INTRODUCTION

THE REDUCTION OF OBSERVATIONS

Not the least important part of a laboratory experiment is the utilisation of the observations made in the course of that experiment. The object of the experiment may be to measure some physical or instrumental constant, such as *g* or the focal length of a lens, or to find out how two or more variable quantities are related to one another, such as the volume, pressure and temperature of a constant mass of gas, or it may be a combination of the two purposes. We may have observed as accurately as we can the value of one of these variables corresponding to an observed value of another, and may have done so for several different values of the latter; the problem remains, how to use these pairs of measurements so as to deduce from them the algebraical relation between them.

The first part of this problem is to discover the *form* of this relation; the two columns of figures representing the crude results of our measurements, the 'pointer-readings' as Eddington called them, are often very unpromising material for this process. For instance, after the wave-lengths of the bright lines in the spectrum of hydrogen had been measured it needed a lot of co-operative effort before any mathematical law connecting them could be discovered; again, the positions of one or two planets at various known times had been fairly accurately measured for many years before Kepler could discover his three general, comparatively simple, laws governing the motion of all planets, from which Newton could deduce his supremely simple law of Universal Gravitation. These are instances of the application of genius to great problems; it may be a descent from the sublime to the ridiculous to point out that in Exp. 22 and again in Exps. 28, 29 and 30 problems of the same nature are set for your solution, but very humble illustrations may be the most illuminating.

The second part of the problem is to discover whether the law thus formulated is obeyed with complete precision throughout the whole range of the variables which, mathematically, it purports to cover. For instance, it was long thought to be established that light travelled to us from the stars in geometrically straight lines, but extremely precise measurements (undertaken because Einstein's work on relativity suggested that it might not be true in all circumstances) showed that a ray of light passing near the sun is deflected, very slightly but measurably, by the mass of the sun. Making again the descent from the sublime to the ridiculous, we shall presently have occasion to consider the way in which a spiral steel spring fails in practice to obey Hooke's law.

There is therefore scope for great ingenuity in reducing tables of observations to a form that will be helpful in each of the foregoing processes. For the first, there is clearly need for more than ingenuity; scientific imagination or intuition of the highest order may be required to suggest the form of the relation. But when it is merely a question of testing the truth or applicability of the relation, of 'verifying' it, a more lowly intelligence may suffice, and it may become merely a matter of choosing the most suitable out of a number of well-recognised manipulations of the crude results of the observations. For example, the mere addition or subtraction of a suitable constant in each case (as in Exp. 22 mentioned above) may make it easy to see the next step in evolving the required law; or it may be useful to take the reciprocal, or the logarithm, of each, and so on.

For this reason the following notes on some of the recognised methods of reduction and presentation may be found helpful.

1. *Computation of a Single Constant*

(*a*) Laboratory work often consists of determining, as accurately as we can, the value of a single physical or instrumental constant, assuming the truth of a relation established by theoretical reasoning. For example, we may have to determine the most probable value of the instrumental constant f of a particular lens, accepting the truth of the general lens formula $1/u + 1/v = 1/f$, or we may have to find by experiments on a simple pendulum the value of the physical constant g, accepting the truth of the theoretical relation $T = 2\pi\sqrt{(l/g)}$.

Taking the former example, we set up the apparatus and measure, for one setting of the object, the values of u and v. We substitute these values in the general relation and compute the value of f. If we repeat this operation with a different setting of the object it is almost a certainty that we shall obtain a different value of f; suppose we do this N times in all, we are left with N different values of f, and are faced with the problem of deducing from them a value which is most likely to be the true value of f. Denote this true value by F, and the various calculated values by f_1, f_2, f_3, etc. Then the error of the first determination was $f_1 - F$, of the second was $f_2 - F$, and so on, and the sum-total of all the errors was $\Sigma f_1 - NF$, where Σf_1 means the sum of all terms like f_1.

Suppose that it is equally probable that any one of the calculated values is correct, and that their number N is infinitely large; then the Theory of Probability shows that for every one that is too large by a certain amount we can find one that is too small by exactly the same amount, and the arithmetic mean of these two will be exactly equal to F, and the sum-total of their errors is zero. Repeating this for all the observations, we see that $\Sigma f_1 - NF$ must be zero, or $F = \Sigma f_1/N$, or F, the true value, is the arithmetic mean of all the calculated values.

We cannot, of course, take an infinite number of observations, but reasons are given in § 4 for taking the arithmetic mean of such calculated values as are available as the most probable value of F.

It may happen that one, or more, of these observations leads to a value outstandingly different from the average value; it is tempting to ignore it out of hand as incorrect. It is clearly sound policy to go back and repeat this observation with great care, to find out whether there has been an error of observation or calculation. If there was, we can obviously ignore the original observation and substitute the revised one; if not, it is on the whole the soundest policy to retain it.

(*b*) This process of computing the arithmetic mean is in many cases quicker than a graphical method of determining the constant; it can be carried to a higher degree of numerical accuracy, and it has the additional merit of leading to a numerical estimate of the degree to which our experiments are trustworthy.

Take, for instance, the numbers in the first and second columns of this table, which are the reciprocals of the values of u and v found in the course of Exp. 25 (B) with a concave lens:

$1/u$	$1/v$	$-1/f$	$-f$	Difference from mean
0·01117	0·07681	0·06564	15·23	0·00
0·01447	0·07987	0·06540	15·29	+0·06
0·02086	0·08681	0·06605	15·14	−0·09
0·02574	0·09076	0·06504	15·37	+0·14
0·02995	0·09506	0·06611	15·13	−0·10
0·03402	0·09982	0·06580	15·20	−0·03
			6|91·36	6|0·42
			15·23	**0·07**

The third column shows the difference between the first and second, which in accordance with the accepted theory is $-1/f$; the fourth shows $-f$ (taken from a table of reciprocals) and the way it leads to the arithmetic mean value for f of 15·23. This is the best value of f which we can get from our observations.

We can consider the fifth column as giving the 'error' of each observation; if we take the arithmetic mean of these numbers, disregarding their signs (as shown), we can consider the result, 0·07, as representing the mean error. It is customary to express these results in the form $f = -15{\cdot}23 \pm 0{\cdot}07$.

Since 0·07 is about $\frac{1}{2}$% of the value of f, it is often said that these experiments are 'accurate to $\frac{1}{2}$%'. This shorthand statement is convenient for purposes of comparison but liable to mislead unless we bear in mind what it really means; for instance, one of these experiments shows an error of almost 1% of the arithmetic mean of the values we have obtained for f.

2. *Graphical Representation; Linear Graphs*

Laboratory work usually involves more complex problems than those dealt with in § 1, where we assumed that the relation between the measured variables was known; the problem is often to determine by experiment what the relation is. Let us assume for the present that our measurements are absolutely accurate; we can afterwards consider how best to allow for observational errors.

Suppose that we plot in the usual way the points represented by the pairs of observations, using such scales along the two axes as will spread the points satisfactorily on the graph paper. Suppose that we find that these points lie on a straight line; the relation between the observed quantities is clearly linear, of the form $ax+by=1$, where x and y are the quantities and a and b are some constants.

If this graph is not a straight line, let us suppose that by some manipulation of the observed quantities (e.g. squaring one or both or taking logs of one or both), and plotting these manipulated quantities we can get a straight-line graph. Then, again, the form of the relation is discovered.

Further, a and b are at once measurable since the intercept on the axis of x, found by putting $y=0$ in the equation $ax+by=1$ to be $1/a$, can be read off as the value of the intersection of the straight-line graph and the axis of x in terms of the scale for that axis. Similarly,

$1/b$ can be found. But it is essential to be on one's guard in carrying out even this familiar procedure.

(*a*) Suppose that a closely coiled spiral steel spring is hung up and loaded with a series of weights (W g.) ranging from 0 to 1300 g., and the corresponding overall lengths (l cm.) of the spring are measured; the points representing l and W are plotted and the full-line graph in Fig. 1 drawn. Let us assume that the graph between A and B is, as it appears to be, a straight line.

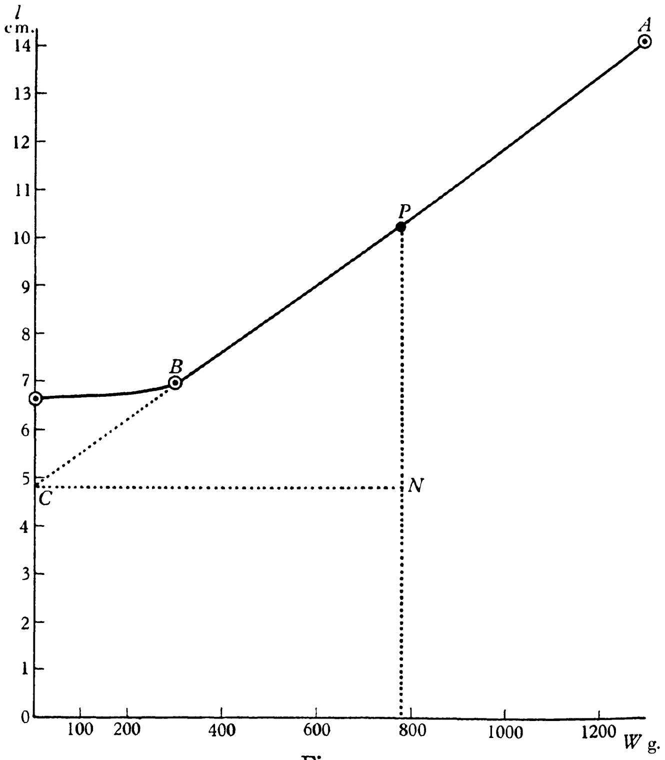

Fig. 1.

It is easy to account for the horizontal part of the graph, since, let us suppose, the experimental set-up showed that at a load of 300 g. consecutive turns of the wire were in contact, and the spring could not shorten further as the load was reduced.

Hence any linear algebraical relation connecting l and W which we may find to be true for loads on this spring between 300 and 1300 g. may be true for greater loads, but it certainly cannot be true for loads less than 300 g.

Nevertheless, any relation between the co-ordinates l and W which holds good for a portion of a straight line must hold good mathematically for the whole of it, produced to infinity both ways. Hence in finding the algebraical relation which holds good physically only for the part AB of the graph we may produce that part geometrically to cut the axis of l at C (as shown by the dotted line) and beyond C to cut the axis of W at D. Hence if the required relation between l and W is

$$al+bW=1,$$

since the intercept on the l-axis is $1/a$, and we see by the graph that the intercept is 4·80, we deduce at once that $a=1/4{\cdot}80$.

The value of the intercept OD on the axis of W cannot be read off so immediately, but can be deduced from the graph as follows.

By similar triangles, $OD/OC=NC/NP$. Here we are assuming that these are geometrical lengths on the graph paper, but the proportion holds good if we replace OD and CN by the quantities of W represented by those lengths, and OC and NP by the quantities of l represented by those lengths. Hence, numerically,

$$\frac{OD}{4{\cdot}80}=\frac{780}{10{\cdot}30-4{\cdot}80}=\frac{780}{5{\cdot}50},$$

so that, since OD represents a negative value of W,

$$b=\frac{1}{OD}=-\frac{5{\cdot}50}{4{\cdot}80\times 780},$$

and the relation between l and W is

$$\frac{l}{4{\cdot}80}=\frac{5{\cdot}50}{4{\cdot}80\times 780}W+1, \quad \text{or} \quad l=\frac{5{\cdot}50}{780}W+4{\cdot}80=\mathbf{0{\cdot}00705\,W+4{\cdot}80},$$

with the proviso that, so far as we know from our experiment, it holds good only for values of W between 300 and 1300.

This is the simplest way of setting out a linear relation. But it is sometimes convenient to write the above relation in the form

$$l=4{\cdot}80(1+0{\cdot}001465W),$$

which is a particular case of the general form $y=y_0(1+\alpha x)$, where y_0 is the value of y when $x=0$. The coefficient of W (that is, 0·001465) is then a constant for all springs of that particular kind, whatever their length may be, and it would be reasonable to call this constant the coefficient of elasticity of such springs.

Note particularly that in this expression for l, y_0 is 4·80, or the intercept of the straight line on the axis of l and not the intercept of the physical graph on that axis, which is 6·60 and represents the actual length of the spring when $W=0$. We may, in fact, call y_0, the value of y when $x=0$, the 'ideal unstretched length', if the relation $y=y_0(1+\alpha x)$ held good when $x=0$ (which it does not).

This has been set down at, perhaps, tedious length, since confusion can easily and often does arise on the point. Hooke's law holds good in this spring only when W is more than 300, and beyond that point the extra extension produced by any additional load is proportional to that additional load, but the total extension is never proportional to the total load. However, the total length l of the spring for any particular value of W can be read off the graph directly, without reference to unstretched lengths and unloaded springs; conversely, of course, the load on the spring can be deduced directly from the graph when the total length l is known.

(*b*) Next, suppose that the periodic times (T sec.) of simple pendulums of various lengths (l cm.) have been measured, producing the first two columns of this table:

l cm.	T sec.	T^2
60	1·55	2·40
80	1·79	3·20
100	2·00	4·00
120	2·19	4·80

Plot the points corresponding to these values of l and T, as (i) in Fig. 2, and draw the best-fit smooth graph among them. This line is so slightly curved that it would be almost excusable to regard it as straight. If we deduce, as in the foregoing example, the relation between T and l we should find it to be

$$T=0{\cdot}01025l+0{\cdot}93.$$

But the above values of l are badly chosen; they do not cover more than a small fraction of the range that can easily be dealt with in the experiment. If we take small values of l, such as 16 and 9, we shall get 0·80 and 0·60 for T, giving the plotted points P and Q. These obviously do not fall on the straight line whose equation has been found; that relation must, therefore, be wrong, and the true relation cannot be linear, of the form $T=al+b$.

If we draw a smooth curve through all these plotted points it is not very illuminating; it is part of a curve which may well pass through the origin, and it looks like a parabola, but that does not enable us to determine at once the relation between l and T, as we could do when the graph was linear.

But the theory of simple harmonic motion, or the practice of Exp. 4, shows that the relation should be $T^2 = Al$, where A is a constant; if then we plot l against T^2 instead of T, we should theoretically get a linear graph, passing through the origin. Filling in the

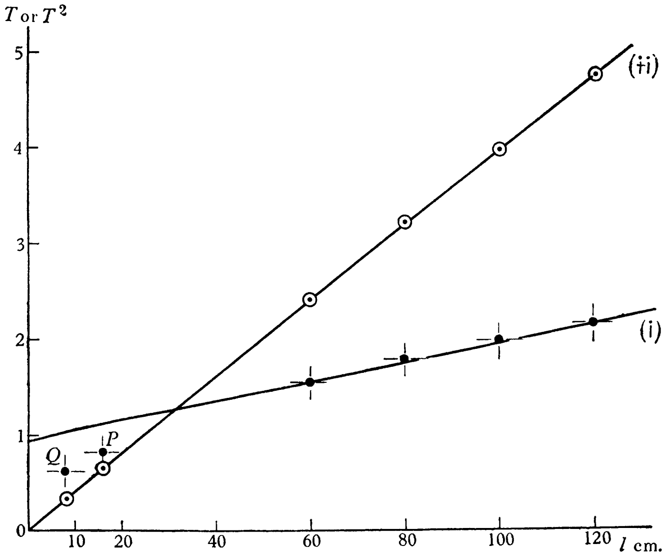

Fig. 2.

third column of the table and taking the squares of 0·80 and 0·60 to correspond with 16·0 and 9·0 for l, we get (ii) in Fig. 2. This may be regarded as confirming the theory by being a straight line through the origin; its slope to the l-axis can be read off the graph as 4 in 100, so the experimental relation is $T^2/l = \frac{4}{100}$, or

$$\mathbf{T^2 = 0{\cdot}04l.}$$

This illustrates the advantages to be gained by manipulating one or both of the measured quantities so that when plotted the corresponding points will, or should, lie on a straight line. To know what sort of manipulation is needed we must know the general form of the relation; theory often furnishes us with this information, as it did in the above instance. But a certain amount of intelligence is often required in addition to theoretical knowledge, for no general rules can be given by which we can dispense with that intelligence.

(*c*) But a good many particular cases fall into one or other of a few categories. Let us denote the experimentally measured quantities by x and y, and unknown constants by a, b, c, etc. The above case of a simple pendulum is a particular case of the general form $\mathbf{y^n = ax + b}$, n in that case being 2 and b being 0. In the general case we must plot y^n against x to get the linear graph from which we determine a and b. Other cases of the same class are given by Boyle's law and Newton's lens formula, and the relation between frequency and length in a monochord; here $b=0$ and $n=-1$, so that we must plot the reciprocal of y against x.

(*d*) Many problems in Optics, and some which are concerned with resistances in parallel, involve the relation $1/x+1/y=\text{const.}$, and this gives a linear graph if we plot the reciprocals of both x and y.

(*e*) Another group, including the temperature of a hot body losing heat to its colder surroundings, the charge in a condenser losing its charge through a resistance, and the speed of a body coming to rest because of 'frictional' resistance proportional to the speed (as in a damped ammeter), involve the relation $y=y_0e^{-ax}$, where x is the elapsed time and y_0 is the value of y when $x=0$, and $e=2{\cdot}718\ldots$. Taking logs of both sides we get $\log y=\log y_0-ax\log e$; since a, $\log e$ and $\log y_0$ are constants, we get a linear graph if we plot $\log y$ against x.

(*f*) Confusion sometimes arises between (A) Hooke's law, Charles's law, etc., where the general relation is $y=y_0(1+ax)$, (B) thermal expansions of solids and liquids, with the same general relation, and (C) the group of cases just mentioned; it may be as well to clear up the position to some extent.

(A) When the change in y may be considerable, as in the thermal expansion of gases under constant pressure, y_1 and y_2 may differ considerably from y_0, and y_0 must therefore be precisely defined.

(B) When the change in y is small in comparison with y, as in the thermal expansion of a solid or liquid, it makes little difference whether we represent the expansion as y_0a, y_1a or y_2a times the change of temperature. This vagueness is harmful only if it leads to the belief that the rate of change of length or volume of the body with temperature is precisely proportional to that length or volume, in which case it should, strictly speaking, come under the next heading (C).

(C) Consider a condenser of capacity C farads, charged with a quantity Q coulombs of electricity which raises the P.D. between the plates to V volts; let the plates be connected through a high resistance R ohms, then the current (i amp.) through the resistance will have a value of V/R or Q/CR, at any time (t) when the charge is Q. Since the current equals the rate at which electricity passes through the resistance, i also equals the time-rate of decrease of Q. If then we plot a graph connecting Q with t, the current at any time will be measured by the slope of the tangent to the graph at the point representing that time. Hence the graph must be of such a form that the slope of the tangent at any point is directly proportional to the ordinate of that point. Mathematics shows (see note to Exp. 16) that in that case the relation between points on the curve must be $Q = Q_0 e^{-at}$, where Q_0 is the value of Q when $t = 0$. From this relation we can get a linear graph, as shown above, in (a).

(g) The experiments in this book furnish many examples of the manipulation of the observed quantities to give linear graphs; it will be seen that in a considerable proportion of them there comes a stage when the results of theoretical reasoning are quoted, expressed as an equation which has already been transformed mathematically so as to be effectively linear (although it may contain powers of the variables higher than the first, or logs of them, etc.). In effect, these transformations may be regarded as part of the laboratory work and not of the theory, which does not itself call for such transformations.

This equation is useful in the practical work in two ways: first, because it may point to the way in which the observations must be manipulated to produce a linear graph so that we may check the *form* of the law, or the relation between the variables, by testing the straightness of the experimental graph; secondly, because we can get accurate numerical values of the two constants a and b in the equation $ax + by = 1$ representing that law, by measuring or calculating the intercepts on the two axes. The theoretical equation gives the values of a and b in terms of the magnitudes of various quantities in the apparatus used in the experiment, which are unchanged throughout that experiment, and we can check the theory in this respect by substituting the measured values of these quantities in our experiment in the theoretical expressions for a and b, and comparing the results with our measured values of a and b.

3. *Distorted Graphs*

If we have measured in a laboratory several pairs of numerical values of two connected physical quantities, each expressed in its appropriate units, A and B say, and wish to represent the results