PART 1 A new approach to teaching probability

Chapter 1

Introduction

1.1 What this book is about

Probability is the branch of mathematics that deals with randomness, chance, unpredictability and risk. These are vital issues for everyone in society – we all need to make decisions in the face of uncertainty. And yet the public's ability to reason with probability is dismally poor. It is also not generally a popular part of the school mathematics syllabus, either for teachers or for students.

However, researchers in risk communication have shown that changing the way in which probability is represented can dramatically improve people's ability to carry out quite complex tasks. Instead of talking about chance or probability in terms of a decimal, percentage or fraction, we look at the **expected frequency** of events in a group of cases. For example, when discussing the risk of a future heart attack or stroke with a patient, medical students are now taught not to say 'a 16% chance', but instead to say 'out of 100 patients like you, we would expect 16 to have a heart attack or stroke in the next ten years'.

This may seem a trivial change, but it has strong implications for the way in which probability is taught in schools. Our aim is not only to enable students to answer the type of probability questions set in examinations, but also to help them handle uncertainty in the world beyond the classroom.

1.2 Probability is important

Life is uncertain. None of us knows what is going to happen so, unless we are prepared to resign ourselves to fate, it seems a good idea to be able to reason about uncertainty. Whether we are deciding about medical treatments, choosing investments, buying insurance, playing games or undertaking a risky activity, we want to be able to weigh up the options in terms of the chances, and consequences, of the good and bad things that might happen. Probability is also the basis for methods used in forecasting the economy, the weather or epidemics, as well as underlying much of physics. When a scientific discovery such as the Higgs boson is claimed, the degree of certainty, or confidence, is expressed in terms of probability since probability forms the basis for statistical inference.

Even without its clear practical importance, probability can be fascinating in its own right and provides a starting point for a wealth of challenging problems and games, as well as (in some cultures) gambling.

There is also the association between probability and fairness. The idea of 'casting lots' as a fair way of making decisions or allocating goods is ancient, and children are sensitised to the link between pure randomness

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and fairness from an early age. Technically, we can identify 'fairness' with the idea that each individual has the same expected gain, which leads us back to the need to understand probability.

1.3 What is probability anyway?

So probability is important, and its applications are all around us. Why, then, do people find it so unintuitive and difficult? Well, after years of working in this area, we have finally concluded that this is because … probability *is* unintuitive and difficult.

People's understanding is not helped by the lack of clarity about what probability actually *is*. We have scales for weight, rulers for length, clocks for time, but where is the probability-meter? Probability, like value, is not directly and objectively measurable. What is worse, philosophers of science have been unable to come up with an agreed definition for probability, and so it is impossible to specify exactly what it is.

Some popular options for the definition of the probability of an event include:

- **a** **Symmetry**: 'The number of outcomes favouring the event, divided by the total number of outcomes, assuming the outcomes are all equally likely.' This is the definition usually taught in school as **theoretical probability**, but it is rather circular as it depends on 'equally likely' being defined. And, it can only be used in nicely balanced situations such as dice, cards or lottery tickets, or when, to use a classic example, picking a coloured sock at random from a drawer. It does not apply, for example, to the probability that you will have a heart attack or stroke in the next ten years.
- **b** **Frequency**: 'The proportion of times, in the long run of identical circumstances, that the event occurs.' This is the idea of an observed **relative frequency** tending to a true probability after sufficient repetitions. This can be fine for situations where there are lots of repeats, but does not seem applicable to unique situations, such as *your* risk of a heart attack.
- **c** **Subjective**: 'My personal confidence that an event will occur, expressed as a number between 0 and 1. When the event either occurs or not, my assessment will be rewarded or penalised according to an appropriate 'scoring rule'.' This definition is one way of formalising the idea that probabilities are purely personal judgements based on available evidence and assumptions.

There are other proposals for understanding probability. Some have suggested it measures an underlying **propensity** for an event to happen – but what is your propensity to have a heart attack or stroke? Or, more imaginatively, we could think of probability as the proportion of **possible futures** in which the event occurs.

For us, the crucial lessons from all this philosophy are:

- **a** We should not claim to have *the* definition of probability – it is a 'virtual' quantity and perhaps best considered in terms of different metaphors depending on circumstances.
- **b** Probabilities are almost inevitably based on judgements and assumptions such as random sampling. They cannot be said to objectively exist, except perhaps for sub-atomic, determined probabilities.
- **c** It is important to emphasise that, despite all these philosophical debates, the mathematics of probability is not controversial.

In this book we primarily adopt a rather hybrid metaphor for probability, based on the expected proportion of times that something will happen in similar circumstances. This is essentially a frequency interpretation of a subjective judgement. Using this idea, we show that complex probability calculations can become remarkably clear.

1.4 People find probability tricky

The language of probability is complex and invites misunderstanding. Suppose you are assessed to have a 16% probability of a heart attack or stroke in the next ten years. Verbal terms are ambiguous and dependent on context and viewpoint: we might personally think that 16% meant this was a 'fairly unlikely' event, although from a medical point of view this could be considered as 'high risk' and perhaps a cholesterol-lowering drug, such as a statin, would be recommended.

Alternatively, we might describe this as around a 1 in 6 chance, but modern advice in risk communication explicitly recommends against this type of expression. For example, a recent population survey by telephone [1] asked:

Which of the following numbers represents the biggest risk of getting a disease: 1 in 100, 1 in 1000, or 1 in 10?

In Germany, 28% of responses were incorrect, and in the USA 25% were wrong. The crucial issue is that *larger* numbers are used to communicate *smaller* risks, so a difficult inversion must be done. This is one reason why flood-risk maps expressed in terms of '1 in 100 year events' are difficult to read and potentially misleading.

The media are also fond of reporting **relative risks**. For example, an American direct-to-consumer advert for a statin to reduce cholesterol declared in large font that there is a '36% reduction' in the risk of heart attack. In very much smaller font it clarifies that this is a reduction, in percentage point terms, from 3% to 2% over five years. This is a reduction of 1 percentage point in the absolute risk, and so 100 such people would have to take the drug every day for five years to prevent one heart attack. This does not sound so impressive.

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People are also not very good at handling relative risks. The same recent survey asked:

If person A's chance of getting a disease is 1 in 100 in 10 years and person B's risk is double that of A, what is B's risk?

46% of respondents in Germany and 43% in the USA could not answer correctly.[1]

[1] *Spoiler alert: the answer is that B's risk is 2 in 100 in 10 years.*

Even if the meaning of the probability statement is clear, there are numerous examples of the trouble people have with even fairly basic probabilistic reasoning.

In 2012, for example, 97 British Members of Parliament were asked:

If you spin a coin twice, what is the probability of getting two Heads?

Only 40% were able to answer correctly [2].

Because a question concerning probability is generally very easy to state (although sometimes ambiguous), people feel the answer should be intuitive. It rarely is. Even when trained, people can find it difficult to match the formal technique to the problem.

The only gut feeling we have about probability is not to trust our gut feelings.

1.5 There's a way to make probability less tricky

Our approach in this book is based on the research of psychologists into the effect different representations have on people's ability to reason with probabilities. The German psychologist, Gerd Gigerenzer, has popularised the idea of 'natural frequencies', which we call 'expected frequencies'. Extensive research [3–5] has shown this helps to prevent confusion and make probability calculations easier and more intuitive.

We have already revealed the basic idea: instead of saying 'the probability of X is 0.20 (or 20%)', we would say 'out of 100 situations like this, we would expect X to occur 20 times'. 'Is that all?', we hear you cry, but this simple re-expression can have a deep impact.

The first point is that it helps clarify what the probability means. When we hear the phrase 'the probability it will rain tomorrow is 30%', what does it mean? That it will rain 30% of the time? That it will rain over 30% of the area? In fact it means that out of 100 computer forecasts in situations like this, we expect rain in 30 of them. By clearly stating the denominator, or **reference class**, ambiguity is avoided.

An explicit reference class might have avoided some other journalistic mistakes, such as when it was reported that '35% of bikers have serious road accidents', when the real statistic was that 35% of serious road accidents involve motorcyclists.

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Any proportion has a numerator and a denominator. Here the numerator is easy: bikers who have accidents. The problem comes with the denominator: in this situation it is not 'all bikers', it is 'all serious accidents'.

Or take the extraordinary headline that in Britain '30% of sex involves under 16s', when the actual claim was that 30% of under-16s have sex. Again the numerator is clearly 'under 16s having sex', but the journalist has taken the denominator as 'all sex' rather than the correct 'all under 16s'.

The crucial question is always to ask 'Out of what?', and then make this reference class explicit.

Even if you are using expected frequencies, such as '20 out of 100', to express risk you must keep in mind that the mathematically equivalent '200 out of 1000' suggests to many people a bigger risk, as the numerator is larger. This is known as **ratio bias**. The following example illustrates the difficulty students (and others) can have [6].

PISA 2003 included the following question.

Consider two boxes A and B. Box A contains three marbles, of which one is white and two are black. Box B contains 7 marbles, of which two are white and five are black. You have to draw a marble from one of the boxes with your eyes covered. From which box should you draw if you want a white marble?

The PISA 2003 Report commented that only 27% of the German school students obtained the correct answer.[2]

Once again, this shows people being misled by focusing on the numerator, where Box B has the larger number of white marbles, 2. Focusing on the fraction (rather than the number) that are white, which is $\frac{1}{3}$ for Box A compared to $\frac{2}{7}$ for Box B, gives the correct answer – Box A. The extreme version of this bias, in which the denominator is ignored completely, is known as **denominator neglect**; the media do this every time they concentrate on a single accident without, for example, mentioning the millions of children who go to school safely each day [7].

Research has shown that, by using expected frequencies, people find it easier to carry out non-intuitive conditional probability calculations. Take a recent newspaper headline saying that eating 50 grams of processed meat each day (e.g. a bacon sandwich) is associated with a 20% increased risk of pancreatic cancer. It turns out that this very serious disease affects only 1 in 80 people. So we want to calculate a 20% increase on a 1 in 80 chance, which is tricky to do.

However, if we imagine 400 people who have an average breakfast each day, we can easily calculate that '1 in 80' means we would expect 5 out of the 400 to get pancreatic cancer. If 400 different people all stuff themselves with a greasy bacon sandwich every day of their lives, this 5 would increase by 20% to 6. This is actually a 1 in 400, or 0.25%,

[2] Box A should be chosen since the chance of winning is $\frac{1}{3}$, which is larger than the chance with Box B, $\frac{2}{7}$. We recommend the following thought experiment to clarify the issue: in 21 replications of the experiment, how many times would you expect to win if you always chose Box A, or always chose Box B? You would expect 7 wins with Box A, and 6 wins with Box B.

increase in absolute risk, which does not seem so important. Note the trick is in identifying 400 as the denominator that will lead to precisely one extra case due to excessive bacon consumption.

As mentioned previously, expected frequency is the standard format taught to medical students for risk communication, and is used extensively in public dialogue. In the advice leaflets for breast cancer screening in the UK, for example, the benefits and risks of screening are communicated in terms of what it means for 200 women being screened for 20 years: we would expect 1 woman to have her early death from breast cancer prevented by screening, at a cost of 3 women with non-threatening cancers being unnecessarily treated [8]. The key idea is that, through using whole numbers, we can think of the information as representing simple summaries of many possible experiences.

Expected frequencies can also be used to answer advanced conditional probability problems of the following classic type.

> Suppose a screening test for doping in sports is claimed to be '95% accurate', meaning that 95% of dopers, and 95% of non-dopers, will be correctly classified. Assume 1 in 50 athletes are truly doping at any time. If an athlete tests positive, what is the probability that they are truly doping?
>
> The way to answer such questions is to think of what we would expect to happen for, say, every 1000 tests conducted. Out of these, 1 in 50 (20) will be true dopers, of which 95% (19) will be correctly detected. But of the 980 non-dopers, 5% (49) will incorrectly test positive. That means a total of 68 positive tests, of which 19 are true dopers. So the probability that someone who tests positive is truly doping is $\frac{19}{68}$ = 28%. So, among the positive tests, the **false-positive** results greatly outnumber the correct detections by around 2.5 to 1.

If you find it difficult to make sense of these numbers, the **expected frequency tree** (Figure 1.1) may help to clarify them:

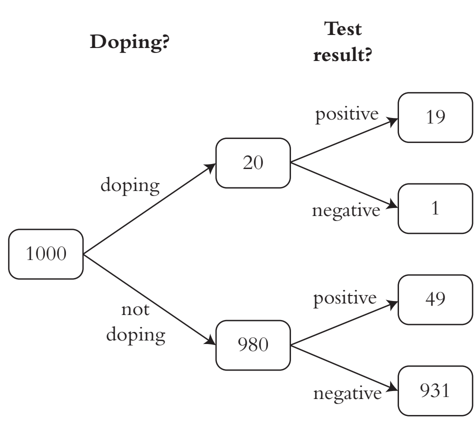

Figure 1.1 Expected frequency tree for doping example, showing that a test that is claimed to be '95% accurate' can still generate more false-positives than true detections: out of 68 positive tests, we would expect only 19 are truly doping

1.6 Teaching probability

Probability is vitally important but poorly understood, and therefore good teaching is essential. It forms part of most secondary school mathematics curricula and is also a component of many science curricula. However, the problems associated with the teaching and learning of probability are well documented. Over 20 years ago, Garfield & Ahlgren [9] noted a number of reasons for this, including difficulty with proportional reasoning and interpreting verbal statements of problems; conflicts between the analysis of probability in the mathematics lesson and experience in real life; and premature exposure to highly abstract, formalised presentations in mathematics lessons. Teacher knowledge may also be an issue, since not all teachers will have studied probability during their own education.

We might also add the continued focus on permutations and combinations, a topic that we (and we believe we are not alone) find intensely tedious. These are often seen as part of a probability curriculum and yet have nothing do with probability itself, being simply tools for counting possible outcomes.

We have sympathy with the struggle for comprehension by both teachers and students. When confronted by a school-level algebra question, students should know the steps required to work through to the answer. Probability questions are different: they always require careful thought, and the precise wording is crucial, as it is easy for it to be ambiguous. We personally try to check answers using at least two different solution methods.

At worst, probability can be taught purely in terms of abstract ideas, for example in this question from a (nameless) examination board.

> Consider three events A, B and C. A and B are independent, B and C are independent and A and C are mutually exclusive. Their probabilities are P(A) = 0.3, P(B) = 0.4 and P(C) = 0.2. Calculate the probability of the following events occurring: (i) Both A and C occur. (ii) Both B and C occur. (iii) At least one of A or B occur.[3]

[3] *For the solution of this horrible question, see Chapter 17 on independent events.*

Here, lack of any connection with the real world means that mistakes are difficult to spot, as it is impossible to apply common-sense ideas of magnitude. Fortunately, most examination boards manage somewhat more engaging questions.

1.7 Experimentation and modelling

Our approach is to teach probability through experimentation, and to use mathematical models to solve contextualised problems. We do not make a big issue about whether probabilities are 'known' or 'unknown'. In real

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life no probabilities are ever known, they are only assumed with more or less justification, and assessing probabilities from data through statistical inference is not the concern of this book.

We regard experimentation as vital to understanding the role of chance and unpredictability. Ideally students should carry out experiments themselves using randomising devices. Our preference is for spinners, either where probabilities are obvious or where they are deliberately concealed – these better reflect real life where nicely balanced situations are rare. We prefer to avoid dice – the choice of outcomes is too restrictive, and numbers may have emotional connotations, quite apart from the practical problems of throwing them in a class.

Having acquainted themselves with spinners as randomising devices in their own right, students can start using these as a way to model real situations – spinners can easily be labelled with specific outcomes, and students then simply count the number of times outcomes occur in a given number of trials. The idea is to use whole numbers initially, and bring in proportions, fractions and probability rules later. We also exploit the strong motivating role of playing competitive non-gambling games of chance, encouraging engagement by clearly making some outcomes more desirable than others.

1.8 What's in the book

Part 1 continues by introducing our approach to teaching probability in Chapter 2, together with our probability curriculum in Chapter 3. We do not follow any specific syllabus, although we have been influenced by the revised GCSE Mathematics (9–1) for England and Wales (2015), but present what we feel is a logical way to develop students' conceptual understanding over a period of four or five years. Our curriculum is sub-divided into three levels, corresponding to the first year or two of secondary teaching, the middle year or two, and then the final year or two.

Part 2 presents a series of detailed classroom activities. The activities in Part 2 can be tackled at more than one level, and detailed notes are provided for this. We consider it advantageous for students to study a good scenario in depth, revisiting it to discover how it can be interpreted in a more advanced way.

Part 3 works through an extensive series of sample assessment questions, with multiple solution methods wherever possible. Inspiration for the style and content of the questions is primarily from the sample assessment material provided by examination boards for the revised GCSE Mathematics (9–1).

Part 4 presents a range of supplementary, extension projects in probability, including both classroom activities and mathematical explanations. We include the 'classics', such as matching birthdays, lotteries, patterns of randomness, and Monty Hall, but we also feature various games, explorations of misconceptions about probability, the idea of 'fairness', psychological attitudes to risk, and in particular the misleading way that risks are often communicated in the media.

1.9 References

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