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Basic convexity

1.1 Convex sets and combinations

A set $A \subset \mathbb{R}^n$ is *convex* if together with any two points x, y it contains the segment $[x, y]$, thus if

$$(1-\lambda)x + \lambda y \in A \quad \text{for } x, y \in A,\ 0 \le \lambda \le 1.$$

Examples of convex sets are obvious; but observe also that $B_0(z, \rho) \cup A$ is convex if A is an arbitrary subset of the boundary of the open ball $B_0(z, \rho)$. As immediate consequences of the definition we note that intersections of convex sets are convex, affine images and pre-images of convex sets are convex, and if A, B are convex, then $A + B$ and λA ($\lambda \in \mathbb{R}$) are convex.

Remark 1.1.1 For $A \subset \mathbb{R}^n$ and $\lambda, \mu > 0$ one trivially has $\lambda A + \mu A \supset (\lambda + \mu)A$. Equality (for all $\lambda, \mu > 0$) holds precisely if A is convex. In fact, if A is convex and $x \in \lambda A + \mu A$, then $x = \lambda a + \mu b$ with $a, b \in A$ and hence

$$x = (\lambda + \mu)\left(\frac{\lambda}{\lambda + \mu}a + \frac{\mu}{\lambda + \mu}b\right) \in (\lambda + \mu)A;$$

thus $\lambda A + \mu A = (\lambda + \mu)A$. If this equation holds, then A is clearly convex.

A set $A \subset \mathbb{R}^n$ is called a *convex cone* if A is convex and nonempty and if $x \in A$, $\lambda \ge 0$ implies $\lambda x \in A$. Thus, a nonempty set $A \subset \mathbb{R}^n$ is a convex cone if and only if A is closed under addition and under multiplication by nonnegative real numbers.

By restricting affine and linear combinations to nonnegative coefficients, one obtains the following two fundamental notions. The point $x \in \mathbb{R}^n$ is a *convex combination* of the points $x_1, \dots, x_k \in \mathbb{R}^n$ if there are numbers $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i \ge 0\ (i = 1, \dots, k),\ \sum_{i=1}^{k} \lambda_i = 1.$$

Similarly, the vector $x \in \mathbb{R}^n$ is a *positive combination* of vectors $x_1, \dots, x_k \in \mathbb{R}^n$ if

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i \geq 0 \ (i = 1, \dots, k).$$

For $A \subset \mathbb{R}^n$, the set of all convex combinations (positive combinations) of any finitely many elements of A is called the *convex hull (positive hull)* of A and is denoted by $\operatorname{conv} A$ ($\operatorname{pos} A$).

Theorem 1.1.2 *If $A \subset \mathbb{R}^n$ is* convex*, then* $\operatorname{conv} A = A$. *For an arbitrary set $A \subset \mathbb{R}^n$,* $\operatorname{conv} A$ *is the intersection of all convex subsets of $\mathbb{R}^n$ containing A. If $A, B \subset \mathbb{R}^n$, then* $\operatorname{conv}(A + B) = \operatorname{conv} A + \operatorname{conv} B$.

Proof Let A be convex. Trivially, $A \subset \operatorname{conv} A$. By induction we show that A contains all convex combinations of any k points of A. For $k = 2$ this holds by the definition of convexity. Suppose that it holds for $k - 1$ and that $x = \lambda_1 x_1 + \cdots + \lambda_k x_k$ with $x_1, \dots, x_k \in A$, $\lambda_1 + \cdots + \lambda_k = 1$ and $\lambda_1, \dots, \lambda_k > 0$, without loss of generality. Then

$$x = (1 - \lambda_k) \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i + \lambda_k x_k \in A,$$

since

$$\frac{\lambda_i}{1 - \lambda_k} > 0, \quad \sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} = 1$$

and hence

$$\sum_{i=1}^{k-1} \frac{\lambda_i}{1 - \lambda_k} x_i \in A,$$

by hypothesis. This proves that $A = \operatorname{conv} A$. For arbitrary $A \subset \mathbb{R}^n$, let $C(A)$ be the intersection of all convex sets $K \subset \mathbb{R}^n$ containing A. Since $A \subset \operatorname{conv} A$ and $\operatorname{conv} A$ is evidently convex, we have $C(A) \subset \operatorname{conv} A$. Every convex set K with $A \subset K$ satisfies $\operatorname{conv} A \subset \operatorname{conv} K = K$, hence $\operatorname{conv} A \subset C(A)$, which proves the equality.

Let $A, B \subset \mathbb{R}^n$. Let $x \in \operatorname{conv}(A + B)$, thus

$$x = \sum_{i=1}^{k} \lambda_i (a_i + b_i) \quad \text{with } a_i \in A,\ b_i \in B,\ \lambda_i \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1$$

and hence $x = \sum \lambda_i a_i + \sum \lambda_i b_i \in \operatorname{conv} A + \operatorname{conv} B$. Let $x \in \operatorname{conv} A + \operatorname{conv} B$, thus

$$x = \sum_i \lambda_i a_i + \sum_j \mu_j b_j$$

with $a_i \in A$, $b_j \in B$, $\lambda_i, \mu_j \geq 0$, $\sum \lambda_i = \sum \mu_j = 1$. We may write

$$x = \sum_{i,j} \lambda_i \mu_j (a_i + b_j)$$

and deduce that $x \in \operatorname{conv}(A + B)$. □

An immediate consequence is that conv (conv A) = conv A.

Theorem 1.1.3 *If $A \subset \mathbb{R}^n$ is a convex cone, then* pos $A = A$. *For a nonempty set $A \subset \mathbb{R}^n$,* pos A *is the intersection of all convex cones in $\mathbb{R}^n$ containing A. If $A, B \subset \mathbb{R}^n$, then* pos $(A + B) \subset$ pos A + pos B.

Proof As above. □

That the last inclusion in the preceding theorem may be strict is shown by the example $A = \{a\}$, $B = \{b\}$ with linearly independent vectors $a, b \in \mathbb{R}^n$.

The following result on the generation of convex hulls is fundamental.

Theorem 1.1.4 (Carathéodory's theorem) *If $A \subset \mathbb{R}^n$ and $x \in$ conv A, then x is a convex combination of affinely independent points of A. In particular, x is a convex combination of $n + 1$ or fewer points of A.*

Proof The point $x \in$ conv A has a representation

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } x_i \in A,\ \lambda_i > 0,\ \sum_{i=1}^{k} \lambda_i = 1,$$

with some $k \in \mathbb{N}$, and we may assume that k is minimal. Suppose that $x_1, \ldots, x_k$ are affinely dependent. Then there are numbers $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We can choose m such that λ_m/α_m is positive and, with this restriction, as small as possible (observe that all λ_i are positive and at least one α_i is positive). In the affine representation

$$x = \sum_{i=1}^{k} \left(\lambda_i - \frac{\lambda_m}{\alpha_m} \alpha_i \right) x_i,$$

all coefficients are nonnegative (trivially, if $\alpha_i \le 0$, otherwise by the choice of m), and at least one of them is zero. This contradicts the minimality of k. Thus, $x_1, \ldots, x_k$ are affinely independent, which implies that $k \le n + 1$. □

The convex hull of finitely many points is called a *polytope*. A *k-simplex* is the convex hull of $k + 1$ affinely independent points, and these points are the *vertices* of the simplex. Thus, Carathéodory's theorem states that conv A is the union of all simplices with vertices in A.

Another equally simple and important result on convex hulls is the following.

Theorem 1.1.5 (Radon's theorem) *Every set of affinely dependent points (in particular, every set of at least $n + 2$ points) in $\mathbb{R}^n$ can be expressed as the union of two disjoint sets whose convex hulls have a common point.*

Proof If $x_1, \dots, x_k$ are affinely dependent, there are numbers $\alpha_1, \dots, \alpha_k \in \mathbb{R}$, not all zero, with

$$\sum_{i=1}^{k} \alpha_i x_i = o \quad \text{and} \quad \sum_{i=1}^{k} \alpha_i = 0.$$

We may assume, after renumbering, that $\alpha_i > 0$ precisely for $i = 1, \dots, j$; then $1 \le j < k$ (at least one α_i is $\neq 0$, say > 0, but not all α_i are > 0). With

$$\alpha := \alpha_1 + \cdots + \alpha_j = -(\alpha_{j+1} + \cdots + \alpha_k) > 0$$

we obtain

$$x := \sum_{i=1}^{j} \frac{\alpha_i}{\alpha} x_i = \sum_{i=j+1}^{k} \left(-\frac{\alpha_i}{\alpha}\right) x_i$$

and thus $x \in \operatorname{conv}\{x_1, \dots, x_j\} \cap \operatorname{conv}\{x_{j+1}, \dots, x_k\}$. The assertion follows. □

From Radon's theorem one easily deduces Helly's theorem, a fundamental and typical result of the combinatorial geometry of convex sets.

Theorem 1.1.6 (Helly's theorem) *Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{R}^n$. If any $n+1$ elements of $\mathcal{M}$ have a common point, then all elements of $\mathcal{M}$ have a common point.*

Proof Let $A_1, \dots, A_k$ be the sets of $\mathcal{M}$. Suppose that $k > n+1$ (for $k < n+1$ there is nothing to prove, and for $k = n+1$ the assertion is trivial) and that the assertion is proved for $k-1$ convex sets. Then for $i \in \{1, \dots, k\}$ there exists a point

$$x_i \in A_1 \cap \cdots \cap \check{A}_i \cap \cdots \cap A_k$$

where $\check{A}_i$ indicates that A_i has been deleted. The $k \ge n+2$ points $x_1, \dots, x_k$ are affinely dependent; hence from Radon's theorem we can infer that, after renumbering, there is a point

$$x \in \operatorname{conv}\{x_1, \dots, x_j\} \cap \operatorname{conv}\{x_{j+1}, \dots, x_k\}$$

for some $j \in \{1, \dots, k-1\}$. Because $x_1, \dots, x_j \in A_{j+1}, \dots, A_k$ we have

$$x \in \operatorname{conv}\{x_1, \dots, x_j\} \subset A_{j+1} \cap \cdots \cap A_k,$$

similarly $x \in \operatorname{conv}\{x_{j+1}, \dots, x_k\} \subset A_1 \cap \cdots \cap A_j$. □

Here is a little example (another one is Theorem 1.3.11) to demonstrate how Helly's theorem can be applied to obtain elegant results of a similar nature.

Theorem 1.1.7 *Let $\mathcal{M}$ be a finite family of convex sets in $\mathbb{R}^n$ and let $K \subset \mathbb{R}^n$ be convex. If any $n+1$ elements of $\mathcal{M}$ are intersected by some translate of K, then all elements of $\mathcal{M}$ are intersected by a suitable translate of K.*

Proof Let $\mathcal{M} = \{A_1, \dots, A_k\}$. To any $n + 1$ elements of $\{1, \dots, k\}$, say $1, \dots, n+1$, there are $t \in \mathbb{R}^n$ and $x_i \in A_i \cap (K+t)$, hence $-t \in K - A_i$, for $i = 1, \dots, n+1$. Thus, any $n+1$ elements of the family $\{K - A_1, \dots, K - A_k\}$ have nonempty intersection. By Helly's theorem, there is a vector $-t \in \mathbb{R}^n$ with $-t \in K - A_i$ and hence $A_i \cap (K+t) \neq \emptyset$ for $i \in \{1, \dots, k\}$. □

Some of these results of combinatorial convexity have 'colourful versions', of which we give a simple example.

Theorem 1.1.8 (Coloured Radon theorem) *Let $F_1, \dots, F_{n+1}$ be two-pointed sets in $\mathbb{R}^n$. Their union has a partition into sets A, B such that each of A, B contains a point from each of the sets $F_1, \dots, F_{n+1}$ ('one of each colour') and the convex hulls of A and B have a common point.*

Proof Let $F_i = \{x_i, y_i\}$, $i = 1, \dots, n + 1$. There is a non-trivial linear relation $\sum_{i=1}^{n+1} \alpha_i (x_i - y_i) = o$. Interchanging the notation for the elements of F_i where necessary, we can assume that $\alpha_i \geq 0$ for all i. After multiplication with a constant, we can also assume that $\sum_{i=1}^{n+1} \alpha_i = 1$. Then the relation

$$\sum_{i=1}^{n+1} \alpha_i x_i = \sum_{i=1}^{n+1} \alpha_i y_i$$

proves the assertion. □

Next we look at the interplay between convexity and topological properties. We start with a simple but useful observation.

Lemma 1.1.9 *Let $A \subset \mathbb{R}^n$ be convex. If $x \in \operatorname{int} A$ and $y \in \operatorname{cl} A$, then $[x, y) \subset \operatorname{int} A$.*

Proof Let $z = (1 - \lambda)y + \lambda x$ with $0 < \lambda < 1$. We have $B(x, \rho) \subset A$ for some $\rho > 0$; put $B(o, \rho) =: U$. First we assume $y \in A$. Let $w \in \lambda U + z$, hence $w = \lambda u + z$ with $u \in U$. Then $x + u \in A$, hence $w = (1 - \lambda)y + \lambda(x + u) \in A$. This shows that $\lambda U + z \subset A$ and thus $z \in \operatorname{int} A$.

Now we assume merely that $y \in \operatorname{cl} A$. Put $V := [\lambda/(1 - \lambda)]U + y$. There is some $a \in A \cap V$. We have $a = [\lambda/(1-\lambda)]u + y$ with $u \in U$ and hence $z = (1-\lambda)a + \lambda(x-u) \in A$. This proves that $[x, y) \subset A$, which together with the first part yields $[x, y) \subset \operatorname{int} A$. □

Theorem 1.1.10 *If $A \subset \mathbb{R}^n$ is convex, then* $\operatorname{int} A$ *and* $\operatorname{cl} A$ *are convex. If $A \subset \mathbb{R}^n$ is open, then* $\operatorname{conv} A$ *is open.*

Proof The convexity of $\operatorname{int} A$ follows from Lemma 1.1.9. The convexity of $\operatorname{cl} A$ for convex A and the openness of $\operatorname{conv} A$ for open A are easy exercises. □

The union of a line and a point not on it is an example of a closed set whose convex hull is not closed. This cannot happen for compact sets, as a first application of Carathéodory's theorem shows.

Theorem 1.1.11 *If $A \subset \mathbb{R}^n$, then* $\operatorname{conv} \operatorname{cl} A \subset \operatorname{cl} \operatorname{conv} A$. *If A is bounded, then* $\operatorname{conv} \operatorname{cl} A = \operatorname{cl} \operatorname{conv} A$. *In particular, the convex hull of a compact set is compact.*

Proof The inclusion $\operatorname{conv}\operatorname{cl} A \subset \operatorname{cl}\operatorname{conv} A$ is easy to see. Let A be bounded. Then

$$\left\{(\lambda_1, \ldots, \lambda_{n+1}, x_1, \ldots, x_{n+1}) : \lambda_i \geq 0,\ x_i \in \operatorname{cl} A,\ \sum_{i=1}^{n+1} \lambda_i = 1\right\}$$

is a compact subset of $\mathbb{R}^{n+1} \times (\mathbb{R}^n)^{n+1}$, hence its image under the continuous map

$$(\lambda_1, \ldots, \lambda_{n+1}, x_1, \ldots, x_{n+1}) \mapsto \sum_{i=1}^{n+1} \lambda_i x_i \in \mathbb{R}^n$$

is compact. By Carathéodory's theorem, this image is equal to $\operatorname{conv}\operatorname{cl} A$. Thus $\operatorname{cl}\operatorname{conv} A \subset \operatorname{cl}\operatorname{conv}\operatorname{cl} A = \operatorname{conv}\operatorname{cl} A$. □

The set $\operatorname{cl}\operatorname{conv} A$, which by Theorem 1.1.10 is convex, is called for short the *closed convex hull* of A. This is also the intersection of all closed convex subsets of $\mathbb{R}^n$ containing A.

To obtain information on the relative interiors of convex hulls, we first consider simplices.

Lemma 1.1.12 *Let $x_1, \ldots, x_k \in \mathbb{R}^n$ be affinely independent; let*

$$S := \operatorname{conv}\{x_1, \ldots, x_k\}$$

and $x \in \operatorname{aff} S$. Then $x \in \operatorname{relint} S$ if and only if in the unique affine representation

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with} \quad \sum_{i=1}^{k} \lambda_i = 1$$

all coefficients λ_i are positive.

Proof Clearly, we may assume that $k = n + 1$. The condition is necessary since otherwise, because the representation is unique, an arbitrary neighbourhood of x would contain points not belonging to S. To prove sufficiency, let x be represented as above with all $\lambda_i > 0$. Since $x_1, \ldots, x_{n+1}$ are affinely independent, the vectors $\tau(x_1), \ldots, \tau(x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ (where $\tau(x) := (x, 1)$) form a linear basis of $\mathbb{R}^n \times \mathbb{R}$, and for $y \in \mathbb{R}^n$ the coefficients $\mu_1, \ldots, \mu_{n+1}$ in the affine representation

$$y = \sum_{i=1}^{n+1} \mu_i x_i \quad \text{with} \quad \sum_{i=1}^{n+1} \mu_i = 1$$

(the 'barycentric coordinates' of y) are just the coordinates of $\tau(y)$ with respect to this basis. Since coordinate functions in $\mathbb{R}^{n+1}$ are continuous, the coefficients $\mu_1, \ldots, \mu_{n+1}$ depend continuously on y. Therefore, a number $\delta > 0$ can be chosen such that $\mu_i > 0$ $(i = 1, \ldots, n+1)$ and thus $y \in S$ for all y with $|y - x| < \delta$. This proves that $x \in \operatorname{int} S$. □

Theorem 1.1.13 *If $A \subset \mathbb{R}^n$ is convex and nonempty, then* $\operatorname{relint} A \neq \emptyset$.

Proof Let $\dim \operatorname{aff} A = k$, then there are $k + 1$ affinely independent points in A. Their convex hull S satisfies $\operatorname{relint} S \neq \emptyset$ by Lemma 1.1.12; furthermore, $S \subset A$ and $\operatorname{aff} S = \operatorname{aff} A$. □

In view of this theorem, it makes sense to define the *dimension*, $\dim A$, of a convex set A as the dimension of its affine hull. The points of $\operatorname{relint} A$ are also called *internal* points of A.

The description of $\operatorname{relint} \operatorname{conv} A$ for an affinely independent set A, given by Lemma 1.1.12, can be extended to arbitrary finite sets.

Theorem 1.1.14 *Let $x_1, \dots, x_k \in \mathbb{R}^n$, let $P := \operatorname{conv}\{x_1, \dots, x_k\}$ and $x \in \mathbb{R}^n$. Then $x \in \operatorname{relint} P$ if and only if x can be represented in the form*

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i > 0 \ (i = 1, \dots, k), \ \sum_{i=1}^{k} \lambda_i = 1.$$

Proof We may clearly assume that $\dim P = n$. Suppose that $x \in \operatorname{int} P$. Put

$$y := \sum_{i=1}^{k} \frac{1}{k} x_i,$$

then $y \in P$. Since $x \in \operatorname{int} P$, we can choose $z \in P$ for which $x \in [y, z)$. There are representations

$$z = \sum_{i=1}^{k} \mu_i x_i \quad \text{with } \mu_i \ge 0, \ \sum_{i=1}^{k} \mu_i = 1,$$

$$x = (1 - \lambda) y + \lambda z \quad \text{with } 0 \le \lambda < 1,$$

which gives

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i = (1 - \lambda)\frac{1}{k} + \lambda \mu_i > 0, \ \sum_{i=1}^{k} \lambda_i = 1.$$

Conversely, suppose that

$$x = \sum_{i=1}^{k} \lambda_i x_i \quad \text{with } \lambda_i > 0, \ \sum_{i=1}^{k} \lambda_i = 1.$$

We may assume that $x_1, \dots, x_{n+1}$ are affinely independent. Put $\lambda_1 + \dots + \lambda_{n+1} =: \lambda$ and

$$y := \sum_{i=1}^{n+1} \frac{\lambda_i}{\lambda} x_i.$$

Lemma 1.1.12 gives $y \in \operatorname{int} \operatorname{conv}\{x_1, \dots, x_{n+1}\} \subset \operatorname{int} P$. If $k = n + 1$, then $x = y \in \operatorname{int} P$. Otherwise, put

$$z := \sum_{i=n+1}^{k} \frac{\lambda_i}{1 - \lambda} x_i.$$

Then $z \in P$ and $x \in [y, z) \subset \operatorname{int} P$, by Lemma 1.1.9. □

Theorem 1.1.15 *Let $A \subset \mathbb{R}^n$ be convex. Then*

(a) $\operatorname{relint} A = \operatorname{relint} \operatorname{cl} A$,
(b) $\operatorname{cl} A = \operatorname{cl} \operatorname{relint} A$,
(c) $\operatorname{relbd} A = \operatorname{relbd} \operatorname{cl} A = \operatorname{relbd} \operatorname{relint} A$.

Proof We may assume that $\dim A = n$. Part (a): trivially, $\operatorname{int} A \subset \operatorname{int} \operatorname{cl} A$. Let $x \in \operatorname{int} \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. There is $z \in \operatorname{cl} A$ with $x \in [y, z)$ and Lemma 1.1.9 shows that $x \in \operatorname{int} A$. Part (b): trivially, $\operatorname{cl} A \supset \operatorname{cl} \operatorname{int} A$. Let $x \in \operatorname{cl} A$. Choose $y \in \operatorname{int} A$. By Lemma 1.1.9 we have $[y, x) \subset \operatorname{int} A$, hence $x \in \operatorname{cl} \operatorname{int} A$. Part (c): $\operatorname{bd} \operatorname{cl} A = \operatorname{cl} (\operatorname{cl} A) \setminus \operatorname{int} (\operatorname{cl} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (a). Then $\operatorname{bd} \operatorname{int} A = \operatorname{cl} (\operatorname{int} A) \setminus \operatorname{int} (\operatorname{int} A) = \operatorname{cl} A \setminus \operatorname{int} A = \operatorname{bd} A$, using (b). □

We end this section with a definition of the central notion of this book. A nonempty, compact, convex subset of $\mathbb{R}^n$ is called a *convex body*. (Thus, in our terminology, a convex body need not have interior points. We warn the reader that many authors reserve the term ‘body’ for sets with interior points. However, we prefer to avoid endless repetitions, in this book, of the expression ‘nonempty, compact, convex subset’.) By $\mathcal{K}^n$ we denote the set of all convex bodies in $\mathbb{R}^n$ and by $\mathcal{K}^n_n$ the subset of convex bodies with interior points (thus, the lower index n stands for the dimension of the bodies). For $\emptyset \neq A \subset \mathbb{R}^n$ we write $\mathcal{K}(A)$ for the set of convex bodies contained in A and $\mathcal{K}_n(A) = \mathcal{K}(A) \cap \mathcal{K}^n_n$. Further, $\mathcal{P}^n$ denotes the set of nonempty polytopes in $\mathbb{R}^n$ and $\mathcal{P}^n_n = \mathcal{P}^n \cap \mathcal{K}^n_n$ is the subset of n-dimensional polytopes.

Notes for Section 1.1

1. The early history of the theorems of Carathéodory, Radon and Helly, and many generalizations, ramifications and analogues of these theorems, forming an essential part of combinatorial convexity, can be studied in the survey article of Danzer, Grünbaum and Klee [464], which is still strongly recommended. Various results related to Carathéodory’s theorem can be found in Reay [1561]. Sufficient conditions on a compact set in $\mathbb{R}^n$ to have Carathéodory number less than $n + 1$, and related results, were given by Bárány and Karasëv [146].

   The proof of the coloured Radon theorem 1.1.8 given here is due to Soberón [1796] (see Bárány and Larman [149] for more history).

   An important extension of Radon’s theorem was proved by Tverberg [1858, 1859]:

   Theorem (Tverberg) Every set of at least $(k - 1)(n + 1) + 1$ points in $\mathbb{R}^n$ (where $k \geq 2$) can be partitioned into k subsets whose convex hulls have a common point.

   A survey is given by Eckhoff [526]. There one also finds hints about versions of the theorems of Carathéodory, Radon and Helly in the abstract setting of so-called convexity spaces. Later surveys on Helly’s and related theorems are due to Eckhoff [528] and Wenger [1958]. For a proof of Tverberg’s theorem and information about later developments, such as the Coloured Tverberg theorem, see Matoušek [1362], §8.3.

   For more recent ‘colourful versions’ of theorems in combinatorial convexity, we refer to Arocha, Bárány, Bracho, Fabila and Montejano [77] and to Blagojević, Matschke and Ziegler [235, 236].

   Another variant of the classical theorems of combinatorial convexity are such ‘with tolerance’, first introduced by Montejano and Oliveros [1447]. The following example is due to Soberón and Strausz [1797]:

Theorem Every set S of $(r+1)(k-1)(n+1)+1$ points in $\mathbb{R}^n$ (where $k \geq 2$) has a partition in k sets $A_1, \dots, A_k$ such that, for any $C \subset S$ of at most r points, the convex hulls of $A_1 \setminus C, \dots, A_k \setminus C$ have a common point.

2. It is clear how a version of Carathéodory's theorem for convex cones is to be formulated and how it can be proved. A common generalization, a version of Carathéodory's theorem for 'convex hulls of points and directions', is given by Rockafellar [1583], Theorem 17.1.

1.2 The metric projection

In this section, $A \subset \mathbb{R}^n$ is a fixed nonempty closed convex set. To each $x \in \mathbb{R}^n$ there exists a unique point $p(A, x) \in A$ satisfying

$$|x - p(A, x)| \leq |x - y| \quad \text{for all } y \in A.$$

In fact, for suitable $\rho > 0$ the set $B(x, \rho) \cap A$ is compact and nonempty, hence the continuous function $y \mapsto |x-y|$ attains a minimum on this set, say at y_0; then $|x-y_0| \leq |x-y|$ for all $y \in A$. If also $y_1 \in A$ satisfies $|x-y_1| \leq |x-y|$ for all $y \in A$, then $z := (y_0+y_1)/2 \in A$ and $|x-z| < |x-y_0|$, except if $y_0 = y_1$. Thus, $y_0 =: p(A, x)$ is unique.

In this way, a map $p(A, \cdot) : \mathbb{R}^n \to A$ is defined; it is called the *metric projection* or *nearest-point map* of A. It will play an essential role in Chapter 4, when the volume of local parallel sets is investigated. It also provides a simple approach to the basic support and separation properties of convex sets (see the next section), as used by Botts [309] and by McMullen and Shephard [1398].

We have $|x - p(A, x)| = d(A, x)$. For $x \in \mathbb{R}^n \setminus A$ we denote by

$$u(A, x) := \frac{x - p(A, x)}{d(A, x)}$$

the unit vector pointing from the nearest point $p(A, x)$ to x and by

$$R(A, x) := \{p(A, x) + \lambda u(A, x) : \lambda \geq 0\}$$

the ray through x with endpoint $p(A, x)$.

Theorem 1.2.1 *The metric projection is contracting, that is,*

$$|p(A, x) - p(A, y)| \leq |x - y| \quad \textit{for } x, y \in \mathbb{R}^n.$$

Proof We may assume that $v := p(A, y) - p(A, x) \neq o$. The function f defined by $f(t) := |x - (p(A, x) + tv)|^2$ for $t \in [0, 1]$ has a minimum at $t = 0$, hence $f'(0) \geq 0$. This gives $\langle x - p(A, x), v \rangle \leq 0$. Similarly we obtain $\langle y - p(A, y), v \rangle \geq 0$. Thus, the segment $[x, y]$ meets the two hyperplanes that are orthogonal to v and that go through $p(A, x)$ and $p(A, y)$, respectively. Now the assertion is obvious. □

Lemma 1.2.2 *Let $x \in \mathbb{R}^n \setminus A$ and $y \in R(A, x)$; then $p(A, x) = p(A, y)$.*

Proof With the notation and auxiliary results of the previous proof, we have $\langle x - p(A,x), v\rangle \leq 0$ and $\langle y - p(A,y), v\rangle \geq 0$. Since $y \in R(A,x)$, the first inequality yields $\langle y - p(A,x), v\rangle \leq 0$ and together with the second this gives $v = o$. □

Lemma 1.2.3 *Let S be the boundary of a ball containing A in its interior. Then* $p(A,S) = \operatorname{bd} A$.

Proof The inclusion $p(A,S) \subset \operatorname{bd} A$ is clear. Let $x \in \operatorname{bd} A$. For $i \in \mathbb{N}$ choose x_i in the ball bounded by S such that $x_i \notin A$ and $|x_i - x| < 1/i$. From Theorem 1.2.1 we have

$$|x - p(A,x_i)| = |p(A,x) - p(A,x_i)| \leq |x - x_i| < \frac{1}{i}.$$

The ray $R(A,x_i)$ meets S in a point y_i and we have $p(A,y_i) = p(A,x_i)$, hence $|x - p(A,y_i)| < 1/i$. A subsequence $(y_{i_j})_{j\in\mathbb{N}}$ converges to a point $y \in S$. From $\lim p(A,y_i) = x$ and the continuity of the metric projection we see that $x = p(A,y)$. Thus $\operatorname{bd} A \subset p(A,S)$. □

The existence of a unique nearest-point map is characteristic of convex sets. We prove this result here to complete the picture, although no use will be made of it.

Theorem 1.2.4 *Let $A \subset \mathbb{R}^n$ be a closed set with the property that to each point of $\mathbb{R}^n$ there is a unique nearest point in A. Then A is convex.*

Proof Suppose A satisfies the assumption but is not convex. Then there are points x, y with $[x,y] \cap A = \{x,y\}$, and one can choose $\rho > 0$ such that the ball $B = B((x+y)/2, \rho)$ satisfies $B \cap A = \emptyset$. By an elementary compactness argument, the family $\mathcal{B}$ of all closed balls B' containing B and satisfying $(\operatorname{int} B') \cap A = \emptyset$ contains a ball C with maximal radius. By this maximality, there is a point $p \in C \cap A$, and, by the assumed uniqueness of nearest points in A, it is unique. If $\operatorname{bd} B$ and $\operatorname{bd} C$ have a common point, let this (unique) point be q; otherwise let q be the centre of B. For sufficiently small $\varepsilon > 0$, the ball $C + \varepsilon(q - p)$ includes B and does not meet A. Hence, the family $\mathcal{B}$ contains an element with greater radius than that of C, a contradiction. □

Note for Section 1.2

1. Theorem 1.2.4 was found independently (in a more general form) by Bunt [354] and Motzkin [1453]; it is usually associated with the name of Motzkin. In general, a subset A of a metric space is called a Chebyshev set if for each point of the space there is a unique nearest point in A. There are several results and interesting open problems concerning the convexity of Chebyshev sets in normed linear spaces. For more information, see Valentine [1866], Chapter VII, Marti [1331], Chapter IX, Vlasov [1894] and §6 of the survey article by Burago and Zalgaller [356].