1

Discrete group actions

The genesis of analytic number theory formally began with the epoch making memoir of Riemann (1859) where he introduced the zeta function,

$$\zeta(s) := \sum_{n=1}^{\infty} n^{-s}, \quad (\Re(s) > 1),$$

and obtained its meromorphic continuation and functional equation

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s), \qquad \Gamma(s) = \int_0^\infty e^{-u}u^s\,\frac{du}{u}.$$

Riemann showed that the Euler product representation

$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1},$$

together with precise knowledge of the analytic behavior of $\zeta(s)$ could be used to obtain deep information on the distribution of prime numbers.

One of Riemann's original proofs of the functional equation is based on the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(ny) = y^{-1}\sum_{n\in\mathbb{Z}} \hat{f}(ny^{-1}),$$

where f is a function with rapid decay as $y \to \infty$ and

$$\hat{f}(y) = \int_{-\infty}^{\infty} f(t)e^{-2\pi ity}\,dt,$$

is the Fourier transform of f. This is proved by expanding the periodic function

$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n)$$

in a Fourier series. If f is an even function, the Poisson summation formula may be rewritten as

$$\sum_{n=1}^{\infty} f(ny^{-1}) = y\sum_{n=1}^{\infty} \hat{f}(ny) - \frac{1}{2}(y\hat{f}(0) - f(0)),$$

from which it follows that for $\Re(s) > 1$,

$$\begin{aligned}\zeta(s)\int_0^\infty f(y)y^s\,\frac{dy}{y} &= \int_0^\infty \sum_{n=1}^{\infty} f(ny)y^s\,\frac{dy}{y}\\ &= \int_1^\infty \sum_{n=1}^{\infty}\big(f(ny)y^s + f(ny^{-1})y^{-s}\big)\,\frac{dy}{y}\\ &= \int_1^\infty \sum_{n=1}^{\infty}\big(f(ny)y^s + \hat{f}(ny)y^{1-s}\big)\,\frac{dy}{y} - \frac{1}{2}\left(\frac{f(0)}{s} + \frac{\hat{f}(0)}{1-s}\right).\end{aligned}$$

If $f(y)$ and $\hat{f}(y)$ have sufficient decay as $y \to \infty$, then the integral above converges absolutely for all complex s and, therefore, defines an entire function of s. Let

$$\tilde{f}(s) = \int_0^\infty f(y)y^s\,\frac{dy}{y}$$

denote the Mellin transform of f, then we see from the above integral representation and the fact that $\hat{\hat{f}}(y) = f(-y) = f(y)$ (for an even function f) that

$$\zeta(s)\tilde{f}(s) = \zeta(1-s)\tilde{\hat{f}}\,(1-s).$$

Choosing $f(y) = e^{-\pi y^2}$, a function with the property that it is invariant under Fourier transform, we obtain Riemann's original form of the functional equation. This idea of introducing an arbitrary test function f in the proof of the functional equation first appeared in Tate's thesis (Tate, 1950).

A more profound understanding of the above proof did not emerge until much later. If we choose $f(y) = e^{-\pi y^2}$ in the Poisson summation formula, then since $\hat{f}(y) = f(y)$, one observes that for $y > 0$,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 y} = \frac{1}{\sqrt{y}}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/y}.$$

This identity is at the heart of the functional equation of the Riemann zeta function, and is a known transformation formula for Jacobi's theta function

$$\theta(z) = \sum_{n=-\infty}^{\infty} e^{2\pi i n^2 z},$$

where $z = x + iy$ with $x \in \mathbb{R}$ and $y > 0$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix with integer coefficients a, b, c, d satisfiying $ad - bc = 1$, $c \equiv 0 \pmod 4$, $c \neq 0$, then the Poisson summation formula can be used to obtain the more general transformation formula (Shimura, 1973)

$$\theta\left(\frac{az+b}{cz+d}\right) = \epsilon_d^{-1}\chi_c(d)(cz+d)^{\frac{1}{2}}\theta(z).$$

Here χ_c is the primitive character of order ≤ 2 corresponding to the field extension $\mathbb{Q}(c^{\frac{1}{2}})/\mathbb{Q}$,

$$\epsilon_d = \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4 \\ i & \text{if } d \equiv -1 \pmod 4, \end{cases}$$

and $(cz + d)^{\frac{1}{2}}$ is the "principal determination" of the square root of $cz + d$, i.e., the one whose real part is > 0.

It is now well understood that underlying the functional equation of the Riemann zeta function are the above transformation formulae for $\theta(z)$. These transformation formulae are induced from the action of a group of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on the upper half-plane $\mathfrak{h} = \{x + iy \mid x \in \mathbb{R},\ y > 0\}$ given by

$$z \mapsto \frac{az+b}{cz+d}.$$

The concept of a group acting on a topological space appears to be absolutely fundamental in analytic number theory and should be the starting point for any serious investigations.

1.1 Action of a group on a topological space

Definition 1.1.1 *Given a topological space X and a group G, we say that G* **acts continuously** *on X (on the left) if there exists a map $\circ : G \to \mathrm{Func}(X \to X)$ (functions from X to X), $g \mapsto g\circ$ which satisfies:*

- $x \mapsto g \circ x$ *is a continuous function of x for all $g \in G$;*
- $g \circ (g' \circ x) = (g \cdot g') \circ x$, *for all $g, g' \in G, x \in X$ where $\cdot$ denotes the internal operation in the group G;*
- $e \circ x = x$, *for all $x \in X$ and $e =$ identity element in G.*

Example 1.1.2 Let G denote the additive group of integers $\mathbb{Z}$. Then it is easy to verify that the group $\mathbb{Z}$ acts continuously on the real numbers $\mathbb{R}$ with group

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action $\circ$ defined by

$$n \circ x := n + x,$$

for all $n \in \mathbb{Z}$, $x \in \mathbb{R}$. In this case $e = 0$.

Example 1.1.3 Let $G = GL(2, \mathbb{R})^+$ denote the group of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$ and determinant $ad - bc > 0$. Let

$$\mathfrak{h} := \left\{ x + iy \,\middle|\, x \in \mathbb{R},\ y > 0 \right\}$$

denote the upper half-plane. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{R})^+$ and $z \in \mathfrak{h}$ define:

$$g \circ z := \frac{az + b}{cz + d}.$$

Since

$$\frac{az+b}{cz+d} = \frac{ac|z|^2 + (ad+bc)x + bd}{|cz+d|^2} + i \cdot \frac{(ad-bc)\cdot y}{|cz+d|^2}$$

it immediately follows that $g \circ z \in \mathfrak{h}$. We leave as an exercise to the reader, the verification that $\circ$ satisfies the additional axioms of a continuous action. One usually extends this action to the larger space $\mathfrak{h}^* = \mathfrak{h} \cup \{\infty\}$, by defining

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \infty = \begin{cases} a/c & \text{if } c \neq 0, \\ \infty & \text{if } c = 0. \end{cases}$$

Assume that a group G acts continously on a topological space X. Two elements $x_1, x_2 \in X$ are said to be equivalent $\pmod G$ if there exists $g \in G$ such that $x_2 = g \circ x_1$. We define

$$Gx := \left\{ g \circ x \,\middle|\, g \in G \right\}$$

to be the equivalence class or orbit of x, and let $G\backslash X$ denote the set of equivalence classes.

Definition 1.1.4 *Let a group G act continuously on a topological space X. We say a subset $\Gamma \subset G$ is* **discrete** *if for any two compact subsets $A, B \subset X$, there are only finitely many $g \in \Gamma$ such that $(g \circ A) \cap B \neq \phi$, where ϕ denotes the empty set.*

Example 1.1.5 The discrete subgroup $SL(2,\mathbb{Z})$. Let

$$\Gamma = SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\},$$

and let

$$\Gamma_\infty := \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \;\middle|\; m \in \mathbb{Z} \right\}$$

be the subgroup of Γ which fixes ∞. Note that $\Gamma_\infty\backslash\Gamma$ is just a set of coset representatives of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where for each pair of relatively prime integers $(c,d)=1$ we choose a unique a,b satisfying $ad-bc=1$. This follows immediately from the identity

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix}.$$

The fact that $SL(2,\mathbb{Z})$ is discrete will be deduced from the following lemma.

Lemma 1.1.6 *Fix real numbers* $0<r$, $0<\delta<1$. *Let* $R_{r,\delta}$ *denote the rectangle*

$$R_{r,\delta} = \left\{ x+iy \;\middle|\; -r \le x \le r,\ 0<\delta \le y \le \delta^{-1} \right\}.$$

Then for every $\epsilon>0$, *and any fixed set* $\mathcal{S}$ *of coset representatives for* $\Gamma_\infty\backslash SL(2,\mathbb{Z})$, *there are at most* $4+(4(r+1)/\epsilon\delta)$ *elements* $g\in\mathcal{S}$ *such that* $\mathrm{Im}(g\circ z)>\epsilon$ *holds for some* $z\in R_{r,\delta}$.

Proof Let $g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then for $z\in R_{r,\delta}$,

$$\mathrm{Im}(g\circ z) = \frac{y}{c^2y^2+(cx+d)^2} < \epsilon$$

if $|c|>(y\epsilon)^{-\frac{1}{2}}$. On the other hand, for $|c|\le(y\epsilon)^{-\frac{1}{2}}\le(\delta\epsilon)^{-\frac{1}{2}}$, we have

$$\frac{y}{(cx+d)^2} < \epsilon$$

if the following inequalities hold:

$$|d| > |c|r + (y\epsilon^{-1})^{\frac{1}{2}} \ge |c|r + (\epsilon\delta)^{-\frac{1}{2}}.$$

Consequently, $\mathrm{Im}(g\circ z)>\epsilon$ only if

$$|c| \le (\delta\epsilon)^{-\frac{1}{2}} \quad \text{and} \quad |d| \le (\epsilon\delta)^{-\frac{1}{2}}(r+1),$$

and the total number of such pairs (not counting $(c,d)=(0,\pm1),(\pm1,0)$) is at most $4(\epsilon\delta)^{-1}(r+1)$. □

It follows from Lemma 1.1.6 that $\Gamma = SL(2,\mathbb{Z})$ is a discrete subgroup of $SL(2,\mathbb{R})$. This is because:

(1) it is enough to show that for any compact subset $A \subset \mathfrak{h}$ there are only finitely many $g \in SL(2,\mathbb{Z})$ such that $(g \circ A) \cap A \neq \phi$;
(2) every compact subset of $A \subset \mathfrak{h}$ is contained in a rectangle $R_{r,\delta}$ for some $r > 0$ and $0 < \delta < \delta^{-1}$;
(3) $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$, except for finitely many $\alpha \in \Gamma_\infty,\;\; g \in \Gamma_\infty\backslash\Gamma$.

To prove (3), note that Lemma 1.1.6 implies that $(g \circ R_{r,\delta}) \cap R_{r,\delta} = \phi$ except for finitely many $g \in \Gamma_\infty\backslash\Gamma$. Let $S \subset \Gamma_\infty\backslash\Gamma$ denote this finite set of such elements g. If $g \notin S$, then Lemma 1.1.6 tells us that it is because $\text{Im}(gz) < \delta$ for all $z \in R_{r,\delta}$. Since $\text{Im}(\alpha g z) = \text{Im}(gz)$ for $\alpha \in \Gamma_\infty$, it is enough to show that for each $g \in S$, there are only finitely many $\alpha \in \Gamma_\infty$ such that $((\alpha g) \circ R_{r,\delta}) \cap R_{r,\delta} \neq \phi$. This last statement follows from the fact that $g \circ R_{r,\delta}$ itself lies in some other rectangle $R_{r',\delta'}$, and every $\alpha \in \Gamma_\infty$ is of the form $\alpha = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ $(m \in \mathbb{Z})$, so that

$$\alpha \circ R_{r',\delta'} = \left\{ x + iy \;\middle|\; -r' + m \le x \le r' + m,\;\; 0 < \delta' \le \delta'^{-1} \right\},$$

which implies $(\alpha \circ R_{r',\delta'}) \cap R_{r,\delta} = \phi$ for $|m|$ sufficiently large.

Definition 1.1.7 *Suppose the group G acts continuously on a connected topological space X. A* **fundamental domain** *for $G\backslash X$ is a connected region $D \subset X$ such that every $x \in X$ is equivalent* $\pmod{G}$ *to a point in D and such that no two points in D are equivalent to each other.*

Example 1.1.8 A fundamental domain for the action of $\mathbb{Z}$ on $\mathbb{R}$ of Example 1.1.2 is given by

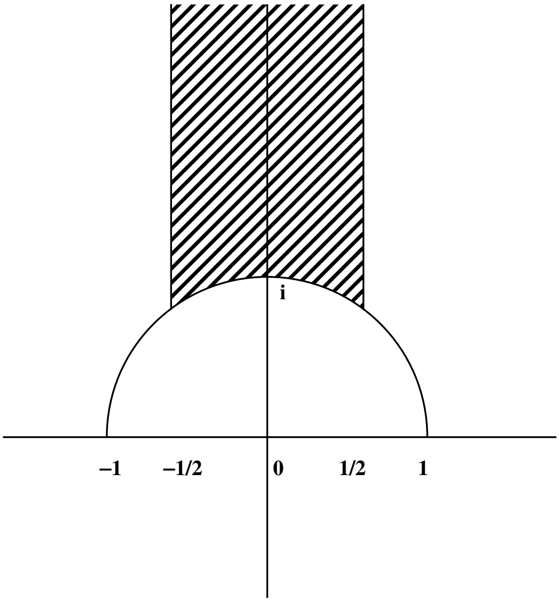
$$\mathbb{Z}\backslash\mathbb{R} = \{0 \le x < 1 \mid x \in \mathbb{R}\}.$$

The proof of this is left as an easy exercise for the reader.

Example 1.1.9 A fundamental domain for $SL(2,\mathbb{Z})\backslash\mathfrak{h}$ can be given as the region $\mathcal{D} \subset \mathfrak{h}$ where

$$\mathcal{D} = \left\{ z \;\middle|\; -\frac{1}{2} \le \text{Re}(z) \le \frac{1}{2},\;\; |z| \ge 1 \right\},$$

with congruent boundary points symmetric with respect to the imaginary axis.



Note that the vertical line $V' := \left\{-\frac{1}{2} + iy \;\middle|\; y \geq \frac{\sqrt{3}}{2}\right\}$ is equivalent to the vertical line $V := \left\{\frac{1}{2} + iy \;\middle|\; y \geq \frac{\sqrt{3}}{2}\right\}$ under the transformation $z \mapsto z + 1$. Furthermore, the arc $A' := \left\{z \;\middle|\; -\frac{1}{2} \leq \text{Re}(z) < 0,\ |z| = 1\right\}$ is equivalent to the reflected arc $A := \left\{z \;\middle|\; 0 < \text{Re}(z) \leq \frac{1}{2},\ |z| = 1\right\}$, under the transformation $z \mapsto -1/z$. To show that $\mathcal{D}$ is a fundamental domain, we must prove:

(1) *For any* $z \in \mathfrak{h}$, *there exists* $g \in SL(2, \mathbb{Z})$ *such that* $g \circ z \in \mathcal{D}$;
(2) *If two distinct points* $z, z' \in \mathcal{D}$ *are congruent* $\pmod{SL(2, \mathbb{Z})}$ *then* $\text{Re}(z) = \pm\frac{1}{2}$ *and* $z' = z \pm 1$, or $|z| = 1$ *and* $z' = -1/z$.

We first prove (1). Fix $z \in \mathfrak{h}$. It follows from Lemma 1.1.6 that for every $\epsilon > 0$, there are at most finitely many $g \in SL(2, \mathbb{Z})$ such that $g \circ z$ lies in the strip

$$D_\epsilon := \left\{ w \;\middle|\; -\frac{1}{2} \leq \text{Re}(w) \leq \frac{1}{2},\quad \epsilon \leq \text{Im}(w) \right\}.$$

Let B_ϵ denote the finite set of such $g \in SL(2, \mathbb{Z})$. Clearly, for sufficiently small ϵ, the set B_ϵ contains at least one element. We will show that there is at least one $g \in B_\epsilon$ such that $g \circ z \in \mathcal{D}$. Among these finitely many $g \in B_\epsilon$, choose one such that $\text{Im}(g \circ z)$ is maximal in D_ϵ. If $|g \circ z| < 1$, then for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$,

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$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and any $m \in \mathbb{Z}$,

$$\text{Im}(T^m S g \circ z) = \text{Im}\left(\frac{-1}{g \circ z}\right) = \frac{\text{Im}(g \circ z)}{|g \circ z|^2} > \text{Im}(g \circ z).$$

This is a contradiction because we can always choose m so that $T^m S g \circ z \in D_\epsilon$. So in fact, $g \circ z$ must be in $\mathcal{D}$.

To complete the verification that $\mathcal{D}$ is a fundamental domain, it only remains to prove the assertion (2). Let $z \in \mathcal{D}$, $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$, and assume that $g \circ z \in \mathcal{D}$. Without loss of generality, we may assume that

$$\text{Im}(g \circ z) = \frac{y}{|cz + d|^2} \geq \text{Im}(z),$$

(otherwise just interchange z and $g \circ z$ and use g^{-1}). This implies that $|cz + d| \leq 1$ which implies that $1 \geq |cy| \geq \frac{\sqrt{3}}{2}|c|$. This is clearly impossible if $|c| \geq 2$. So we only have to consider the cases $c = 0, \pm 1$. If $c = 0$ then $d = \pm 1$ and g is a translation by b. Since $-\frac{1}{2} \leq \text{Re}(z), \text{Re}(g \circ z) \leq \frac{1}{2}$, this implies that either $b = 0$ and $z = g \circ z$ or else $b = \pm 1$ and $\text{Re}(z) = \pm\frac{1}{2}$ while $\text{Re}(g \circ z) = \mp\frac{1}{2}$. If $c = 1$, then $|z + d| \leq 1$ implies that $d = 0$ unless $z = e^{2\pi i/3}$ and $d = 0, 1$ or $z = e^{\pi i/3}$ and $d = 0, -1$. The case $d = 0$ implies that $|z| \leq 1$ which implies $|z| = 1$. Also, in this case, $c = 1, d = 0$, we must have $b = -1$ because $ad - bc = 1$. Then $g \circ z = a - \frac{1}{z}$. It follows that $a = 0$. If $z = e^{2\pi i/3}$ and $d = 1$, then we must have $a - b = 1$. It follows that $g \circ e^{2\pi i/3} = a - \frac{1}{1 + e^{2\pi i/3}} = a + e^{2\pi i/3}$, which implies that $a = 0$ or 1. A similar argument holds when $z = e^{\pi i/3}$ and $d = -1$. Finally, the case $c = -1$ can be reduced to the previous case $c = 1$ by reversing the signs of a, b, c, d.

1.2 Iwasawa decomposition

This monograph focusses on the general linear group $GL(n, \mathbb{R})$ with $n \geq 2$. This is the multiplicative group of all $n \times n$ matrices with coefficients in $\mathbb{R}$ and non-zero determinant. We will show that every matrix in $GL(n, \mathbb{R})$ can be written as an upper triangular matrix times an orthogonal matrix. This is called the Iwasawa decomposition (Iwasawa, 1949).

The Iwasawa decomposition, in the special case of $GL(2, \mathbb{R})$, states that every $g \in GL(2, \mathbb{R})$ can be written in the form:

$$g = \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} \tag{1.2.1}$$

where $y > 0$, $x, d \in \mathbb{R}$ with $d \neq 0$ and

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in O(2, \mathbb{R}),$$

where

$$O(n, \mathbb{R}) = \left\{ g \in GL(n, \mathbb{R}) \;\middle|\; g \cdot {}^t g = I \right\}$$

is the orthogonal group. Here I denotes the identity matrix on $GL(n, \mathbb{R})$ and ${}^t g$ denotes the transpose of the matrix g. The matrix $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$ in the decomposition (1.2.1) is actually uniquely determined. Furthermore, the matrices $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ and $\begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ are uniquely determined up to multiplication by $\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$.

Note that explicitly,

$$O(2, \mathbb{R}) = \left\{ \begin{pmatrix} \pm\cos t & -\sin t \\ \pm\sin t & \cos t \end{pmatrix} \;\middle|\; 0 \leq t \leq 2\pi \right\}.$$

We shall shortly give a detailed proof of (1.2.1) for $GL(n, \mathbb{R})$ with $n \geq 2$.

The decomposition (1.2.1) allows us to realize the upper half-plane

$$\mathfrak{h} = \left\{ x + iy \;\middle|\; x \in \mathbb{R},\ y > 0 \right\}$$

as the set of two by two matrices of type

$$\left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R},\ y > 0 \right\},$$

or by the isomorphism

$$\mathfrak{h} \equiv GL(2, \mathbb{R}) \big/ \langle O(2, \mathbb{R}), Z_2 \rangle, \tag{1.2.2}$$

where

$$Z_n = \left\{ \begin{pmatrix} d & & 0 \\ & \ddots & \\ 0 & & d \end{pmatrix} \;\middle|\; d \in \mathbb{R},\ d \neq 0 \right\}$$

is the center of $GL(n, \mathbb{R})$, and $\langle O(2, \mathbb{R}), Z_2 \rangle$ denotes the group generated by $O(2, \mathbb{R})$ and Z_2.

The isomorphism (1.2.2) is the starting point for generalizing the classical theory of modular forms on $GL(2, \mathbb{R})$ to $GL(n, \mathbb{R})$ with $n > 2$. Accordingly, we define the generalized upper half-plane $\mathfrak{h}^n$ associated to $GL(n, \mathbb{R})$.

Definition 1.2.3 *Let $n \geq 2$. The* **generalized upper half-plane** $\mathfrak{h}^n$ *associated to $GL(n, \mathbb{R})$ is defined to be the set of all $n \times n$ matrices of the form $z = x \cdot y$ where*

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & & x_{1,n} \\ & 1 & x_{2,3} & \cdots & & x_{2,n} \\ & & \ddots & & & \vdots \\ & & & & 1 & x_{n-1,n} \\ & & & & & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y'_{n-1} & & & & \\ & y'_{n-2} & & & \\ & & \ddots & & \\ & & & y'_1 & \\ & & & & 1 \end{pmatrix},$$

with $x_{i,j} \in \mathbb{R}$ for $1 \leq i < j \leq n$ and $y'_i > 0$ for $1 \leq i \leq n-1$.

To simplify later formulae and notation in this book, we will always express y in the form:

$$y = \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

with $y_i > 0$ for $1 \leq i \leq n-1$. Note that this can always be done since $y'_i \neq 0$ for $1 \leq i \leq n-1$.

Explicitly, x is an upper triangular matrix with 1s on the diagonal and y is a diagonal matrix beginning with a 1 in the lowest right entry. Note that x is parameterized by $n \cdot (n-1)/2$ real variables $x_{i,j}$ and y is parameterized by $n-1$ positive real variables y_i.

Example 1.2.4 The generalized upper half plane $\mathfrak{h}^3$ is the set of all matrices $z = x \cdot y$ with

$$x = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}$, $y_1, y_2 > 0$. Explicitly, every $z \in \mathfrak{h}^3$ can be written in the form

$$z = \begin{pmatrix} y_1 y_2 & x_{1,2} y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 1.2.5 The generalized upper half-plane $\mathfrak{h}^3$ does not have a complex structure. Thus $\mathfrak{h}^3$ is quite different from $\mathfrak{h}^2$, which does have a complex structure.