Introduction

Let G be a finite group, p a prime, and S a Sylow p-subgroup of G. Subsets of S are said to be *fused* in G if they are conjugate under some element of G. The term "fusion" seems to have been introduced by Brauer in the fifties, but the general notion has been of interest for over a century. For example, in his text *The Theory of Groups of Finite Order* [Bu] (first published in 1897), Burnside proved that if S is abelian then the normalizer in G of S controls fusion in S. (A subgroup H of G is said to *control fusion* in S if any pair of tuples of elements of S which are conjugate in G are also conjugate under H.)

Initially, information about fusion was usually used in conjunction with transfer, as in the proof of the normal p-complement theorems of Burnside and Frobenius, which showed that, under suitable hypotheses on fusion, G possesses a *normal p-complement*: a normal subgroup of index $|S|$ in G. But in the sixties and seventies more sophisticated results on fusion began to appear, such as Alperin's Fusion Theorem [Al1], which showed that the family of normalizers of suitable subgroups of S control fusion in S, and Goldschmidt's Fusion Theorem [Gd3], which determined the groups G possessing a nontrivial abelian subgroup A of S such that no element of A is fused into $S \smallsetminus A$.

In the early nineties, Lluis Puig abstracted the properties of G-fusion in a Sylow subgroup S, in his notion of a *Frobenius category* on a finite p-group S, by discarding the group G and focusing instead on isomorphisms between subgroups of S. (But even earlier in 1976 in [P1], Puig had already considered the standard example $\mathcal{F}_S(G)$ of a Frobenius category, defined below.) Puig did not publish his work until his 2006 paper [P6] and his 2009 book [P7]. Meanwhile, his approach was taken up and extended by others, and in the process, alternate terminology evolved which is now commonly used, and which we therefore have adopted here. In particular, Puig's Frobenius categories are now referred to as "saturated fusion systems" in most of the literature.

A *fusion system* $\mathcal{F}$ on a finite p-group S is a category whose objects are the subgroups of S, with the set $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ of morphisms from P to Q consisting of monomorphisms from P into Q, and such that some weak axioms are satisfied (see Definition I.2.1 for the precise conditions). The standard example of a fusion system is the category $\mathcal{F}_S(G)$, where G is a finite group, $S \in \mathrm{Syl}_p(G)$, and the morphisms are those induced by conjugation in G. A fusion system $\mathcal{F}$ is *saturated* if it satisfies two more axioms (Definition I.2.2), which hold in $\mathcal{F}_S(G)$ as a consequence of Sylow's Theorem.

Many classic results on fusion in a Sylow subgroup S of a finite group G can be interpreted as results about the fusion system $\mathcal{F} = \mathcal{F}_S(G)$. Burnside's Fusion Theorem becomes $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$ when S is abelian. Alperin's Fusion Theorem says that $\mathcal{F}$ is generated by certain "local" subcategories of $\mathcal{F}$ (cf. Theorem I.3.5). Goldschmidt's Fusion Theorem says that an abelian subgroup A of S is "normal" in $\mathcal{F}$ (cf. Definition I.4.1) when no element of A is fused into $S \smallsetminus A$, and goes on to use this fact to determine G.

Puig created his theory of Frobenius categories largely as a tool in modular representation theory, motivated in part by work of Alperin and Broue in [AB]. Later, homotopy theorists used this theory to provide a formal setting for, and prove results about, the p-completed classifying spaces of finite groups. As part of this process, objects called *p-local finite groups* associated to abstract fusion systems were introduced by Broto, Levi and Oliver in [BLO2]; these also possess interesting p-completed classifying spaces. Finally, local finite group theorists became interested in fusion systems, in part because methods from local group theory proved to be effective in the study of fusion systems, but also because certain results in finite group theory seem to be easier to prove in the category of saturated fusion systems.

These three themes — the application of fusion systems in modular representation theory, homotopy theory, and finite group theory — together with work on the foundations of the theory of saturated fusion systems, remain the focus of interest in the subject. And these are the four themes to which this volume is devoted.

This book grew out of a workshop on fusion systems at the University of Birmingham in July–August of 2007, sponsored by the London Mathematical Society and organized by Chris Parker. At that workshop there were three series of talks, one each on the role of fusion systems in modular representation theory, homotopy theory, and finite group theory, given by Kessar, Oliver, and Aschbacher, respectively. It was Chris Parker's idea to use those talks as the point of departure for this book, although he unfortunately had to pull out of the project before it was completed. We have extracted material on the foundations of the theory of fusion systems from the various series and collected them in Part I of the book, where we have also included proofs of many of the most basic results. Then the talks have been updated and incorporated in Parts II through IV of the book, which describe the state of the art of the role of fusion systems in each of the three areas.

David Craven has also written a book on fusion systems [Cr2], which also can trace its origins to the 2007 workshop in Birmingham, and which should appear at about the same time as this one. The two books are very different in style — for example, his is intended more as a textbook and ours as a survey — and also very different in the choice of topics. In this way, we expect that the two books will complement each other.

The theory of fusion systems is an emerging area of mathematics. As such, its foundations are not yet firmly established, and the frontiers of the subject are receding more rapidly than those of more established areas. With this in mind, we have two major goals for this volume: first, collect in one place the various definitions, notation, terminology, and basic results which constitute the foundation of the theory of fusion systems, but are currently spread over a number of papers in the literature. In the process we also seek to reconcile differences in notation, terminology, and even basic concepts among papers in the literature. In particular, there is a discussion of the three existing notions of a "normal subsystem" of a saturated fusion system. Second, we seek to present a snapshot of the important theorems and open problems in our four areas of emphasis at this point in time. Our hope is that the book will serve both as a basic reference on fusion systems and as an introduction to the field, particularly for students and other young mathematicians.

The book is organized as follows. Part I contains foundational material about fusion systems, including the most basic definitions, notation, concepts, and lemmas. Then Parts II, III, and IV discuss the role of fusion systems in local finite group theory, homotopy theory, and modular representation theory, respectively. Finally the book closes with an appendix which records some of the basic material on finite groups which is well known to specialists, but perhaps not to those who approach fusion systems from the point of view of representation theory or homotopy theory.

We have received help from a large number of people while working on this project. In particular, we would like to thank Kasper Andersen, David Craven, and Ellen Henke for reading large parts of the manuscript in detail and making numerous suggestions and corrections. Many of the others who have assisted us will be acknowledged in the introductions to the individual Parts.

Notation: We close this introduction with a list of some of the basic notation involving finite groups used in all four Parts of the book. Almost all of this notation is fairly standard.

For $x, g \in G$, we write ${}^g x = gxg^{-1}$ for the *conjugate* of x under g, and let $c_g \colon G \to G$ be *conjugation by* g, defined by $c_g(x) = {}^g x$. Set $x^g = g^{-1}xg$ and for $X \subseteq G$, set ${}^g X = c_g(X)$ and $X^g = c_{g^{-1}}(X)$. Let $N_G(X) = \{g \in G \,|\, {}^g X = X\}$ be the *normalizer* in G of X and $C_G(X) = \{g \in G \,|\, xg = gx \text{ for all } g \in G\}$ be the *centralizer* in G of X. Write $\langle X \rangle$ for the subgroup of G generated by X.

Similar notation will be used when conjugating by an isomorphism of (possibly distinct) groups. For example (when group homomorphisms are composed from right to left), if $\varphi\colon G \xrightarrow{\cong} H$ is an isomorphism of groups, and $\alpha \in \mathrm{Aut}(G)$ and $\beta \in \mathrm{Aut}(H)$, we write ${}^{\varphi}\alpha = \varphi\alpha\varphi^{-1} \in \mathrm{Aut}(H)$ and $\beta^{\varphi} = \varphi^{-1}\beta\varphi \in \mathrm{Aut}(G)$.

We write $H \leq G$, $H < G$, or $H \trianglelefteq G$ to indicate that H is a subgroup, proper subgroup, or normal subgroup of G, respectively. Observe, for $H \leq G$, that $c\colon g \mapsto c_g$ is a homomorphism from $N_G(H)$ into $\mathrm{Aut}(H)$ with kernel $C_G(H)$; we write $\mathrm{Aut}_G(H)$ for the image $c(N_G(H))$ of H under this homomorphism. Thus $\mathrm{Aut}_G(H)$ is the *automizer* in G of H and $\mathrm{Aut}_G(H) \cong N_G(H)/C_G(H)$. The *inner automorphism group* of H is $\mathrm{Inn}(H) = \mathrm{Aut}_H(H) = c(H)$, and the *outer automorphism group* of H is $\mathrm{Out}(H) = \mathrm{Aut}(H)/\mathrm{Inn}(H)$.

We write $\mathrm{Syl}_p(G)$ for the set of Sylow p-subgroups of G. When π is a set of primes, a *π-subgroup* of G is a subgroup whose order is divisible only by primes in π. We write $O_\pi(G)$ for the largest normal π-subgroup of G, and $O^\pi(G)$ for the smallest normal subgroup H of G such that G/H is a π-group. We write p' for the set of primes distinct from p; we will be particularly interested in the groups $O_p(G)$, $O_{p'}(G)$, $O^p(G)$, and $O^{p'}(G)$. Sometimes we write $O(G)$ for $O_{2'}(G)$.

As usual, when P is a p-group for some prime p, we set $\Omega_1(P) = \langle g \in P \mid g^p = 1 \rangle$.

As for specific groups, C_n denotes a (multiplicative) cyclic group of order n, and D_{2^k}, SD_{2^k}, Q_{2^k} denote dihedral, semidihedral, and quaternion groups of order 2^k. Also, $A_n \leq S_n$ denote alternating and symmetric groups on n letters.

Throughout the book, p is always understood to be a fixed prime. All p-groups are assumed to be finite.

Part I. Introduction to fusion systems

This part is intended as a general introduction to the book, where we describe the properties of fusion systems which will be used throughout. We begin with the basic definitions of fusion systems of finite groups and abstract fusion systems, and give some versions of Alperin's fusion theorem in this setting. Afterwards, we discuss various topics such as normal and central subgroups of fusion systems, constrained fusion systems, normal fusion subsystems, products of fusion systems, the normalizer and centralizer fusion subsystems of a subgroup, and fusion subsystems of p-power index or of index prime to p.

1. The fusion category of a finite group

For any group G and any pair of subgroups $H, K \le G$, we define

$$\mathrm{Hom}_G(H,K) = \bigl\{\varphi \in \mathrm{Hom}(H,K) \bigm| \varphi = c_g \text{ for some } g \in G \text{ such that } {}^gH \le K\bigr\}.$$

In other words, $\mathrm{Hom}_G(H,K)$ is the set of all (injective) homomorphisms from H to K which are induced by conjugation in G. Similarly, we write $\mathrm{Iso}_G(H,K)$ for the set of elements of $\mathrm{Hom}_G(H,K)$ which are isomorphisms of groups.

Definition 1.1 Fix a finite group G, a prime p, and a Sylow p-subgroup $S \in \mathrm{Syl}_p(G)$. The *fusion category* of G over S is the category $\mathcal{F}_S(G)$ whose objects are the subgroups of S, and which has morphism sets

$$\mathrm{Mor}_{\mathcal{F}_S(G)}(P,Q) = \mathrm{Hom}_G(P,Q).$$

Many concepts and results in finite group theory can be stated in terms of this category. We list some examples of this here. In all cases, G is a finite group and $S \in \mathrm{Syl}_p(G)$.

- Alperin's fusion theorem [Al1], at least in some forms, is the statement that $\mathcal{F}_S(G)$ is generated by automorphism groups of certain subgroups of S, in the sense that each morphism in $\mathcal{F}_S(G)$ is a composite of restrictions of automorphisms of those subgroups.
- Glauberman's Z^*-theorem [Gl1] says that when $p = 2$ and $O_{2'}(G) = 1$, then $Z(G) = Z(\mathcal{F}_S(G))$, where $Z(\mathcal{F}_S(G))$ is the "center" of the fusion category in a sense which will be made precise later (Definition 4.3).

- A subgroup $H \le G$ which contains S *controls fusion in* S if $\mathcal{F}_S(H) = \mathcal{F}_S(G)$. Thus Burnside's fusion theorem states that when S is abelian, $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, and that every morphism in $\mathcal{F}_S(G)$ extends to an automorphism of S.
- By a theorem of Frobenius (cf. [A4, 39.4]), G has a normal p-complement (a subgroup $H \trianglelefteq G$ of p-power index and order prime to p) if and only if $\mathcal{F}_S(G) = \mathcal{F}_S(S)$.
- The focal subgroup theorem says that

$$S \cap [G,G] = \mathfrak{foc}(\mathcal{F}_S(G)) \stackrel{\text{def}}{=} \langle x^{-1}y \,|\, x,y \in S,\ x = {}^g y \text{ for some } g \in G \rangle,$$

and is thus described in terms of the category $\mathcal{F}_S(G)$. By the hyperfocal subgroup theorem of Puig [P5, § 1.1], $S \cap O^p(G)$ can also be described in terms of the fusion category $\mathcal{F}_S(G)$ (see Section 7).

The following lemma describes some of the properties of these fusion categories, properties which help to motivate the definition of abstract fusion systems in the next section.

Lemma 1.2 *Fix a finite group G and a Sylow p-subgroup $S \in \mathrm{Syl}_p(G)$.*

(a) *For each $P \le S$, there is $Q \le S$ such that Q is G-conjugate to P and $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$. For any such Q, $C_S(Q) \in \mathrm{Syl}_p(C_G(Q))$ and $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_G(Q))$.*

(b) *Fix $P, Q \le S$ and $g \in G$ such that ${}^gP = Q$. Assume $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$. Set*

$$N = \{x \in N_S(P) \,|\, {}^g x \in N_S(Q)\cdot C_G(Q)\} .$$

Then there is $h \in C_G(Q)$ such that ${}^{hg}N \le S$.

Proof. **(a)** Fix $T \in \mathrm{Syl}_p(N_G(P))$. Since T is a p-subgroup of G, there is $g \in G$ such that ${}^gT \le S$. Set $Q = {}^gP$. Then $Q \trianglelefteq {}^gT \le S$, and ${}^gT \in \mathrm{Syl}_p(N_G(Q))$ since $c_g \in \mathrm{Aut}(G)$. Also, ${}^gT \le N_S(Q)$, and since ${}^gT \in \mathrm{Syl}_p(N_G(Q))$, ${}^gT = N_S(Q)$. The last statement now holds by Lemma A.3, upon identifying $\mathrm{Aut}_X(Q)$ with $N_X(Q)/C_X(Q)$ for $X = G, S$.

(b) Since $N_S(Q)$ normalizes $C_G(Q)$, $N_S(Q)\cdot C_G(Q)$ is a subgroup of G. By assumption, ${}^gN \le N_S(Q)\cdot C_G(Q)$. Since $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$ and $N_S(Q)\cdot C_G(Q) \le N_G(Q)$, $N_S(Q)$ is also a Sylow subgroup of $N_S(Q)\cdot C_G(Q)$. Hence there is $h \in C_G(Q)$ such that ${}^h({}^gN) \le N_S(Q)$; i.e., ${}^{hg}N \le N_S(Q)$. □

2. Abstract fusion systems

The notion of an abstract fusion system is due to Puig. The definitions we give here are modified versions of Puig's definitions (given in [P6]), but equivalent to them. The following is what he calls a "divisible S-category".

Definition 2.1 ([P6], [BLO2]) A fusion system over a p-group S is a category $\mathcal{F}$, where $\mathrm{Ob}(\mathcal{F})$ is the set of all subgroups of S, and which satisfies the following two properties for all $P, Q \leq S$:

- $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Mor}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$; and
- each $\varphi \in \mathrm{Mor}_{\mathcal{F}}(P, Q)$ is the composite of an $\mathcal{F}$-isomorphism followed by an inclusion.

Composition in a fusion system $\mathcal{F}$ is always given by composition of homomorphisms. We usually write $\mathrm{Hom}_{\mathcal{F}}(P, Q) = \mathrm{Mor}_{\mathcal{F}}(P, Q)$ to emphasize that the morphisms in $\mathcal{F}$ actually are group homomorphisms, and also set $\mathrm{Aut}_{\mathcal{F}}(P) = \mathrm{Hom}_{\mathcal{F}}(P, P)$. Note that a fusion system over $\mathcal{F}$ contains all inclusions incl_P^Q, for $P \leq Q \leq S$, by the first condition (it is conjugation by $1 \in S$). The second condition means that for each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$, $\varphi\colon P \longrightarrow \varphi(P)$ and $\varphi^{-1}\colon \varphi(P) \longrightarrow P$ are both morphisms in $\mathcal{F}$.

Fusion systems as defined above are too general for most purposes, and additional conditions are needed for them to be very useful. This leads to the concept of what we call a "saturated fusion system": a fusion system satisfying certain axioms which are motivated by properties of fusion in finite groups. The following version of these axioms is due to Roberts and Shpectorov [RS].

Definition 2.2 Let $\mathcal{F}$ be a fusion system over a p-group S.

- Two subgroups $P, Q \leq S$ are *$\mathcal{F}$-conjugate* if they are isomorphic as objects of the category $\mathcal{F}$. Let $P^{\mathcal{F}}$ denote the set of all subgroups of S which are $\mathcal{F}$-conjugate to P.
- A subgroup $P \leq S$ is *fully automized* in $\mathcal{F}$ if $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
- A subgroup $P \leq S$ is *receptive* in $\mathcal{F}$ if it has the following property: for each $Q \leq S$ and each $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$, if we set
$$N_\varphi = N_\varphi^{\mathcal{F}} = \{g \in N_S(Q) \mid {}^{\varphi}c_g \in \mathrm{Aut}_S(P)\},$$
then there is $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_Q = \varphi$.
- A fusion system $\mathcal{F}$ over a p-group S is *saturated* if each subgroup of S is $\mathcal{F}$-conjugate to a subgroup which is fully automized and receptive.

We also say that two elements $x, y \in S$ are $\mathcal{F}$-conjugate if there is an isomorphism $\varphi \in \mathrm{Iso}_{\mathcal{F}}(\langle x \rangle, \langle y \rangle)$ such that $\varphi(x) = y$, and let $x^{\mathcal{F}}$ denote the $\mathcal{F}$-conjugacy class of x.

The fusion category $\mathcal{F}_S(G)$ of a finite group G clearly satisfies the conditions in Definition 2.1, and thus is a fusion system. It also satisfies the saturation conditions by Lemma 1.2.

Theorem 2.3 (Puig) *If G is a finite group and $S \in \mathrm{Syl}_p(G)$, then $\mathcal{F}_S(G)$ is a saturated fusion system.*

Proof. By Lemma 1.2(a), each subgroup $P \leq S$ is G-conjugate to a subgroup $Q \leq S$ such that $N_S(Q) \in \mathrm{Syl}_p(N_G(Q))$, and each such subgroup Q is fully automized in $\mathcal{F}_S(G)$. By Lemma 1.2(b), Q is also receptive in $\mathcal{F}_S(G)$, and thus $\mathcal{F}_S(G)$ is saturated. □

A saturated fusion system $\mathcal{F}$ over a p-group S will be called *realizable* if $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G with $S \in \mathrm{Syl}_p(G)$, and will be called *exotic* otherwise. Examples of exotic fusion systems will be described in Section III.6.

There are several, equivalent definitions of saturated fusion systems in the literature. We discuss here the definition of saturation which was given in [BLO2]. Two other definitions, the original one given by Puig and another one by Stancu, will be described and shown to be equivalent to these in Section 9 (Proposition 9.3).

In order to explain the definition in [BLO2], and compare it with the one given above, we first need to define two more concepts.

Definition 2.4 Let $\mathcal{F}$ be a fusion system over a p-group S.

- A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.
- A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(Q)|$ for all $Q \in P^{\mathcal{F}}$.

For example, when $\mathcal{F} = \mathcal{F}_S(G)$ for some finite group G and some $S \in \mathrm{Syl}_p(G)$, then by Lemma 1.2(a), a subgroup $P \leq S$ is fully normalized (centralized) in $\mathcal{F}_S(G)$ if and only if $N_S(P) \in \mathrm{Syl}_p(N_G(P))$ ($C_S(P) \in \mathrm{Syl}_p(C_G(P))$).

Definition 2.4 is different from the definition of "fully normalized" and "fully centralized" in [P6, 2.6], but it is equivalent to Puig's definition when working in saturated fusion systems. This will be discussed in much more detail in Section 9.

The following equivalent condition for a fusion system to be saturated, stated in terms of fully normalized and fully centralized subgroups, was given as the definition of saturation in [BLO2, Definition 1.2].

Proposition 2.5 ([RS, Theorem 5.2]) *Let $\mathcal{F}$ be a fusion system over a p-group S. Then $\mathcal{F}$ is saturated if and only if the following two conditions hold.*

- (I) (Sylow axiom) *Each subgroup $P \leq S$ which is fully normalized in $\mathcal{F}$ is also fully centralized and fully automized in $\mathcal{F}$.*
- (II) (Extension axiom) *Each subgroup $P \leq S$ which is fully centralized in $\mathcal{F}$ is also receptive in $\mathcal{F}$.*

Proposition 2.5 is an immediate consequence of the following lemma.

Lemma 2.6 ([RS]) *The following hold for any fusion system $\mathcal{F}$ over a p-group S.*

- (a) *Every receptive subgroup of S is fully centralized.*
- (b) *Every subgroup of S which is fully automized and receptive is fully normalized.*
- (c) *Assume $P \leq S$ is fully automized and receptive. Then for each $Q \in P^{\mathcal{F}}$, there is a morphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), N_S(P))$ such that $\varphi(Q) = P$. Furthermore, Q is fully centralized if and only if it is receptive, and is fully normalized if and only if it is fully automized and receptive.*

Proof. **(a)** Assume $P \leq S$ is receptive. Fix any $Q \in P^{\mathcal{F}}$ and any $\varphi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$. Since P is receptive, φ extends to some $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$, where N_φ contains $C_S(Q)$ by definition. Thus $\bar{\varphi}$ sends $C_S(Q)$ injectively into $C_S(P)$, and so $|C_S(P)| \geq |C_S(Q)|$. Since this holds for all $Q \in P^{\mathcal{F}}$, P is fully centralized in $\mathcal{F}$.

(b) Now assume P is fully automized and receptive, and fix $Q \in P^{\mathcal{F}}$. Then $|C_S(Q)| \leq |C_S(P)|$ by (a), and $|\mathrm{Aut}_S(Q)| \leq |\mathrm{Aut}_S(P)|$ since $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\mathcal{F}}(P)$ and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$. Thus

$$|N_S(Q)| = |C_S(Q)|\cdot|\mathrm{Aut}_S(Q)| \leq |C_S(P)|\cdot|\mathrm{Aut}_S(P)| = |N_S(P)|.$$

Since this holds for all $Q \in P^{\mathcal{F}}$, P is fully normalized in $\mathcal{F}$.

(c) Assume P is fully automized and receptive, and fix $Q \in P^{\mathcal{F}}$. Choose $\psi \in \mathrm{Iso}_{\mathcal{F}}(Q, P)$. Then ${}^{\psi}\mathrm{Aut}_S(Q)$ is a p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$, and hence is $\mathrm{Aut}_{\mathcal{F}}(P)$-conjugate to a subgroup of $\mathrm{Aut}_S(P)$ since P is fully automized. Fix $\alpha \in \mathrm{Aut}_{\mathcal{F}}(P)$ such that ${}^{\alpha\psi}\mathrm{Aut}_S(Q)$ is contained in $\mathrm{Aut}_S(P)$. Then $N_{\alpha\psi} = N_S(Q)$ (see Definition 2.2), and so $\alpha\psi$ extends to some homomorphism $\varphi \in \mathrm{Hom}_{\mathcal{F}}(N_S(Q), S)$. Since $\varphi(Q) = \alpha\psi(Q) = P$, $\mathrm{Im}(\varphi) \leq N_S(P)$.

If Q is fully centralized, then $\varphi(C_S(Q)) = C_S(P)$. Fix $R \le S$ and $\beta \in \mathrm{Iso}_{\mathcal{F}}(R,Q)$. For $g \in N_\beta$, ${}^{\beta}c_g \in \mathrm{Aut}_S(Q)$ implies ${}^{\alpha\psi\beta}c_g \in \mathrm{Aut}_S(P)$ since $\alpha\psi$ extends to a homomorphism defined on $N_S(Q)$, and thus $g \in N_{\alpha\psi\beta}$. Since P is receptive, $\alpha\psi\beta$ extends to a homomorphism $\chi \in \mathrm{Hom}_{\mathcal{F}}(N_\beta, N_S(P))$. For each $g \in N_\beta$, ${}^{\beta}c_g = c_h$ for some $h \in N_S(Q)$ by definition of N_β, so $c_{\varphi(h)} = {}^{\varphi}c_h = c_{\chi(g)}$, and thus $\chi(g) \in \mathrm{Im}(\varphi){\cdot}C_S(P) = \mathrm{Im}(\varphi)$. Thus $\mathrm{Im}(\chi) \le \mathrm{Im}(\varphi)$, so χ factors through some $\bar\beta \in \mathrm{Hom}_{\mathcal{F}}(N_\beta, N_S(Q))$ with $\bar\beta|_R = \beta$, and this proves that Q is receptive.

If Q is fully normalized, then φ is an isomorphism. Hence φ sends $C_S(Q)$ onto $C_S(P)$, so Q is fully centralized and hence receptive. Also, $\mathrm{Aut}_S(Q) \cong N_S(Q)/C_S(Q)$ is isomorphic to $\mathrm{Aut}_S(P) \cong N_S(P)/C_S(P)$, and $\mathrm{Aut}_{\mathcal{F}}(Q) \cong \mathrm{Aut}_{\mathcal{F}}(P)$ since $Q \in P^{\mathcal{F}}$. So $\mathrm{Aut}_S(Q) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(Q))$ since $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$, and Q is fully automized. □

We end this section with an example, which describes how to list all possible saturated fusion systems over one very small 2-group.

Example 2.7 Assume $S \cong D_8$: the dihedral group of order 8. Fix generators $a, b \in S$, where $|a| = 4$, $|b| = 2$, and $|ab| = 2$. Set $T_0 = \langle a^2, b\rangle$ and $T_1 = \langle a^2, ab\rangle$: these are the only subgroups of S isomorphic to $C_2 \times C_2$. Then the following hold for any saturated fusion system $\mathcal{F}$ over S.

(a) Since S is fully automized and $\mathrm{Aut}(S)$ is a 2-group, $\mathrm{Aut}_{\mathcal{F}}(S) = \mathrm{Inn}(S)$.

(b) If $P = \langle a\rangle$ and $Q = P$ or S, then $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_S(P,Q) = \mathrm{Hom}(P,Q)$.

(c) The subgroups T_0 and T_1 are both fully normalized in $\mathcal{F}$, and hence are fully automized and receptive. So if $T_1 \in T_0^{\mathcal{F}}$, then by Lemma 2.6(c), there is $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S)$ such that $\alpha(T_0) = T_1$. Since this contradicts (a), T_0 and T_1 cannot be $\mathcal{F}$-conjugate.

(d) Set $P = \langle a^2\rangle$, and let $Q \le S$ be any subgroup of order 2. Since $P \trianglelefteq S$, P is fully normalized in $\mathcal{F}$ and hence fully automized and receptive. So if $Q \in P^{\mathcal{F}}$ (and $Q \ne P$), then by Lemma 2.6(c) again, there is some $\varphi \in \mathrm{Hom}_{\mathcal{F}}(T_i, S)$, where $T_i = N_S(Q)$ ($i = 0$ or 1), such that $\varphi(Q) = P$. Also, $\varphi(T_i) = T_i$ by (c).

(e) By (a–d), $\mathcal{F}$ is completely determined by $\mathrm{Aut}_{\mathcal{F}}(T_0)$ and $\mathrm{Aut}_{\mathcal{F}}(T_1)$. Also, for each i, $\mathrm{Aut}_S(T_i) \le \mathrm{Aut}_{\mathcal{F}}(T_i) \le \mathrm{Aut}(T_i)$, and hence $\mathrm{Aut}_{\mathcal{F}}(T_i)$ has order 2 or 6.

Thus there are at most four saturated fusion systems over S. Denote these fusion systems $\mathcal{F}_{ij}$, where $i = 0$ if $|\mathrm{Aut}_{\mathcal{F}}(T_0)| = 2$, $i = 1$ if $|\mathrm{Aut}_{\mathcal{F}}(T_0)| = 6$, and similarly $j = 0, 1$ depending on $|\mathrm{Aut}_{\mathcal{F}}(T_1)|$. Then $\mathcal{F}_{00}$ is the fusion system of $S \cong D_8$ itself, $\mathcal{F}_{01} \cong \mathcal{F}_{10}$ are isomorphic to the