PRACTICAL PHYSICS

A Collection of Experiments
for Upper Forms of Schools & Colleges
together with the Relevant Theory

BY

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NOTE

The Local Examinations Syndicate was fortunate in securing Sir Cyril Ashford's help and interest after his retirement from Dartmouth in the preparation of practical physics exercises for the Higher School Certificate Examination. Most of the problems in this book were set in that examination between 1932 and 1947. They thus combine the originality and interest which they owe to their author with the merit of having been worked over by moderating examiners in the laboratory and at meetings. In their present form they have also benefited from the fact that the author was able to judge their suitability from the successful and unsuccessful solutions sent in by many examination candidates, and from the comments of their teachers. The Syndicate welcomes Sir Cyril Ashford's re-use of these practical problems as the basis of a teaching text-book which will make available in Schools and Universities the results of the author's pioneering efforts over a period of 15 years.

J. L. BRERETON

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PREFACE

In the early stages of his science course a boy lacks reasoning power and first-hand acquaintance with the properties and behaviour of materials, so that he cannot fully comprehend a logically connected course of lecture-demonstrations. During those stages, laboratory work should therefore, so far as is practicable, go hand-in-hand with the lectures. By the time the boy enters what is commonly called the post-certificate stage the conditions have changed; he can then follow lectures and demonstrations without having previously handled the apparatus himself, and consciousness of the true relation between theory and practice in an inductive science is beginning to dawn on him. The need for individual experimentation is certainly not lessened at this stage, but its main purpose becomes for the time being the verification of theory which has been, or can be, developed by deductive logic from earlier, more fundamental, experiments, and the practical testing of hypotheses, sprung perhaps from his own, perhaps from a more mature, scientific intuition.

The emphasis should now be laid on the best way of using his available instrumental equipment to establish or reject a particular theoretical result, and on the extent to which his experiments do establish it. Laboratory technique and manipulative skill in general, the setting up of apparatus and the precautions for accuracy of observation in a particular case, and the reduction of those observations, all enter into this procedure. New powers of a general kind, applicable to any particular problem, have to be developed, so that there is no longer any strong reason for close relation between lecture and laboratory work; indeed, laboratory work may almost be kept in a separate compartment, to be developed concurrently with his growing body of theoretical knowledge, whereby each may be ready to come to the assistance of the other as need arises. Hence the exercises forming a laboratory text-book for this stage need not have any logical sequence and are therefore capable of being taken in any order (which is a great convenience in a school laboratory with its necessarily limited equipment), nor do they need to cover the whole of the lecture course.

There is already available a considerable body of traditional experiments designed to illustrate the lectures appropriate to this stage; to propose new ones calls for justification. Many of those included in this book, which is primarily designed for the last year of the school course, are intentionally concerned with theory which the schoolboy is unlikely to have studied, and the verification of such theory by experiment is a procedure in line with the foregoing arguments. In others the results to be tested can be deduced from theoretical work with which he is more or less familiar, but the general lay-out of the experiments and the methods of reducing the observations in such a way as to check the theory effectively may be unfamiliar to him. In every case the aim has been to present the problem as one in Practical Physics, with the laboratory functioning at least as the Court of Appeal rather than as a place where lectures can be revised at leisure, and preferably as a real laboratory where a boy can try out hypotheses and determine physical or instrumental constants, free from authoritarian influences.

The outlines of relevant theory with which experiments are prefaced in many text-books are often worded in such a way as to convey the impression that that theory settles the matter by its own authority, and that the boy succeeds in his experiments only in so far as his results agree with it; that he is to do the experiment for the sake of practice in experimenting, and that it is his manipulative skill, not the theory, which is being put to the test. It is in the belief that this is to deny Physics its true status as an inductive science that a vigorous attempt has been made here to banish any such impression from the boy's mind and to make him realise that teachers and text-books alike 'abide our question'.

Most of these exercises are based on questions set during the last 15 years in the Higher School Certificate papers of the Cambridge University Local Examinations Syndicate, and grateful acknowledgement is made to the Syndics for their permission to use them in this form. The exercises have of course been radically transformed and amplified to fit them to form part of a teaching course, and care has been taken to increase rather than diminish the extent to which their original design was influenced by the foregoing considerations. The use of examination questions as a basis for a teaching course has the merit that the problems set for solution in a practical examination must be so designed that they can be carried through in every one of a wide variety of laboratories, some at any rate possessed of a very slender

sequence, all the experiments in

exception, require no apparatus that does not already exist in every laboratory where the book is likely to be used. This puts into effect another general principle which the author believes to be of considerable importance, that a student should learn to rely on his own powers rather than on elaborate equipment; in C. V. Boys's words, not 'to bow down to the brazen image that the instrument maker has set up', but to make the very best use of what is to hand, however simple it may be.

In some of these practical problems the boy cannot be expected to know, to have been taught, or even to have access to books or publications containing the relevant theory. Notes on this theory have therefore been appended to individual exercises.

It is by no means essential, though it is desirable, for him to master the substance of these notes; they may in some cases be too advanced for him, but it will do him good to see how much of them he can follow at the stage he has reached in his theoretical work, and to realise that he can safely and properly explore by experiment ground which is beyond his present range of theory. From that point of view there is really no need for him even to glance at these notes, but they are at hand in case he is moved to do so. Their presence may be rather intimidating to a student unless he understands that this is their purpose, and that for him the actual experiment is the all-important matter; it is hoped that this will be made plain by the teacher and that he may not himself be misled into overrating the difficulty of the course. The fact that most of the experiments have been carried out under examination conditions by hundreds of VIth Form boys, with a measure of success represented by normal distribution curves, should dispose of such misapprehensions.

An Introduction has been provided in which a number of practical points in the reduction of observations is set forth. Teachers usually prefer that their pupils should follow local customs in the setting-up of apparatus and the choice of precautions for accuracy; they would be unlikely to welcome outside interference in these matters. On the other hand, the art of reducing observations may almost be said to be one and indivisible; it would take too long for a demonstrator to teach it separately to each pupil, but it is of such practical importance that the demonstrator may well wish to be able to refer the student to some treatise on the subject; such treatises are not usually to be found in elementary text-books; it therefore seemed desirable to attempt to meet the need here, even in this restricted form. It will be seen that graphical methods, with emphasis on linear equations, have been given the preponderance that is now common in teaching laboratories.

In the Introduction, appendices and some of the notes on theory a few unpublished pieces of theoretical work have been included; the author can only express the hope that they will withstand criticism and plead that he has at least offered his critics facilities for testing their truth by designing experiments for that purpose, in accordance with the principles set out in this preface.

INTRODUCTION

THE REDUCTION OF OBSERVATIONS

Not the least important part of a laboratory experiment is the utilisation of the observations made in the course of that experiment. The object of the experiment may be to measure some physical or instrumental constant, such as g or the focal length of a lens, or to find out how two or more variable quantities are related to one another, such as the volume, pressure and temperature of a constant mass of gas, or it may be a combination of the two purposes. We may have observed as accurately as we can the value of one of these variables corresponding to an observed value of another, and may have done so for several different values of the latter; the problem remains, how to use these pairs of measurements so as to deduce from them the algebraical relation between them.

The first part of this problem is to discover the *form* of this relation; the two columns of figures representing the crude results of our measurements, the 'pointer-readings' as Eddington called them, are often very unpromising material for this process. For instance, after the wave-lengths of the bright lines in the spectrum of hydrogen had been measured it needed a lot of co-operative effort before any mathematical law connecting them could be discovered; again, the positions of one or two planets at various known times had been fairly accurately measured for many years before Kepler could discover his three general, comparatively simple, laws governing the motion of all planets, from which Newton could deduce his supremely simple law of Universal Gravitation. These are instances of the application of genius to great problems; it may be a descent from the sublime to the ridiculous to point out that in Exp. 22 and again in Exps. 28, 29 and 30 problems of the same nature are set for your solution, but very humble illustrations may be the most illuminating.

The second part of the problem is to discover whether the law thus formulated is obeyed with complete precision throughout the whole range of the variables which, mathematically, it purports to cover. For instance, it was long thought to be established that light travelled to us from the stars in geometrically straight lines, but extremely precise measurements (undertaken because Einstein's work on relativity suggested that it might not be true in all circumstances) showed that a ray of light passing near the sun is deflected, very slightly but measurably, by the mass of the sun. Making again the descent from the sublime to the ridiculous, we shall presently have occasion to consider the way in which a spiral steel spring fails in practice to obey Hooke's law.

There is therefore scope for great ingenuity in reducing tables of observations to a form that will be helpful in each of the foregoing processes. For the first, there is clearly need for more than ingenuity; scientific imagination or intuition of the highest order may be required to suggest the form of the relation. But when it is merely a question of testing the truth or applicability of the relation, of 'verifying' it, a more lowly intelligence may suffice, and it may become merely a matter of choosing the most suitable out of a number of well-recognised manipulations of the crude results of the observations. For example, the mere addition or subtraction of a suitable constant in each case (as in Exp. 22 mentioned above) may make it easy to see the next step in evolving the required law; or it may be useful to take the reciprocal, or the logarithm, of each, and so on.

For this reason the following notes on some of the recognised methods of reduction and presentation may be found helpful.

1. *Computation of a Single Constant*

(*a*) Laboratory work often consists of determining, as accurately as we can, the value of a single physical or instrumental constant, assuming the truth of a relation established by theoretical reasoning. For example, we may have to determine the most probable value of the instrumental constant f of a particular lens, accepting the truth of the general lens formula $1/u + 1/v = 1/f$, or we may have to find by experiments on a simple pendulum the value of the physical constant g, accepting the truth of the theoretical relation $T = 2\pi\sqrt{(l/g)}$.

Taking the former example, we set up the apparatus and measure, for one setting of the object, the values of u and v. We substitute these values in the general relation and compute the value of f. If we repeat this operation with a different setting of the object it is almost a certainty that we shall obtain a different value of f; suppose we do this N times in all, we are left with N different values of f, and are faced with the problem of deducing from them a value which is most likely to be the true value of f. Denote this true value by F, and the various calculated values by f_1, f_2, f_3, etc. Then the error of the first determination was $f_1 - F$, of the second was $f_2 - F$, and so on, and the sum-total of all the errors was $\Sigma f_1 - NF$, where Σf_1 means the sum of all terms like f_1.

Suppose that it is equally probable that any one of the calculated values is correct, and that their number N is infinitely large; then the Theory of Probability shows that for every one that is too large by a certain amount we can find one that is too small by exactly the same amount, and the arithmetic mean of these two will be exactly equal to F, and the sum-total of their errors is zero. Repeating this for all the observations, we see that $\Sigma f_1 - NF$ must be zero, or $F = \Sigma f_1/N$, or F, the true value, is the arithmetic mean of all the calculated values.

We cannot, of course, take an infinite number of observations, but reasons are given in § 4 for taking the arithmetic mean of such calculated values as are available as the most probable value of F.

It may happen that one, or more, of these observations leads to a value outstandingly different from the average value; it is tempting to ignore it out of hand as incorrect. It is clearly sound policy to go back and repeat this observation with great care, to find out whether there has been an error of observation or calculation. If there was, we can obviously ignore the original observation and substitute the revised one; if not, it is on the whole the soundest policy to retain it.

(*b*) This process of computing the arithmetic mean is in many cases quicker than a graphical method of determining the constant; it can be carried to a higher degree of numerical accuracy, and it has the additional merit of leading to a numerical estimate of the degree to which our experiments are trustworthy.

Take, for instance, the numbers in the first and second columns of this table, which are the reciprocals of the values of u and v found in the course of Exp. 25 (B) with a concave lens:

$1/u$	$1/v$	$-1/f$	$-f$	Difference from mean
0·01117	0·07681	0·06564	15·23	0·00
0·01447	0·07987	0·06540	15·29	+0·06
0·02086	0·08681	0·06605	15·14	−0·09
0·02574	0·09076	0·06504	15·37	+0·14
0·02995	0·09506	0·06611	15·13	−0·10
0·03402	0·09982	0·06580	15·20	−0·03
			6|91·36	6|0·42
			15·23	**0·07**

The third column shows the difference between the first and second, which in accordance with the accepted theory is $-1/f$; the fourth shows $-f$ (taken from a table of reciprocals) and the way it leads to the arithmetic mean value for f of 15·23. This is the best value of f which we can get from our observations.

We can consider the fifth column as giving the 'error' of each observation; if we take the arithmetic mean of these numbers, disregarding their signs (as shown), we can consider the result, 0·07, as representing the mean error. It is customary to express these results in the form $f = -15{\cdot}23 \pm 0{\cdot}07$.

Since 0·07 is about $\frac{1}{2}$% of the value of f, it is often said that these experiments are 'accurate to $\frac{1}{2}$%'. This shorthand statement is convenient for purposes of comparison but liable to mislead unless we bear in mind what it really means; for instance, one of these experiments shows an error of almost 1% of the arithmetic mean of the values we have obtained for f.

2. *Graphical Representation; Linear Graphs*

Laboratory work usually involves more complex problems than those dealt with in § 1, where we assumed that the relation between the measured variables was known; the problem is often to determine by experiment what the relation is. Let us assume for the present that our measurements are absolutely accurate; we can afterwards consider how best to allow for observational errors.

Suppose that we plot in the usual way the points represented by the pairs of observations, using such scales along the two axes as will spread the points satisfactorily on the graph paper. Suppose that we find that these points lie on a straight line; the relation between the observed quantities is clearly linear, of the form $ax+by=1$, where x and y are the quantities and a and b are some constants.

If this graph is not a straight line, let us suppose that by some manipulation of the observed quantities (e.g. squaring one or both or taking logs of one or both), and plotting these manipulated quantities we can get a straight-line graph. Then, again, the form of the relation is discovered.

Further, a and b are at once measurable since the intercept on the axis of x, found by putting $y=0$ in the equation $ax+by=1$ to be $1/a$, can be read off as the value of the intersection of the straight-line graph and the axis of x in terms of the scale for that axis. Similarly,

$1/b$ can be found. But it is essential to be on one's guard in carrying out even this familiar procedure.

(*a*) Suppose that a closely coiled spiral steel spring is hung up and loaded with a series of weights (W g.) ranging from 0 to 1300 g., and the corresponding overall lengths (l cm.) of the spring are measured; the points representing l and W are plotted and the full-line graph in Fig. 1 drawn. Let us assume that the graph between A and B is, as it appears to be, a straight line.

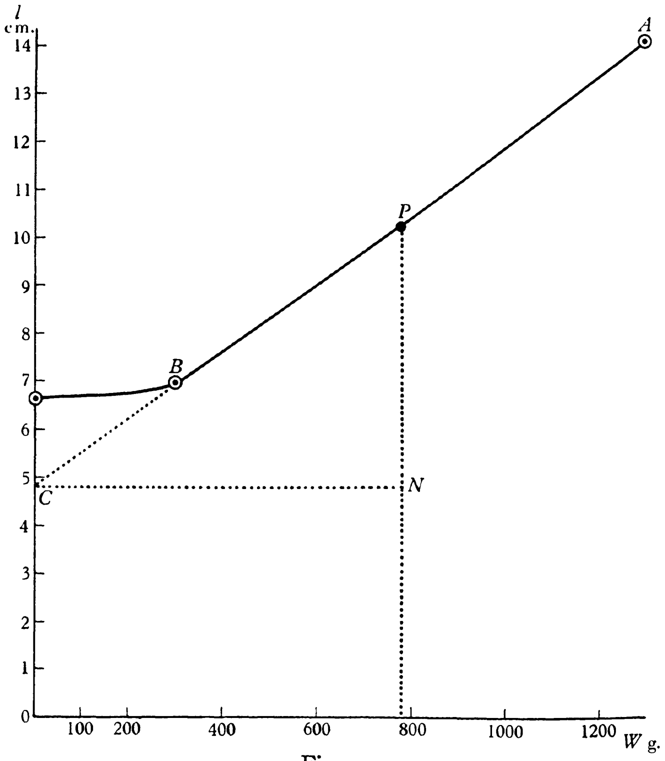

Fig. 1.

It is easy to account for the horizontal part of the graph, since, let us suppose, the experimental set-up showed that at a load of 300 g. consecutive turns of the wire were in contact, and the spring could not shorten further as the load was reduced.

Hence any linear algebraical relation connecting l and W which we may find to be true for loads on this spring between 300 and 1300 g. may be true for greater loads, but it certainly cannot be true for loads less than 300 g.

Nevertheless, any relation between the co-ordinates l and W which holds good for a portion of a straight line must hold good mathematically for the whole of it, produced to infinity both ways. Hence in finding the algebraical relation which holds good physically only for the part AB of the graph we may produce that part geometrically to cut the axis of l at C (as shown by the dotted line) and beyond C to cut the axis of W at D. Hence if the required relation between l and W is

$$al+bW=1,$$

since the intercept on the l-axis is $1/a$, and we see by the graph that the intercept is 4·80, we deduce at once that $a=1/4{\cdot}80$.

The value of the intercept OD on the axis of W cannot be read off so immediately, but can be deduced from the graph as follows.

By similar triangles, $OD/OC=NC/NP$. Here we are assuming that these are geometrical lengths on the graph paper, but the proportion holds good if we replace OD and CN by the quantities of W represented by those lengths, and OC and NP by the quantities of l represented by those lengths. Hence, numerically,

$$\frac{OD}{4{\cdot}80}=\frac{780}{10{\cdot}30-4{\cdot}80}=\frac{780}{5{\cdot}50},$$

so that, since OD represents a negative value of W,

$$b=\frac{1}{OD}=-\frac{5{\cdot}50}{4{\cdot}80\times 780},$$

and the relation between l and W is

$$\frac{l}{4{\cdot}80}=\frac{5{\cdot}50}{4{\cdot}80\times 780}W+1, \quad \text{or} \quad l=\frac{5{\cdot}50}{780}W+4{\cdot}80=\mathbf{0{\cdot}00705\,W+4{\cdot}80},$$

with the proviso that, so far as we know from our experiment, it holds good only for values of W between 300 and 1300.

This is the simplest way of setting out a linear relation. But it is sometimes convenient to write the above relation in the form

$$l=4{\cdot}80(1+0{\cdot}001465W),$$

which is a particular case of the general form $y=y_0(1+\alpha x)$, where y_0 is the value of y when $x=0$. The coefficient of W (that is, 0·001465) is then a constant for all springs of that particular kind, whatever their length may be, and it would be reasonable to call this constant the coefficient of elasticity of such springs.

Note particularly that in this expression for l, y_0 is 4·80, or the intercept of the straight line on the axis of l and not the intercept of the physical graph on that axis, which is 6·60 and represents the actual length of the spring when $W=0$. We may, in fact, call y_0, the value of y when $x=0$, the 'ideal unstretched length', if the relation $y=y_0(1+\alpha x)$ held good when $x=0$ (which it does not).

This has been set down at, perhaps, tedious length, since confusion can easily and often does arise on the point. Hooke's law holds good in this spring only when W is more than 300, and beyond that point the extra extension produced by any additional load is proportional to that additional load, but the total extension is never proportional to the total load. However, the total length l of the spring for any particular value of W can be read off the graph directly, without reference to unstretched lengths and unloaded springs; conversely, of course, the load on the spring can be deduced directly from the graph when the total length l is known.

(*b*) Next, suppose that the periodic times (T sec.) of simple pendulums of various lengths (l cm.) have been measured, producing the first two columns of this table:

l cm.	T sec.	T^2
60	1·55	2·40
80	1·79	3·20
100	2·00	4·00
120	2·19	4·80

Plot the points corresponding to these values of l and T, as (i) in Fig. 2, and draw the best-fit smooth graph among them. This line is so slightly curved that it would be almost excusable to regard it as straight. If we deduce, as in the foregoing example, the relation between T and l we should find it to be

$$T=0{\cdot}01025l+0{\cdot}93.$$

But the above values of l are badly chosen; they do not cover more than a small fraction of the range that can easily be dealt with in the experiment. If we take small values of l, such as 16 and 9, we shall get 0·80 and 0·60 for T, giving the plotted points P and Q. These obviously do not fall on the straight line whose equation has been found; that relation must, therefore, be wrong, and the true relation cannot be linear, of the form $T=al+b$.

If we draw a smooth curve through all these plotted points it is not very illuminating; it is part of a curve which may well pass through the origin, and it looks like a parabola, but that does not enable us to determine at once the relation between l and T, as we could do when the graph was linear.

But the theory of simple harmonic motion, or the practice of Exp. 4, shows that the relation should be $T^2 = Al$, where A is a constant; if then we plot l against T^2 instead of T, we should theoretically get a linear graph, passing through the origin. Filling in the

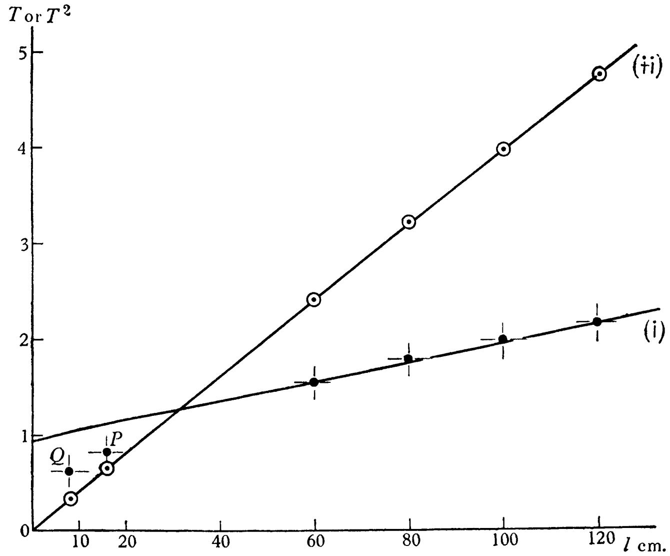

Fig. 2.

third column of the table and taking the squares of 0·80 and 0·60 to correspond with 16·0 and 9·0 for l, we get (ii) in Fig. 2. This may be regarded as confirming the theory by being a straight line through the origin; its slope to the l-axis can be read off the graph as 4 in 100, so the experimental relation is $T^2/l = \frac{4}{100}$, or

$$\mathbf{T^2 = 0{\cdot}04l.}$$

This illustrates the advantages to be gained by manipulating one or both of the measured quantities so that when plotted the corresponding points will, or should, lie on a straight line. To know what sort of manipulation is needed we must know the general form of the relation; theory often furnishes us with this information, as it did in the above instance. But a certain amount of intelligence is often required in addition to theoretical knowledge, for no general rules can be given by which we can dispense with that intelligence.

(*c*) But a good many particular cases fall into one or other of a few categories. Let us denote the experimentally measured quantities by x and y, and unknown constants by a, b, c, etc. The above case of a simple pendulum is a particular case of the general form $\mathbf{y^n = ax + b}$, n in that case being 2 and b being 0. In the general case we must plot y^n against x to get the linear graph from which we determine a and b. Other cases of the same class are given by Boyle's law and Newton's lens formula, and the relation between frequency and length in a monochord; here $b = 0$ and $n = -1$, so that we must plot the reciprocal of y against x.

(*d*) Many problems in Optics, and some which are concerned with resistances in parallel, involve the relation $1/x + 1/y = \text{const.}$, and this gives a linear graph if we plot the reciprocals of both x and y.

(*e*) Another group, including the temperature of a hot body losing heat to its colder surroundings, the charge in a condenser losing its charge through a resistance, and the speed of a body coming to rest because of 'frictional' resistance proportional to the speed (as in a damped ammeter), involve the relation $y = y_0 e^{-ax}$, where x is the elapsed time and y_0 is the value of y when $x = 0$, and $e = 2{\cdot}718\ldots$. Taking logs of both sides we get $\log y = \log y_0 - ax \log e$; since a, $\log e$ and $\log y_0$ are constants, we get a linear graph if we plot $\log y$ against x.

(*f*) Confusion sometimes arises between (A) Hooke's law, Charles's law, etc., where the general relation is $y = y_0(1 + ax)$, (B) thermal expansions of solids and liquids, with the same general relation, and (C) the group of cases just mentioned; it may be as well to clear up the position to some extent.

(A) When the change in y may be considerable, as in the thermal expansion of gases under constant pressure, y_1 and y_2 may differ considerably from y_0, and y_0 must therefore be precisely defined.

(B) When the change in y is small in comparison with y, as in the thermal expansion of a solid or liquid, it makes little difference whether we represent the expansion as $y_0 a$, $y_1 a$ or $y_2 a$ times the change of temperature. This vagueness is harmful only if it leads to the belief that the rate of change of length or volume of the body with temperature is precisely proportional to that length or volume, in which case it should, strictly speaking, come under the next heading (C).

(C) Consider a condenser of capacity C farads, charged with a quantity Q coulombs of electricity which raises the P.D. between the plates to V volts; let the plates be connected through a high resistance R ohms, then the current (i amp.) through the resistance will have a value of V/R or Q/CR, at any time (t) when the charge is Q. Since the current equals the rate at which electricity passes through the resistance, i also equals the time-rate of decrease of Q. If then we plot a graph connecting Q with t, the current at any time will be measured by the slope of the tangent to the graph at the point representing that time. Hence the graph must be of such a form that the slope of the tangent at any point is directly proportional to the ordinate of that point. Mathematics shows (see note to Exp. 16) that in that case the relation between points on the curve must be $Q=Q_0e^{-at}$, where Q_0 is the value of Q when $t=0$. From this relation we can get a linear graph, as shown above, in (a).

(g) The experiments in this book furnish many examples of the manipulation of the observed quantities to give linear graphs; it will be seen that in a considerable proportion of them there comes a stage when the results of theoretical reasoning are quoted, expressed as an equation which has already been transformed mathematically so as to be effectively linear (although it may contain powers of the variables higher than the first, or logs of them, etc.). In effect, these transformations may be regarded as part of the laboratory work and not of the theory, which does not itself call for such transformations.

This equation is useful in the practical work in two ways: first, because it may point to the way in which the observations must be manipulated to produce a linear graph so that we may check the *form* of the law, or the relation between the variables, by testing the straightness of the experimental graph; secondly, because we can get accurate numerical values of the two constants a and b in the equation $ax+by=1$ representing that law, by measuring or calculating the intercepts on the two axes. The theoretical equation gives the values of a and b in terms of the magnitudes of various quantities in the apparatus used in the experiment, which are unchanged throughout that experiment, and we can check the theory in this respect by substituting the measured values of these quantities in our experiment in the theoretical expressions for a and b, and comparing the results with our measured values of a and b.

3. *Distorted Graphs*

If we have measured in a laboratory several pairs of numerical values of two connected physical quantities, each expressed in its appropriate units, A and B say, and wish to represent the results graphically, the simplest way to do so is to use a sheet of squared paper, divided into inches (say) and tenths, and to take an inch horizontally to represent A and an inch vertically to represent B, and number each axis progressively from the origin. The point (such as P_1) corresponding to each observation is then plotted as usual, and a smooth curve drawn to fit these points as nearly as possible; it is assumed in Fig. 3*a* to be a straight line. Let us call such a curve the *true* graph of the experiment; it is what a mathematician means by the phrase 'the curve represented by an equation'.

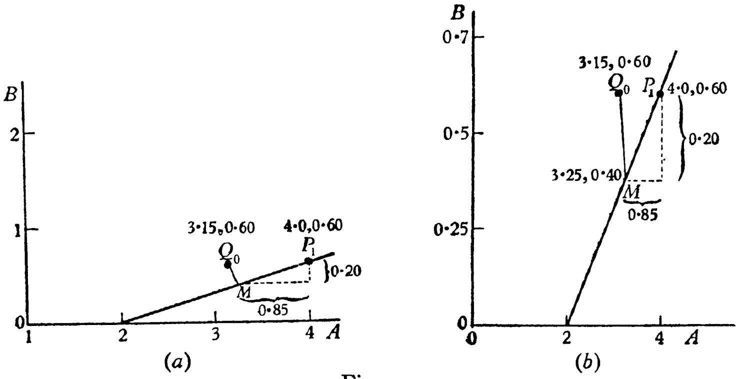

Fig. 3.

This graph may be badly spaced on the paper; we usually rectify this by changing the scales on one or both of the axes, making an inch on the axis represent many times, or a fraction of, A or B; the two axes are then numbered off according to these new scales, as in Fig. 3*b*, and the points are plotted and the smooth curve drawn exactly as before.

What we have done in effect is to stretch, or shrink, the paper carrying the graphs and all lines connected with it; the stretching or shrinking takes place in the direction of the axes, and is usually different in the two directions. Comparison of the two parts of Fig. 3 shows that this procedure produces a geometrical distortion of the true graph; let us then call the result the *distorted* graph. In particular, lines such as Q_0M and P_1M which are at right angles in the true graph are far from perpendicular to one another in the distorted graph. A tangent at a point of a curve in the true graph is still a tangent in the distorted graph, but the normal which is at right angles to that tangent in the true graph looks absurd in the distorted graph; a circle distorts into an ellipse, and so on.

The only way in which the beginner is likely to be troubled by the distortion is in the matter of measuring the inclination of a straight line to one of the axes; that inclination is obviously increased by any stretching perpendicular to that axis. He is therefore taught not to measure the angle of inclination by a protractor and take the tangent of that angle, either on the distorted or true graph, but to measure what he learns to call the 'slope' of the line, by finding by means of the graduations on the vertical axis the increase, for two points on the line, of the distance from the horizontal axis, and the corresponding increase of distance from the vertical axis and dividing the former by the latter.

It will be seen that this is in effect merely a process of 'referring back' to the true graph, since this procedure applied to a line on the true graph gives the trigonometrical tangent of the actual inclination to the axis, and as will be seen by comparing the two parts of Fig. 3 the same numbers are used in the same way when measuring the slope of the line on both graphs. It is an application of the general rule to work by the numbered graduations along the axes in any graph, and disregard the lengths on the graph paper, especially when those lengths are not parallel to either of the axes.

4. *Most Probable Value of a Single Constant*

The problem of deducing the most probable final result, whether that be a single constant or a linear graph, from the results of a few experiments when those results are not all identical was shirked in the earlier parts of this Introduction; it is so important that it must now be dealt with.

The general treatment will be much easier to follow if we first analyse in detail a simple concrete case. Suppose that we have made three sets of measurements of a single quantity such as the focal length of a certain lens, and have got 20·60, 20·20 and 20·10 cm. Let us denote by x_0 the most probable value of the quantity, 'most probable value' meaning the closest approximation to the *true* value that we can deduce from our observations.

These observations may be incorrect for either or both of two reasons. First, the apparatus may be faulty; for instance, it may contain a paper scale of lengths which expands when damp, or there may be an undetermined zero error; errors on this account are often termed 'systematic errors' and strictly speaking are not covered by this section. Secondly, the setting of a moveable component or the reading taken may be faulty, for example in estimating tenths of a scale division; these are usually termed 'casual errors of experiment' and will for brevity be called 'errors' in this section.

The 'error' of any observation is properly the difference between the observed and the true values; since we can never know the absolutely true value we will take the error to be the amount by which the observed value exceeds x_0, the most probable value deduced in some way from our observations. Hence the error may be positive or negative.

It is reasonable to assume that the observed values are on the whole grouped closely round the most probable value x_0.

(*a*) It would be reasonable to choose x_0 so that it makes the aggregate of all the errors, that is, their numerical sum without regard to their signs, as small as possible. Now in this case, for such close grouping, x_0 must lie somewhere between the largest, 20·60, and the smallest, 20·10, and the numerical values of the errors of these two observations will be $20{\cdot}60 - x_0$ and $x_0 - 20{\cdot}10$. Hence whatever value x_0 may have between 20·60 and 20·10, their numerical sum will be $20{\cdot}60 - x_0 + x_0 - 20{\cdot}10$ or **0·50**. To this must be added, in order to get the aggregate, the numerical value of the second error, which is $20{\cdot}20 \sim x_0$, without regard to sign; whatever its sign, this addition will be smallest when x_0 equals 20·20, for this error is then zero. Hence on this assumption **20·20** is the most probable value of x_0, and the numerical sum of the errors is $0{\cdot}40 + 0{\cdot}00 + 0{\cdot}10$ or **0·50**.

(*b*) Another possible assumption for getting x_0 is that the *algebraic* sum of the errors should be zero.

This leads to $20{\cdot}60 - x_0 + 20{\cdot}20 - x_0 + 20{\cdot}10 - x_0 = 0$ or

$$x_0 = \tfrac{1}{3} \times 60{\cdot}90 = \mathbf{20{\cdot}30}.$$

This is obviously equivalent to adopting the arithmetic mean of the observed values as the most probable value, which is the usual custom. It brings out a different value for x_0, 20·30 instead of 20·20 as in (*a*); the numerical sum of the errors is now 0·6 instead of 0·5.

(*c*) A third possible assumption avoids all trouble with signs, by dealing with squares, which are always positive. It is, that the sum of the squares of the errors should be a minimum.

Let us try, without using anything but simple arithmetic, how well the above values of x_0 (20·20 and 20·30) agree with this assumption. If we calculate the sums of the squares of the errors when x_0

is 20·195, 20·200 and 20·205 we find them to be 0·1730, 0·1700 and 0·1670 respectively. So the sum decreases continuously, and is not a minimum when $x_0 = 20{\cdot}20$. Repeating this for $x_0 = 20{\cdot}30$, the sums of the squares when x_0 is 20·295, 20·300 and 20·305 are 0·14009, 0·14000 and 0·14009 respectively, showing a minimum at $x_0 = 20{\cdot}30$.

Hence this assumption leads to the same results as the Arithmetic Mean method of (*b*), but not as (*a*).

(*d*) Each of these three assumptions is reasonable; how are we to choose between them? They have the common merit that, as shown in the Theory of Probability, they all give identically the same result when applied to an infinitely large number of observations; but in practice we are concerned only with a small number of observations. However, there is one distinct difference between the assumptions; the first does not lend itself at all to mathematical treatment, the second does so, and the third does so more widely and in some cases (one of which we shall soon come across) it is the only one that can be used. This third one is called the Method of Least Squares.

Hence, when dealing with a small number of observations of a single constant we will adopt the result of the second and third assumptions, that *the most probable value is the arithmetic mean of the observed values.*

(*e*) Replacing the particular case of (*b*) and (*c*) by a more general treatment, if we denote by ΣE_1 the algebraic sum of the errors in all the N observations which gave observed values x_1, x_2, x_3, etc., the assumption in (*b*) gives

$$0 = \Sigma E_1 = \Sigma(x_1 - x_0) = \Sigma x_1 - \Sigma x_0 = \Sigma x_1 - N x_0.$$

But $\dfrac{\Sigma x_1}{N}$ is the arithmetic mean of x_1, x_2, etc., so $x_0 = \dfrac{\Sigma x_1}{N}$ = the arithmetic mean of the observed values.

Again, the assumption in (*c*) is that $(x_1 - x_0)^2 + (x_2 - x_0)^2 +$ etc. is a minimum for changes in x_0. Books on the differential calculus show that this happens when $\dfrac{d}{dx_0}\{(x_1 - x_0)^2 + (x_2 - x_0)^2 + \text{etc.}\} = 0$,

or when $$-2(x_1 - x_0) - 2(x_2 - x_0) - \text{etc.} = 0,$$

or when $$-2\{\Sigma x_1 - N x_0\} = 0,$$

or when

$$x_0 = \frac{\Sigma x_1}{N} = \text{arithmetic mean of the observed values as before.}$$

(5) *Most Probable Linear Equation, or best-fit straight line*

It may perhaps be advisable to preface this section with the words inscribed over the entrance to Plato's Academy, ἀγεωμέτρητος μηδεὶς εἰσίτω, although it is the quantity rather than the advanced quality of the mathematics in it that may deter him.

If we want to find the most probable position of the straight-line graph among a number of plotted points representing observations, which we may call the 'best-fit line', we have a less simple task, for this line may be displaced broadside, or may turn about a fixed point, or both simultaneously.

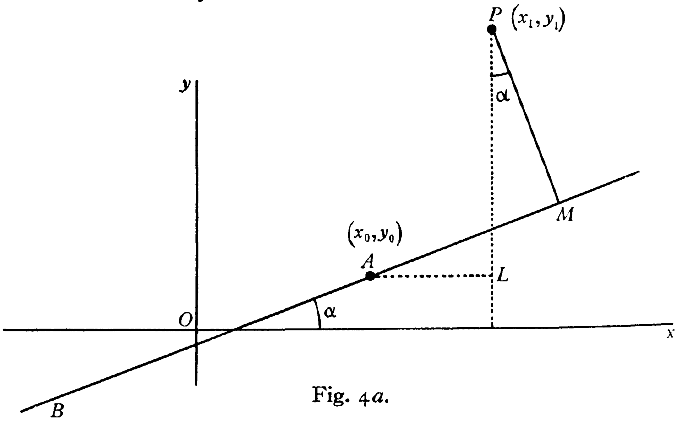

Fig. 4*a*.

Let us make the reasonable assumption that the error of any of the observations can be measured by the length of the perpendicular, from the point P on the graph paper representing the observation, dropped on the best-fit line.

Hence in Fig. 4*a*, if AB is the best-fit line, the error of observation P is PM.

Let x_1 and y_1, x_2 and y_2, etc. denote the pairs of observations, represented on the graph by points P_1, P_2, etc.; we want to find the best-fit straight line among these points, or the linear equation which is most nearly satisfied by x_1 and y_1, etc. It should be noted that these values of x_1, x_2, etc. probably differ widely in magnitude and perhaps in sign, unlike the x_1, x_2, etc. of § 4 (*c*) which are all nearly equal in magnitude and almost certainly of the same sign. Hence Σx_1 represents the algebraic sum of x_1, etc., each keeping its appropriate sign; $\frac{\Sigma x_1}{N}$ may therefore differ widely from any particular value of x.

Denote the co-ordinates of some point A on the line by x_0, y_0; then if AL and PL are parallel to the axis of x and y, $AL = x_1 - x_0$ and $PL = y_1 - y_0$. Suppose that AB makes an angle α with the axis of x; this is the inclination of the graph when the scales along the two axes are equal, not the actual angle on a distorted graph (see §3). From Fig. 4*a* we see that the error

$$\begin{aligned} PM &= PL \cos\alpha - AL \sin\alpha \\ &= (y_1 - y_0)\cos\alpha - (x_1 - x_0)\sin\alpha. \qquad (1) \end{aligned}$$

(*a*) Let us first apply condition (*b*) of §4, that the algebraic sum of all these errors is zero if AB is the best-fit line. Then the algebraic sum of the errors is

$$\Sigma\{(y_1 - y_0)\cos\alpha - (x_1 - x_0)\sin\alpha\}$$

so we must have

$$\Sigma\{(y_1 - y_0)\cos\alpha - (x_1 - x_0)\sin\alpha\} = 0 \qquad (2)$$

as the equation from which to determine α and the fixed point x_0, y_0 on the best-fit line. This equation can be written, since α and x_0 and y_0 are the same for all the N observations,

$$\cos\alpha\Sigma(y_1 - y_0) - \sin\alpha\Sigma(x_1 - x_0) = 0$$

or

$$\cos\alpha(\Sigma y_1 - Ny_0) - \sin\alpha(\Sigma x_1 - Nx_0) = 0. \qquad (3)$$

Denote $\frac{\Sigma x_1}{N}$ by $\bar{x}$, and $\frac{\Sigma y_1}{N}$ by $\bar{y}$. Hence, since neither $\cos\alpha$ nor $\sin\alpha$ can be infinitely large, this equation is satisfied if $y_0 = \bar{y}$ and $x_0 = \bar{x}$. Hence the best-fit line must pass through the point $\bar{x}, \bar{y}$.

The equation (3) is satisfied by these values of x_0 and y_0, whatever be the value of α; hence condition (*b*) of §4 is satisfied by any straight line through $\bar{x}, \bar{y}$, and it does not suffice to fix completely the best-fit line. We must therefore try another, more effective, condition; let us try the Method of Least Squares, the condition (*c*) of § 4.

(*b*) This condition is that ΣPM^2 must be a minimum when AB is the best-fit line.

Now

$$\begin{aligned} PM^2 &= \{(y_1 - y_0)\cos\alpha - (x_1 - x_0)\sin\alpha\}^2 \\ &= (y_1 - y_0)^2\cos^2\alpha + (x_1 - x_0)^2\sin^2\alpha - 2(y_1 - y_0)(x_1 - x_1)\sin\alpha\cos\alpha. \end{aligned}$$

Hence

$$\begin{aligned} \Sigma PM^2 = \Sigma(y_1 - y_0)^2\cos^2\alpha + \Sigma(x_1 - x_0)^2\sin^2\alpha \\ - 2\Sigma(y_1 - y_0)(x_1 - x_0)\sin\alpha\cos\alpha. \end{aligned}$$

Denote this expression by u for brevity.

Books on the differential calculus show that u is a maximum or a minimum for changing values of x_0 and y_0 and α when

$$\frac{du}{dx_0}=0, \quad \frac{du}{dy_0}=0 \quad \text{and} \quad \frac{du}{d\alpha}=0$$

respectively.

Now for a constant value of α

$$u=\cos^2\alpha\Sigma(y_1-y_0)^2+\sin^2\alpha\Sigma(x_1-x_0)^2-2\sin\alpha\cos\alpha\Sigma(y_1-y_0)(x_1-x_0). \tag{4}$$

Then if y_0 alone varies

$$\frac{du}{dy_0}=-2\cos^2\alpha\Sigma(y_1-y_0)+2\sin\alpha\cos\alpha\Sigma(x_1-x_0)$$
$$=-2\cos^2\alpha(\Sigma y_1-\Sigma y_0)+2\sin\alpha\cos\alpha(\Sigma x_1-\Sigma x_0).$$

We have denoted Σy_1 by $N\bar{y}$ and Σx_1 by $N\bar{x}$, and since x_0 and y_0, whatever they may be, are the same for all the N observations,

$$\Sigma y_0=Ny_0 \quad \text{and} \quad \Sigma x_0=Nx_0.$$

Hence

$$\frac{du}{dy_0}=-2\cos^2\alpha(N\bar{y}-Ny_0)+2\sin\alpha\cos\alpha(N\bar{x}-Nx_0). \tag{5}$$

Similarly

$$\frac{du}{dx_0}=-2\sin^2\alpha(N\bar{x}-Nx_0)+2\sin\alpha\cos\alpha(N\bar{y}-Ny_0). \tag{6}$$

Both (5) and (6) vanish if $y_0=\bar{y}$ and $x_0=\bar{x}$. ΣPM^2 will therefore be a maximum or minimum for any particular value of α if the line AB passes through $\bar{x}$, $\bar{y}$. This is what we found before, but we have now a further condition in hand by which we may determine α.

In using this condition we can substitute $\bar{x}$ for x_0 and $\bar{y}$ for y_0 in (4). Then if α varies, (4) gives

$$\frac{du}{d\alpha}=-2\cos\alpha\sin\alpha\Sigma(y_1-\bar{y})^2+2\sin\alpha\cos\alpha\Sigma(x_1-\bar{x})^2$$
$$-2(\cos^2\alpha-\sin^2\alpha)\Sigma(y_1-\bar{y})(x_1-\bar{x}). \tag{7}$$

Hence the condition determining α for a maximum or minimum of ΣPM^2 is that (7) equals zero. Dividing by $-2\cos^2\alpha$ and putting $m=\tan\alpha$, we must have

$$0=m\{\Sigma(y_1-\bar{y})^2-\Sigma(x_1-\bar{x})^2\}+(1-m^2)\Sigma(y_1-\bar{y})(x_1-\bar{x}).$$

Solving this quadratic,

$$m=A\pm\sqrt{(A^2+1)}, \quad \text{where} \quad A=\frac{\Sigma(y_1-\bar{y})^2-\Sigma(x_1-\bar{x})^2}{2\Sigma(y_1-\bar{y})(x_1-\bar{x})}. \tag{8}$$

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Hence the straight line which we seek passes through the point $\bar{x}, \bar{y}$ and is inclined to the axis of x at an angle α whose tangent m is given by (8).

But (8) gives two possible values for m; denote them by m_1 and m_2. Then the equation of the best-fit line is either

$$y-\bar{y}=m_1(x-\bar{x}) \tag{9}$$

or

$$y-\bar{y}=m_2(x-\bar{x}). \tag{10}$$

It is easy to determine which of these two is the best-fit line, by substituting a pair of observed values, say $x_1 y_1$, which is far from $\bar{x}, \bar{y}$, in (9) and (10). Only one equation will be nearly satisfied, and this contains the value of m to be adopted in the equation to the best-fit line.

It is also easy to see what the other equation represents; from (8)

$$m_1 m_2=\{A+\sqrt{(A^2+1)}\}\{A-\sqrt{(A^2+1)}\}=A^2-(A^2+1)=-1.$$

This shows that the straight lines represented by (9) and (10) are at right angles to one another; in fact, if one is the best-fit line, the other is the worst-fit line, corresponding to the condition that the sum of the squares of the perpendiculars on it is a *maximum*.

(*c*) A worked-out numerical example will help to make this general theoretical reasoning more clear, and it may be a sufficient guide to those who wish to make use of the method without understanding the whole or any part of the reasoning on which the method is based.

Suppose that we have obtained the five pairs of values x and y shown in the first two columns of this table, either by direct observation or by manipulating direct observations in such a way as to get a linear graph; some poor observations have been purposely included, as will be seen in Fig. 4*b*.

x	y	$x-\bar{x}$	$(x-\bar{x})^2$	$y-\bar{y}$	$(y-\bar{y})^2$	$(x-\bar{x})(y-\bar{y})$ +	$(x-\bar{x})(y-\bar{y})$ −
7·30	3·40	−2·330	5·429	−6·302	39·72	14·68	
8·23	7·00	−1·400	1·960	−2·702	7·301	3·782	
9·33	11·91	−0·300	0·090	2·208	4·875	—	0·662
11·42	10·89	1·790	3·204	1·188	1·411	2·127	
11·87	15·31	2·240	5·018	5·608	31·45	12·56	
5 | 48·15	5 | 48·51		15·701		84·757	33·149	
						0·662	
9·630	9·702					32·487	
$\bar{x}$	$\bar{y}$		$\Sigma(x_1-\bar{x})^2$		$\Sigma(y_1-\bar{y})^2$	$\Sigma(x_1-\bar{x})(y_1-\bar{y})$	

In this table $\bar{x}$ and $\bar{y}$ are found as the arithmetic means of the numbers in each of the first two columns in the usual way. The third column contains the values of $x-\bar{x}$ with the appropriate signs, and the fourth the values of $(x-\bar{x})^2$, and consequently of $\Sigma(x_1-\bar{x})^2$, the sum of all the terms like $(x_1-\bar{x})^2$.

In the fifth and sixth, the same is done for $\Sigma(y_1-y)^2$; the seventh and eighth give the products of the third and fourth, paying due regard to signs; these give the algebraical sum $\Sigma(x_1-\bar{x})(y_1-\bar{y})$.

In this case A in (8) becomes

$$\frac{84{\cdot}757-15{\cdot}701}{2\times 32{\cdot}487} \text{ or } 1{\cdot}063, \text{ and therefore } A^2=1{\cdot}130.$$

Substituting these in $m=A\pm\sqrt{(A^2+1)}$ we get

$$m=1{\cdot}063\pm\sqrt{(2{\cdot}130)}=1{\cdot}063\pm1{\cdot}459$$
$$=2{\cdot}522 \text{ or } -0{\cdot}396. \qquad (11)$$

Hence from the theoretical reasoning in § 5(*b*) the best-fit line must pass through the point $\bar{x}$, $\bar{y}$ or 9·630, 9·702 and must have a slope to the axis of x of either 2·522 or −0·396. That is, the relation between x and y in the best-fit line must be either

$$y-9{\cdot}702=2{\cdot}522(x-9{\cdot}630) \qquad (12)$$

or

$$y-9{\cdot}702=-0{\cdot}396(x-9{\cdot}630). \qquad (13)$$

To find out which of these it is, substitute a pair of observed values of x and y from the table for x and y in each of these equations. Thus

$$3{\cdot}40-9{\cdot}702=2{\cdot}522(7{\cdot}30-9{\cdot}630) \quad \text{or} \quad -6{\cdot}302=-2{\cdot}522\times2{\cdot}330,$$
$$3{\cdot}40-9{\cdot}702=-0{\cdot}396(7{\cdot}30-9{\cdot}630) \quad \text{or} \quad -6{\cdot}302=+0{\cdot}396\times2{\cdot}330.$$

It is obvious that the first equation is nearly satisfied, and the second is not, so it is (12) that is the 'most probable relation' between x and y, deduced from the observations.

It may be pointed out at this point that the foregoing procedure is simply a matter of obtaining the 'constants' in (12) by computation; graphical methods do not enter into it, any more than they do in obtaining the arithmetic mean of a number of observed values of a single quantity.

Computation is usually a more accurate process than graphical methods, even in the case of a linear equation which is most favourable to the latter; but it is not so illuminating, or perhaps so convincing. So it may be helpful to show graphically the final result of the above computation.

The two straight lines represented by equations (12) and (13) are shown in Fig. 4*b*; since the vertical and horizontal scales are not equal this is a distorted graph (see §3), but if equal scales had been used the two lines would have been at right angles to one another.

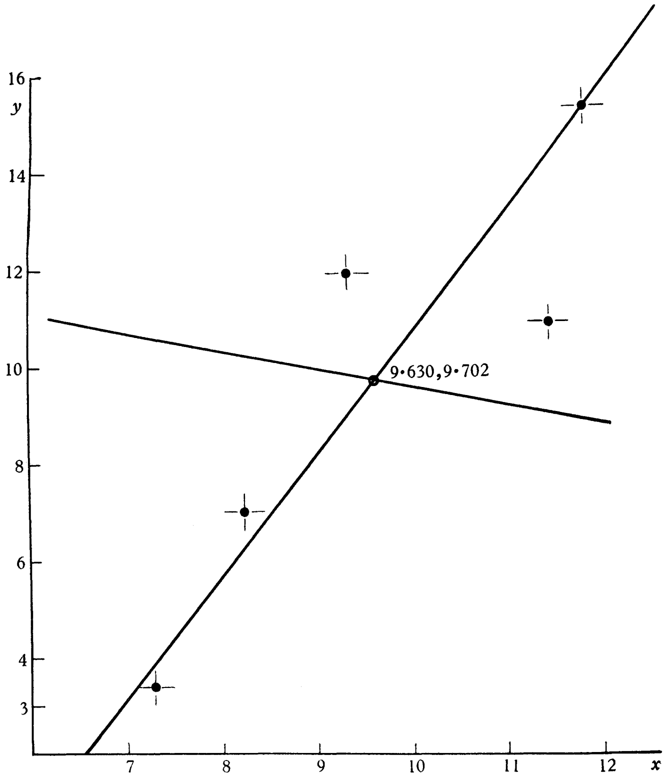

Fig. 4*b*.

It is obvious that the preparation of this table and the deduction of (12) from it is more laborious than plotting the observations, as in Fig. 4*b*, and drawing the best-fit straight line among them 'by eye'. But the result is far more accurate, even if the observed points lie nearly on a straight line; it is foolish to take elaborate precautions for accuracy in the setting up and conduct of the experiment and then to 'spoil the ship for a haporth of tar' by neglecting to use the most accurate method of getting the final result from the observations.

PART I. MECHANICS

Exp. 1. *Stretched elastic string, I*

We know from experience that if we fix a string, rope, wire, etc., to two points at the same level, and stretch it as tightly as we like, and then hang a weight on its centre, it will sag to a certain extent. The purpose of this and the next experiment is to find out how the amount of this sagging depends on the length of the string, its extensibility, the tension we put in it before hanging on the weight, and the magnitude of that weight.

(*a*) Attach a light scale-pan, whose weight (which should be less than 7 g.) you have measured, to a spiral steel spring, which should be not less than 7 cm. or more than 20 cm. long and should give an extension between 0·5 and 1·5 cm. per 100 g. wt. Hang the spring vertically and plot a graph connecting the length (l cm.) of the spring and the load (T g. wt.) on it, including the weight of the scale-pan, up to about 60% extension. l must be measured accurately between two well-defined points on the spring.

(*b*) Drive stout nails, up to half their length, into the edge at A and B of a board about a metre long, and fix the board with its face horizontal. Attach the spring to A and B by strong thread (fine wire, such as S.W.G. 26, will serve) which is strong enough to carry 1 kg., stretching the spring to about $1\frac{1}{2}$ times its unstretched length. Guard against any possibility of the attachments to the nails slipping, by taking several tight turns round the nail before fastening off the thread. Measure the length ($2L$ cm.) of AB, and the stretched length (l_1 cm.) of the spring; deduce from your graph the tension (T_1 g. wt.) of the string and thread corresponding to this length l_1 cm.

Hang the scale-pan, by a loop of cotton through which the thread passes, from a point B on the thread vertically below C, the mid-point of AB, and fix a vertical millimetre scale with its edge passing through C. Measure the vertical displacements (y cm.) of P for a series of loads (w g. wt.). The loop of cotton must be moved along the thread as required to keep P vertically below C. y should not exceed 4 or 5 cm.

Plot a graph with w as ordinates and y as abscissae.

Theory (see appended note) shows that for very small values of y/L this graph should be a straight line through the origin; determine from your graph the slope to the axis of y of this line, if it is a straight line, or if the graph is curved the slope of its tangent at the origin. This 'slope' must be measured, not by a protractor, but by using the scales along the axes; see Introduction (§ 3).

Compare your value of this slope with its theoretical value $2T_1/L$.

(*c*) Drive another nail into another part of the board's edge, about $\frac{1}{3}L$ from B, and repeat part (*b*) of the experiment.

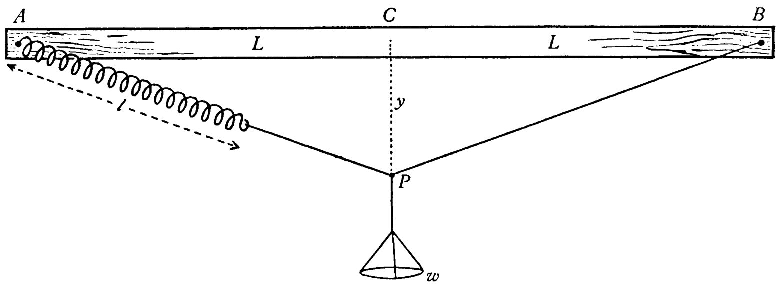

Fig. 5.

If in each case your results agree (except for differences that can legitimately be ascribed to casual errors of experiment) with the theoretical relationship $w = \frac{2T_1}{L}y$, they may be taken to establish that relationship, since the values of T_1 and L were taken at random.

In that case you have proved experimentally that the force needed to displace transversely, through a certain small distance, the centre of a stretched elastic string is directly proportional to its tension before displacement, and inversely proportional to its length, and that if all other things remain unchanged the force is directly proportional to the displacement.

Note on the Theory

Suppose that the spiral spring used in the experiment, when the tension T g. wt. exceeds a certain amount, obeys the law $l = aT + b$, so that an additional tension of 1 g. wt. causes an additional extension of a cm. If the initial tension of the thread is T_1 g. wt., and its tension when P is displaced y cm. is T g. wt., the spring will have increased in length by $a(T - T_1)$ cm. and the length of PA or PB will be $L + \frac{1}{2}a(T - T_1)$ cm.

Denote the angles PAB and PBA in Fig. 5 by β; then

$$y=\left\{L+\frac{a}{2}(T-T_1)\right\}\sin\beta$$

$$=\left(L-\frac{aT_1}{2}\right)\sin\beta+\frac{a}{2}T\sin\beta.$$

Now since P is in equilibrium

$$w=2T\sin\beta.$$

Hence
$$y=\left(L-\frac{aT_1}{2}\right)\sin\beta+\frac{aw}{4},$$

or
$$\sin\beta=\frac{4y-aw}{2(2L-aT_1)}.$$

But $\tan\beta=\frac{y}{L}$, so that $\sin\beta=\frac{y}{\sqrt{(L^2+y^2)}}$. Expanding this by the binomial theorem

$$\sin\beta=\frac{y}{L}\left(1+\frac{y^2}{L^2}\right)^{-\frac{1}{2}}=\frac{y}{L}\left(1-\frac{y^2}{2L^2}+\frac{y^4}{8L^4}-\right), \text{ etc.}$$

Hence
$$\frac{4y-aw}{2(2L-aT_1)}=\frac{y}{L}-\frac{y^3}{2L^3}\quad\text{nearly},$$

or
$$4y-aw=2(2L-aT_1)\frac{y}{L}-(2L-aT_1)\frac{y^3}{L^3}\quad\text{nearly},$$

or
$$w=\frac{2T_1}{L}y+\frac{2L-aT_1}{aL^3}y^3\quad\text{nearly}$$

$$=\frac{2T_1}{L}y\quad\text{if }\frac{y}{L}\text{ is very small.}$$

It will be seen that there is no need here to take into account the difference between the real and the ideal unstretched length of the spring, dealt with in the Introduction (§ 2 (*a*)), since the constant in the final equation involves only L and the initial tension T_1 which is read off the first graph.

Exp. 2. *Stretched elastic string, II*

Take the set-up of Exp. 1, but make $2L$ about 60 cm., use a spring which stretches about 0·5 or 0·8 cm. per 100 g. wt., and adjust the initial tension (T_1 g. wt.) so that it is not much greater than is required to separate all the consecutive turns of the spiral spring.

Measure $2L$, determine T_1 as in Exp. 1 and the extension (a cm.) per g. wt. for loads more than sufficient to separate consecutive turns, by hanging up the spring as in part (a) of Exp. 1, loading it with a succession of weights w, drawing a graph, and measuring its slope to the axis of w, which gives a.

Get the smooth graph connecting w with y as in part (b) of Exp. 1, carrying the value of y from 0 to 7 or 8 cm.

Draw tangents to this smoothed graph at points corresponding to values of y of 6, 5, 4, 3 and 2 cm. Measure the slope to the axis of y of the tangent in each case (not by a protractor but by using the scales along the axes; see Introduction (§ 3)), and plot a graph connecting these values as ordinates with the corresponding values of y^2 (*not* of y) as abscissae.

The appended note on theory shows that the graph should be a straight line, making an intercept on the vertical axis of $2T_1/L$, and sloped to the axis of y^2 at $3 \times \frac{2L - aT_1}{aL^3}$.

Calculate the intercept and slope of your second experimental graph, and compare them with these theoretical values, after substituting the measured values of L, T_1 and a.

Note on the Theory

In the note appended to Exp. 1 the theoretical relation between w and y was shown to be $w = \frac{2T_1}{L} y + \frac{2L - aT_1}{aL^3} y^3$ for small values of y/L.

If we keep T_1, L and a unchanged, and differentiate both sides of this equation with respect to y, we get for the slope (dw/dy) to the axis of y of the curve, at the point corresponding to y,

$$\frac{dw}{dy} = \frac{2T_1}{L} + 3 \times \frac{2L - aT_1}{aL^3} y^2.$$

Hence if we measure this slope at a number of points on the graph, and plot it as abscissa against the corresponding value of y^2 as ordinate, we should get a straight line, which should make an intercept $2T_1/L$ on the axis of dw/dy, and should be inclined to the axis of y^2 at a slope of $3 \times \frac{2L - aT_1}{aL^3}$.

This is a device for getting a linear graph which is not mentioned in the Introduction.

Exp. 3. *Rigid pendulum*

The 'ideal simple pendulum' is found only in text-books; the simple pendulum found in laboratories is chiefly of use as a cheap and convenient time-measurer. The pendulum of a clock, or any rigid object that can swing in a vertical plane about a fixed horizontal axis under the action of gravity, can be of almost any shape, and the purpose of this experiment is to find out something of the way in which its time of swing depends on that shape and on the distance between its centre of gravity and its point of support.

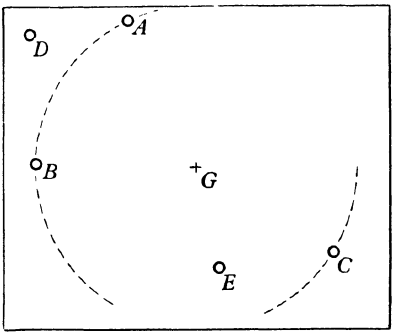

Fig. 6.

Take as the object to be used as the rigid pendulum a piece of cardboard of any shape and size, the larger the better, but a rectangle with sides of 20 and 24 cm. is convenient. Locate its centre of gravity, *G*, by balancing it on the point of a needle. Punch holes, about 3–5 mm. in diameter, in it as in Fig. 6, the centres *A*, *B* and *C* of three of them being equidistant from *G*, and *D* and *E* wherever you like. Stick the needle through a small cork and support it horizontally by fixing the cork in a retort stand. Pass the needle through *A*, so that the cardboard can oscillate freely in its own plane like a pendulum; measure the distance (l_1 cm.) of *G* from the point of support, estimating tenths of a millimetre.

Set up in front of the cardboard a simple pendulum, with its thread held between two pieces of wood in the jaws of the same or another retort stand, so that its length (l_2 cm.) from the centre of gravity of its bob to its point of support can be easily varied and accurately measured. Adjust l_2 until the two pendulums, when displaced from their equilibrium positions and released simultaneously, have the same periods of oscillation, so that they continue to swing together until the oscillations die out.

Measure l_2, the length of the so-called 'Simple Equivalent Pendulum'. Keeping l_2 unchanged, use B and C in succession as points of support of the cardboard, and see whether the cardboard still oscillates in time with the simple pendulum of length l_2 cm.

If so, calculate the value of a^2, where

$$l_1(l_2 - l_1) = a^2. \tag{1}$$

Find in the same way the lengths of the simple equivalent pendulums when the holes D and E are used; calculate a^2 in each case, changing l_1 and l_2 in (1) to suit each case.

If, as is probable, the value of a^2 is found to be about constant, calculate the arithmetic mean of its observed values. Then from (1)

$$l_2 = l_1 + \frac{a^2}{l_1},$$

or, in words, the length of the simple equivalent pendulum exceeds the distance (l_1) between the centre of gravity and the point of support of the rigid pendulum by a^2/l_1, where a^2 is a constant.

If the cardboard is a uniform rectangle and the effect of the holes is ignored, this constant a^2 should theoretically equal one-twelfth of the sum of the squares of the length and breadth of the cardboard. Check this in your case.

Note on the Theory

Text-books of Mechanics show that the period of oscillation of a rigid pendulum equals $2\pi\sqrt{\frac{m(l_1^2+k^2)}{ml_1 g}}$, where m is its mass, l_1 the distance of its centre of gravity from its point of support, and mk^2 is its moment of inertia about an axis through its centre of gravity perpendicular to its plane of oscillation. So a simple pendulum of length l_2 has a period of oscillation of

$$2\pi\sqrt{\frac{l_2^2}{l_2 g}} \quad \text{or} \quad 2\pi\sqrt{\frac{l_2}{g}}.$$

Hence in this case

$$\frac{l_1^2+k^2}{l_1 g} = \frac{l_2}{g} \quad \text{or} \quad k^2 = l_1 l_2 - l_1^2 = l_1(l_2 - l_1) \quad \text{or} \quad l_2 = l_1 + \frac{k^2}{l_1}.$$

These text-books also show that in the case of a rectangle of sides b and c, $k^2 = \frac{1}{12}(b^2+c^2)$.

Exp. 4. *Simple pendulum*

The object of this experiment is to investigate the relation between the length (l) and the period (T) of a simple pendulum, with the help of a graduated ruler, but without using a stop-watch or any theory of mechanics in general or S.H.M. in particular.

Set up a simple pendulum of any length (l_0), but it is convenient to take it about 20 cm.; measure l_0. This pendulum will serve as a standard, and its period as a time-unit, for the experiment.

Set up another simple pendulum, of length between 78 and 82 cm., with its bob at about the same level as the bob of the first and an inch or two in front of it; the line joining the two bobs, when they are hanging at rest, should be roughly perpendicular to the planes in which they will swing. This pendulum can best be supported by passing its thread between two thin pieces of wood gripped between the jaws of a retort stand, so that the length (l) of the pendulum can be adjusted by sliding the thread between the pieces of wood when the jaws are slightly opened.

Start the two pendulums simultaneously by 'kicking them off' gently with the face of a ruler held against them while they hang freely at rest. Adjust the length of the front pendulum so that the two pendulums pass simultaneously through their initial central positions after one complete, to-and-fro, swing of the front pendulum and two complete swings of the back one.

If you have made this adjustment with absolute precision, the two pendulums will coincide after every whole number of swings of the longer pendulum, and you should improve your adjustment accordingly; in fact, you can in this way compare the periods of the two pendulums more accurately than you can compare their lengths. It corresponds to timing a large number of swings when using a stop-watch. Measure l, the length of this front pendulum.

Repeat the experiment, adjusting the length of the front pendulum so that it makes two complete swings while the standard one makes three swings; expressing it otherwise, it has to make n swings while the standard one makes $n+1$ swings, n being two in this case, and one in the first case.

Repeat the experiment with n equal to, say, 3 and 4.

Make a table of your corresponding values of n and l. These quantities must be connected by some law; the next step is to discover it. This can best be done by assuming various forms which the law is likely to take, and testing each by applying it to the four pairs of values of n and l which you have obtained.

The simplest hypothesis is that $l = An^p$, where A and p are constants. If this is so,

$$\log l = \log A + p \log n,$$

and the graph connecting $\log l$ and $\log n$ should be a straight line. Test this by plotting the graph connecting your values of $\log l$ and $\log n$; you will probably get a curved graph, and if so you must reject this hypothesis and try out another.

We have found that one pendulum, of variable length l, makes n swings in the same time as another, of fixed length l_0, makes $n+1$ swings; a possible form of the law would seem to be that

$$ln^p = l_0(n+1)^p \quad \text{or} \quad \frac{l}{l_0} = \left(\frac{n+1}{n}\right)^p$$

or

$$\log \frac{l}{l_0} = p \log\left(\frac{n+1}{n}\right),$$

where p is some fixed number.

In that case, the graph connecting $\log l/l_0$ and $\log\left(\frac{n+1}{n}\right)$ should be a straight line through the origin, with a slope of p to the axis of $\log\left(\frac{n+1}{n}\right)$.

You can test this by your experimental results; add to your table columns showing $\log l - \log l_0$ and $\log(n+1) - \log n$, and plot the graph connecting the last two columns. You will probably find that this hypothesis is confirmed, and that the most probable value of p is 2·0.

If so, the relation is

$$\log \frac{l}{l_0} = 2 \log\left(\frac{n+1}{n}\right)$$

or

$$\frac{l}{l_0} = \left(\frac{n+1}{n}\right)^2. \qquad (1)$$

Since you took n more or less at random and (1) holds good in all the cases you tried, it is safe to assume that it will hold good for any value of n which is a whole number.

But we want the relation between the length and period of any pendulum, not the relation between l and n under these conditions; so a further step is required.

Denote by T sec. the period of any of the pendulums for which (1) holds good, and by T_0 sec. the period of the standard (back) pendulum. Since the front pendulum makes n swings while the standard pendulum makes $n+1$ swings, nT must equal $(n+1)T_0$, or

$$\frac{n+1}{n}=\frac{T}{T_0}, \tag{2}$$

where n is any whole number. Substituting this in (1) we get

$$\frac{l}{l_0}=\left(\frac{T}{T_0}\right)^2$$

or

$$\frac{T^2}{l}=\frac{T_0^2}{l_0}=A, \tag{3}$$

where A is some constant which does not depend on T or l.

In this form, (3) is a general law connecting T and l in any simple pendulum. But it is very important to realise that you have not really 'verified' this law by your experiments; all that you have done is to discover and verify it in a limited number of cases. Even in those cases you did not actually measure T, but arrived at its value, quite legitimately, by comparison with T_0.

Strictly speaking there is the same imperfection in the ordinary laboratory method of verifying the law by measuring the periods of simple pendulums of various lengths by means of a stop-watch. For the second hand of a watch does not move uniformly, but by a series of jerks; it will only measure an interval of time that begins and ends with these jerks. It is most unlikely, if l is taken at random, that the corresponding T will be exactly an interval of this kind. For a theoretically sound verification we must use a time-measurer such as a chronograph which moves at uniform speed. In practice, the error caused by using a stop-watch is so small that it is less serious than 'casual' errors of observation, and the theoretical imperfection of this practical method of verifying the law is overlooked.

But it is too large in this case to be overlooked. The difficulty cannot be overcome by the argument, which in many cases is valid, that you took l_0 at random, so the law may be expected to hold good for any value of l_0 and therefore of l. For it is quite possible that the constant A in (3) may depend on the mass or length or period of the standard pendulum, and thus may not be the same for different groups of front pendulums. However, the appended note on theory shows that this is not the case, if we can trust the theoretical reasoning, and this argument applies also to the verification of the law by means of a stop-watch. Hence we can claim in both cases alike that we can verify the law by a combination of experiment and theoretical reasoning.

Note on the Theory

We have shown experimentally that for any simple pendulum in a certain group, of length l and period T, $T^2=Al$, where A does not depend on T or l; the only other physical quantities on which A for any group can depend are the acceleration (g) of gravity at the place of experiment and the period (T_0) and the mass (m_0) of the bob of the standard pendulum of its particular group. We can therefore represent A by $Nm_0^q g^r T_0^s$, where N, q, r and s are numbers.

Then $$T^2=Nm_0^q g^r T_0^s l \qquad \text{(i)}$$

is an equation which holds good for every simple pendulum.

Now it is theoretically necessary that in such an equation every term should have the same dimensions in mass, length and time, separately. The dimension of the left-hand term of (i) in mass is zero, and of the right-hand term is q; hence $\mathbf{q=0}$.

The dimension of the left-hand term in length is zero, and of the right-hand term is $r+1$; hence $r+1=0$, or $\mathbf{r=-1}$.

The dimension of the left-hand term in time is 2, and of the right-hand side is $-2r+s$; hence $-2r+s=2$, or since $r=-1$, $\mathbf{s=0}$.

Consequently, $A=N/g$ and does not depend on T_0 or m_0; so it is the same for every group, and every simple pendulum obeys the same law

$$T^2=\frac{Nl}{g},$$

where N is some number.

Exp. 5. *Bending of a lath, I*

The object of this and the two following experiments is to check the theoretical reasoning (which is far from simple) by which we can determine the behaviour of a straight elastic beam, supported horizontally at two points and loaded at a third point with sufficient weight to bend it to a moderate extent. If you did not have this theory as a guide, any experiments you might make would probably not be very illuminating; but using it as a guide your experiments should establish the truth of a rather complicated and useful formula.

The following apparatus will be needed in this and the next two experiments; two short glass tubes or rods supported horizontally and parallel to one another on piles of wood blocks at equal heights of about 1 ft. above the bench; the tubes should be prevented from moving on the blocks by means of tin-tacks, and the blocks must give them firm support which does not yield appreciably when loaded. A boxwood metre ruler, lying face upwards on the tubes; the ruler must not rock on the tubes. A scale-pan, which may be made of a piece of plywood about 6 in. square, which can be hung by a thread passed over the ruler at any desired point P; a set of weights up to a total of 1000 or 1500 g.; a pin which can be fixed

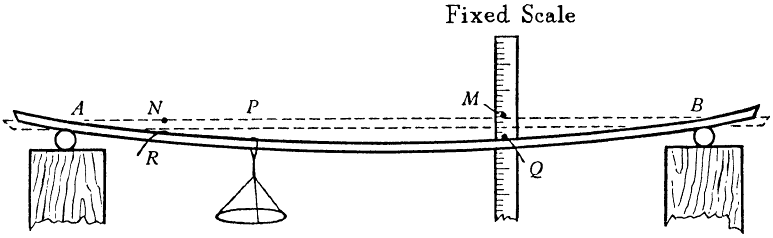

Fig. 7.

at a point Q on the upper face of the ruler by a piece of soft wax so that it overhangs the edge of the ruler, and a scale held vertically in a retort stand, with which to measure the vertical displacement (y cm.) of Q below its position (M) when the scale-pan carries no load (tenths of a millimetre to be estimated).

Fix A and B about 94 cm. apart; hang the scale-pan from a point P on the ruler, about 35 cm. from A, and keep P unchanged throughout the experiment. Fix a pin at a point Q, about 55 cm. from A, keeping Q unchanged throughout the experiment. Load the scale-pan with an increasing, and then decreasing, weight W g., disregarding the weight of the scale-pan, and observe the corresponding values of y up to a maximum of 2 or 3 cm. Plot a graph connecting y and W. It will probably be a straight line through the origin.

Repeat the operation, hanging the scale-pan from Q and measuring vertical displacements of P. Plot these observations of y and W on the same axes; the corresponding graph will probably be almost identical with the former graph. If so, P and Q are interchangeable; the 'deflection' of the ruler at any point P caused by a given load on the ruler at any other point Q is the same as the deflection at Q caused by the same load at P.