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AUTOMORPHIC FORMS AND L-FUNCTIONS FOR THE GROUP $GL(n, \mathrm{R})$

L-functions associated with automorphic forms encode all classical number theoretic information. They are akin to elementary particles in physics. This book provides an entirely self-contained introduction to the theory of L-functions in a style accessible to graduate students with a basic knowledge of classical analysis, complex variable theory, and algebra. Also within the volume are many new results not yet found in the literature. The exposition provides complete detailed proofs of results in an easy-to-read format using many examples and without the need to know and remember many complex definitions. The main themes of the book are first worked out for GL(2,R) and GL(3,R), and then for the general case of GL(n,R). In an appendix to the book, a set of *Mathematica*® functions is presented, designed to allow the reader to explore the theory from a computational point of view.

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Automorphic Forms and L-Functions for the Group $GL(n, \mathbb{R})$

DORIAN GOLDFELD

Columbia University

With an Appendix by Kevin A. Broughan

University of Waikato

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Dedicated to Ada, Dahlia, and Iris

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Introduction

The theory of automorphic forms and L-functions for the group of $n \times n$ invertible real matrices (denoted $GL(n, \mathbb{R})$) with $n \geq 3$ is a relatively new subject. The current literature is rife with 150+ page papers requiring knowledge of a large breadth of modern mathematics making it difficult for a novice to begin working in the subject. The main aim of this book is to provide an essentially self-contained introduction to the subject that can be read by someone with a mathematical background consisting only of classical analysis, complex variable theory, and basic algebra – groups, rings, fields. Preparation in selected topics from advanced linear algebra (such as wedge products) and from the theory of differential forms would be helpful, but is not strictly necessary for a successful reading of the text. Any Lie or representation theory required is developed from first principles.

This is a low definition text which means that it is not necessary for the reader to memorize a large number of definitions. While there are many definitions, they are repeated over and over again; in fact, the book is designed so that a reader can open to almost any page and understand the material at hand without having to backtrack and awkwardly hunt for definitions of symbols and terms.

The philosophy of the exposition is to demonstrate the theory by simple, fully worked out examples. Thus, the book is restricted to the action of the discrete group $SL(n, \mathbb{Z})$ (the group of invertible $n \times n$ matrices with integer coefficients) acting on $GL(n, \mathbb{R})$. The main themes are first developed for $SL(2, \mathbb{Z})$ then repeated again for $SL(3, \mathbb{Z})$, and yet again repeated in the more general case of $SL(n, \mathbb{Z})$ with $n \geq 2$ arbitrary. All of the proofs are carefully worked out over the real numbers $\mathbb{R}$, but the knowledgeable reader will see that the proofs will generalize to any local field. In line with the philosophy of understanding by simple example, we have avoided the use of adeles, and as much as possible the theory of representations of Lie groups. This very explicit language appears particularly useful for analytic number theory where precise growth estimates of L-functions and automorphic forms play a major role.

The theory of L-functions and automorphic forms is an old subject with roots going back to Gauss, Dirichlet, and Riemann. An L-function is a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where the coefficients $a_n, \;\; n = 1, 2, \ldots,$ are interesting number theoretic functions. A simple example is where a_n is the number of representations of n as a sum of two squares. If we knew a lot about this series as an analytic function of s then we would obtain deep knowledge about the statistical distribution of the values of a_n. An automorphic form is a function that satisfies a certain differential equation and also satisfies a group of periodicity relations. An example is given by the exponential function $e^{2\pi ix}$ which is periodic (i.e., it has the same value if we transform $x \to x + 1$) and it satisfies the differential equation $\frac{d^2}{dx^2}e^{2\pi ix} = -4\pi^2 e^{2\pi ix}$. In this example the group of periodicity relations is just the infinite additive group of integers, denoted $\mathbb{Z}$. Remarkably, a vast theory has been developed exposing the relationship between L-functions and automorphic forms associated to various infinite dimensional Lie groups such as $GL(n, \mathbb{R})$.

The choice of material covered is very much guided by the beautiful paper (Jacquet, 1981), titled *Dirichlet series for the group $GL(n)$*, a presentation of which I heard in person in Bombay, 1979, where a classical outline of the theory of L-functions for the group $GL(n, \mathbb{R})$ is presented, but without any proofs. Our aim has been to fill in the gaps and to give detailed proofs. Another motivating factor has been the grand vision of Langlands' philosophy wherein L-functions are akin to elementary particles which can be combined in the same way as one combines representations of Lie groups. The entire book builds upon this underlying hidden theme which then explodes in the last chapter.

In the appendix a set of Mathematica functions is presented. These have been designed to assist the reader to explore many of the concepts and results contained in the chapters that go before. The software can be downloaded by going to the website given in the appendix.

This book could not have been written without the help I have received from many people. I am particularly grateful to Qiao Zhang for his painstaking reading of the entire manuscript. Hervé Jacquet, Daniel Bump, and Adrian Diaconu have provided invaluable help to me in clarifying many points in the theory. I would also like to express my deep gratitude to Xiaoqing Li, Elon Lindenstrauss, Meera Thillainatesan, and Akshay Venkatesh for allowing me to include their original material as sections in the text. I would like to especially thank

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Dorian Goldfeld