) := (x, 1)$; then $x_1, \dots, x_k \in \mathbb{R}^n$ are affinely independent if and only if $\tau(x_1), \dots, \tau(x_k)$ are linearly independent.

For $x, y \in \mathbb{R}^n$ we write

$$[x, y] := \{(1 - \lambda)x + \lambda y : 0 \le \lambda \le 1\}$$

for the *closed segment* and

$$[x, y) := \{(1 - \lambda)x + \lambda y : 0 \le \lambda < 1\}$$

for a *half-open segment*, both with endpoints x, y. For $A, B \subset \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ we define

$$A + B := \{a + b : a \in A,\ b \in B\}, \qquad \lambda A := \{\lambda a : a \in A\},$$

and we write $-A$ for $(-1)A$, $A - B$ for $A + (-B)$ and $A + x$ for $A + \{x\}$, where $x \in \mathbb{R}^n$. The set $A + B$ is written $A \oplus B$ and called the *direct sum* of A and B if A and B are contained in complementary affine subspaces of $\mathbb{R}^n$. A set A is called *o-symmetric* (or *centred*) if $A = -A$.

By $\operatorname{cl} A$, $\operatorname{int} A$, $\operatorname{bd} A$ we denote, respectively, the closure, interior and boundary of a subset A of a topological space. For $A \subset \mathbb{R}^n$, the sets $\operatorname{relint} A$, $\operatorname{relbd} A$ are the relative interior and relative boundary, that is, the interior and boundary of A relative to its affine hull.

The scalar product in $\mathbb{R}^n$ will often be used to describe hyperplanes and halfspaces. A *hyperplane* of $\mathbb{R}^n$ can be written in the form

$$H_{u,\alpha} = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha\}$$

with $u \in \mathbb{R}^n \setminus \{o\}$ and $\alpha \in \mathbb{R}$; here $H_{u,\alpha} = H_{v,\beta}$ if and only if $(v, \beta) = (\lambda u, \lambda\alpha)$ with $\lambda \neq 0$. (We warn the reader that for 'supporting hyperplane' we shall often say 'support plane', for short.) We say that u is a *normal vector* of $H_{u,\alpha}$. The hyperplane $H_{u,\alpha}$ bounds the two *closed halfspaces*

$$H^-_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \leq \alpha\}, \qquad H^+_{u,\alpha} := \{x \in \mathbb{R}^n : \langle x, u \rangle \geq \alpha\}.$$

Occasionally we also use $\langle \cdot, \cdot \rangle$ to denote the scalar product on $\mathbb{R}^n \times \mathbb{R}$ which is given by

$$\langle (x, \xi), (y, \eta) \rangle = \langle x, y \rangle + \xi\eta.$$

An affine subspace of $\mathbb{R}^n$ is often called a *flat*, and the intersection of a flat with a closed halfspace meeting the flat but not entirely containing it will be called a *half-flat*. A one-dimensional flat is a *line* and a one-dimensional half-flat a *ray*.

Concerning set-theoretic notation, we point out that we use the inclusion symbol $\subset$ always in the meaning of $\subseteq$, and that for a subset $A \subset \mathbb{R}^n$ we must well distinguish between its *characteristic function*, defined by

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{for } x \in A, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus A, \end{cases}$$

and its *indicator function*, given by

$$I^\infty_A(x) := \begin{cases} 0 & \text{for } x \in A, \\ \infty & \text{for } x \in \mathbb{R}^n \setminus A. \end{cases}$$

If $P(x)$ is an assertion about the elements x of a set A, we use also the notation

$$\mathbf{1}\{P(x)\} := \begin{cases} 1 & \text{if } P(x) \text{ is true}, \\ 0 & \text{if } P(x) \text{ is false}. \end{cases}$$

The following metric notions will be used. For $x, y \in \mathbb{R}^n$ and $\emptyset \neq A \subset \mathbb{R}^n$, $|x - y|$ is the *distance* between x and y and

$$d(A, x) := \inf\{|x - a| : a \in A\}$$

is the distance of x from A. For a bounded set $\emptyset \neq A \subset \mathbb{R}^n$,

$$\operatorname{diam} A := \sup\{|x - y| : x, y \in A\}$$

is the *diameter* of A. It is also denoted by $D(A)$. We write

$$B(z, \rho) := \{x \in \mathbb{R}^n : |x - z| \leq \rho\}$$

and

$$B_0(z, \rho) := \{x \in \mathbb{R}^n : |x - z| < \rho\},$$

respectively, for the closed ball and the open ball with centre $x \in \mathbb{R}^n$ and radius $\rho > 0$. The set $B^n := B(o, 1)$ is the *unit ball* and

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$$

is the *unit sphere* of $\mathbb{R}^n$.

By $\mathcal{H}^k$ we denote the k-dimensional Hausdorff (outer) measure on $\mathbb{R}^n$, where $0 \leq k \leq n$. If A is a Borel subset of a k-dimensional flat E^k or a k-dimensional sphere $\mathbb{S}^k$ in $\mathbb{R}^n$, then $\mathcal{H}^k(A)$ coincides, respectively, with the k-dimensional Lebesgue measure of A computed in E^k, or with the k-dimensional spherical Lebesgue measure of A computed in $\mathbb{S}^k$. Hence, all integrations with respect to these Lebesgue measures can be expressed by means of the Hausdorff measure $\mathcal{H}^k$. In integrals with respect to $\mathcal{H}^n$ we often abbreviate $\mathrm{d}\mathcal{H}^n(x)$ by $\mathrm{d}x$. Similarly, in integrals over the unit sphere $\mathbb{S}^{n-1}$, instead of $\mathrm{d}\mathcal{H}^{n-1}(u)$ we write $\mathrm{d}u$. Occasionally, spherical Lebesgue measure on $\mathbb{S}^{n-1}$ is denoted by σ. The n-dimensional measure of the unit ball in $\mathbb{R}^n$ is denoted by κ_n and its surface area by ω_n, thus

$$\kappa_n = \mathcal{H}^n(B^n) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(1 + \frac{n}{2}\right)}, \quad \omega_n = \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\kappa_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}.$$

(In choosing this notation, we follow Bonnesen and Fenchel [284], which has, unfortunately, not set a standard.) We use the definition $\kappa_p = \pi^{p/2}/\Gamma(1 + p/2)$ for arbitrary $p \geq 0$, not just for positive integers.

Linear maps, affine maps and isometries between Euclidean spaces are defined as usual. In particular, a map $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a *translation* if $\varphi(x) = x + t$ for $x \in \mathbb{R}^n$ with some fixed vector $t \in \mathbb{R}^n$, the *translation vector*. The set $A + t$ is called the *translate* of A by t. The map φ is a *dilatation* if $\varphi(x) = \lambda x$ for $x \in \mathbb{R}^n$ with some $\lambda > 0$. The set λA with $\lambda > 0$ is called a *dilatate* of A. The map φ is a *homothety* if $\varphi(x) = \lambda x + t$ for $x \in \mathbb{R}^n$ with some $\lambda > 0$ and some $t \in \mathbb{R}^n$. The set $\lambda A + t$ with $\lambda > 0$ is called a *homothet* of A. Sets A, B are called *positively homothetic* if $A = \lambda B + t$ with $t \in \mathbb{R}^n$ and $\lambda > 0$, and *homothetic* if either they are positively homothetic or one of them is a singleton (a one-pointed set). A *rigid motion* of $\mathbb{R}^n$ is an isometry of $\mathbb{R}^n$ onto itself, and it is a *rotation* if it is an isometry fixing the origin. Every rigid motion is the composition of a rotation and a translation. A rigid motion is called *proper* if it preserves the orientation of $\mathbb{R}^n$; otherwise it is called *improper*. A rotation of $\mathbb{R}^n$ is a linear map; it preserves the scalar product and can be represented, with respect to an orthonormal basis, by an orthogonal matrix; this matrix has determinant 1 if and only if the rotation is proper. The composition of a rigid motion and a dilatation is called a *similarity*.

By $\mathrm{GL}(n)$ we denote the general linear group of $\mathbb{R}^n$ and by $\mathrm{SL}(n)$ the subgroup of volume-preserving and orientation-preserving linear mappings. The subgroup $\mathrm{SO}(n)$ is the group of proper rotations. With the topology induced by the usual matrix norm, both are topological groups, and $\mathrm{SO}(n)$ is compact. The group of proper rigid motions of $\mathbb{R}^n$ is denoted by G_n and topologized as usual. Also, the Grassmannian $G(n,k)$ of k-dimensional linear subspaces of $\mathbb{R}^n$ and the set $A(n,k)$ of k-dimensional affine subspaces of $\mathbb{R}^n$ are endowed with their standard topologies.

The Haar measures on $\mathrm{SO}(n)$, G_n, $G(n,k)$, $A(n,k)$ are denoted, respectively, by ν, μ, ν_k, μ_k. We normalize ν by $\nu(\mathrm{SO}(n)) = 1$ and ν_k by $\nu_k(G(n,k)) = 1$. The normalizations of the measures μ and μ_k will be fixed in Section 4.4 when they are needed.

For an affine subspace E of $\mathbb{R}^n$, we denote by proj_E the orthogonal projection from $\mathbb{R}^n$ onto E. We often write $\mathrm{proj}_E A =: A \,|\, E$ for $A \subset \mathbb{R}^n$ (since A is a set, no confusion with the restriction of a function, for example $f|E$, can arise).

Some final remarks are in order. Since any k-dimensional affine subspace E of $\mathbb{R}^n$ is the image of $\mathbb{R}^k$ under some isometry, it is clear (and common practice without mention) that all notions and results that have been established for $\mathbb{R}^k$ and are invariant under isometries can be applied in E; similarly for affine-invariant notions and results.

The following notational conventions will be useful at several places. If f is a homogeneous function on $\mathbb{R}^n$, then $\bar{f}$ denotes its restriction to the unit sphere $\mathbb{S}^{n-1}$. Very often, mappings of the type $f : \mathcal{K} \times \mathbb{R}^n \to M$ will occur where $\mathcal{K}$ is some class of subsets of $\mathbb{R}^n$. In this case we usually abbreviate, for fixed $K \in \mathcal{K}$, the function $f(K,\cdot) : \mathbb{R}^n \to M$ by f_K.

We wish to point out that in definitions the word ‘if’ is always understood as ‘if and only if’.

Finally, a remark about citations. When we list several publications consecutively, particularly in the chapter notes, we usually order them chronologically, and not in the order in which they appear in the list of references.