LECTURES ON THE LUNAR THEORY.

[LECTURES on the Lunar Theory were given by Adams from 1860 with few intermissions until 1889. Originally their aim was to illustrate geometrically the analytical processes and thereby render them more comprehensible, and they included some elegant theorems on the geometry of conics which have since become common property; but every year several lectures were rewritten, and thus the whole fabric gradually changed into the form in which it is here presented,—the form, practically, in which he gave them last.

Perhaps it is superfluous to say that these Lectures stand upon a different footing to treatises that are intended to form the basis of Tables. With such, completeness is the first object and manner of presentation is secondary. Immense as is the labour of forming a treatise of this description, there exist several that leave little to desire in respect to fulness of detail. Indeed it may be suspected that their very perfection in the quality they profess has stifled to some degree the proper development of the subject, because at first sight it suggests that there is little left to do in the Lunar Theory, unless one is prepared to track down the inconsiderable errors that have eluded his Masters. This seems a mistake; the methods most suitable for the whole task adapt themselves comparatively ill to each detail of it, and there seems much that remains to be done in respect to inventing methods suitable for attacking separately, as far as they permit of separate attack, the many difficulties into which the theory divides at the outset, and thence perhaps approximating to a more adequate knowledge than we now possess of the relative motion of Three Bodies. So far, with the notable exception of Dr G. W. Hill and those that have followed him, we have seen comparatively little effort in this direction.

This was the cardinal feature of Adams's plan, and his lectures shew the methods he had gradually elaborated to accomplish it. They separate the inequalities from one another as far as possible, and are content with indicating the manner in which these separate inequalities afterwards combine. To shew that, with so slight an apparatus and within so small a compass, the result is no mere sketch, we need but set side by side the coefficients of longitude found in these Lectures and the corresponding terms in Delaunay's *Théorie*.

		Adams.	Delaunay.
		"	"
Variation, coeff. of	$\sin 2D$	2106·4	2106·25
	$\sin 4D$	8·74	8·75
Parallactic inequality,	$\sin D$	$-$124·90*	$-$127·62
	$\sin 3D$	0·73	0·84
	$\sin 5D$	0·01	0·01
Annual equation,	$\sin l'$	$-$658·9	$-$659·23
	$\sin(2D-l')$	152·09	152·11
	$\sin(2D+l')$	$-$21·57	$-$21·63
Evection,	$\sin(2D-l)$	4596·6	4607·77
	$\sin(2D+l)$	175·1	174·87
Further,			
Motion of Apse,	$1-c$	·008554	·008572
Motion of Node,	$g-1$	·003997	·003999

For those to whom the difficulties of the Lunar Theory are known, these numbers need no comment.

No Manuscript exists of Lecture I. It is taken substantially from my own notes of 1889.]

* With Delaunay's value of the Sun's Parallax, viz. 8″·75.

LECTURE I.

HISTORICAL SKETCH.

[THE Lunar Theory may be said to have had its commencement with Newton. Many irregularities in the Moon's motion were known before his time, but it was he that first explained the cause of those irregularities and calculated their amounts from theory.

Of the inequalities which are due to the action of the Sun, the first,—which is called the Evection,—was discovered by Ptolemy, who lived at Alexandria in the first half of the second century of our era, under the reigns of Hadrian and Antoninus Pius. At a very early period the relative distance of the Moon at different times could be told from the angle it subtended, and its orbit could thus be mapped out. By such means Ptolemy found that its form was not the same from month to month, and that the longer axis moved continually though not uniformly in one direction. He represented this change by a motion of the centre of the ellipse, as we would put it, in an epicycle round the focus, obtaining thus a variable motion for the longer axis and a variable eccentricity.

The representation of position by means of epicycles is intimately related to the modern method of developing the coordinates in harmonic series; thus if we have

$$x = A_1 \cos (n_1 t + \alpha_1) + A_2 \cos (n_2 t + \alpha_2) + \dots$$

$$y = A_1 \sin (n_1 t + \alpha_1) + A_2 \sin (n_2 t + \alpha_2) + \dots$$

the motion of the point (x, y) is that on a circle of radius A_1 with angular velocity n_1, around a centre which moves on a circle of radius A_2 with angular velocity n_2, and so on; and if, more generally, we have

$$x = A_1 \cos(n_1 t + \alpha_1) + \dots$$
$$y = B_1 \sin(n_1 t + \alpha_1) + \dots$$

we may reduce this case to the former by rewriting

$$x = \frac{1}{2}(A_1 + B_1)\cos(n_1 t + \alpha_1) + \frac{1}{2}(A_1 - B_1)\cos(-n_1 t - \alpha_1) + \dots,$$

$$y = \frac{1}{2}(A_1 + B_1)\sin(n_1 t + \alpha_1) + \frac{1}{2}(A_1 - B_1)\sin(-n_1 t - \alpha_1) + \dots.$$

Probably we have here the reason why circular motions and epicycles were first employed.

Tycho Brahe (1546—1601) discovered the existence of another inequality in the Moon's Longitude quite different from the Elliptic Inequality and the Evection. He found it bore reference to the position of the Sun with regard to the Moon; so that when the Sun and the Moon were in conjunction or opposition or quadratures the position of the Moon was quite well represented by the existing theory, but from conjunction to the quadrature following, her position was more advanced than the place assigned to it, reaching a maximum of some 35′ about half-way; and in the second quadrant it was just as much behind. This inequality he called the Variation; it was the first that Newton accounted for theoretically, and if we were to suppose the Moon and Sun to move, except for mutual disturbance, in pure circles in the same plane, it is the only one that would present itself.

The next significant step was made by Horrox (1619—1641) who represented the Evection geometrically by motion in a variable ellipse, and gave very approximately the law of variation of the eccentricity and the motion of the apse. He supposed the focus of the orbit to move in an epicycle about its mean place.

Newton's *Principia* did not profess to be and was not intended for a complete exposition of the Lunar Theory. It was fragmentary; its object was to shew that the more prominent irregularities admitted of explanation on his newly discovered theory of universal gravitation. He explained the Variation completely, and traced its effects in Radius Vector as well as in Longitude; and he also saw clearly that the change of eccentricity and motion of the apse that constitute the Evection could be explained on his principles, but he did not give the investigation in the *Principia*, even to the extent to which he had actually carried it. The approximations are more difficult in this case than in that of the Variation, and require to be carried further in order to furnish results of the same accuracy as had already been obtained by Horrox from observation. He was more successful in dealing with the motion of the node and the law of change of inclination. He shewed that when Sun and Node were in conjunction, then for nearly a month the Moon moved in a plane very approximately, and that the inclination of the orbit then reached its maximum, namely, 5° 17′ about; but as the Sun moved away from the Node the latter also began to move, attaining its greatest rate when the separation was a quadrant, and that at this instant the inclination was 5° very nearly. He also assigned the law for intermediate positions. The fact that there was no motion when the Sun was at the Node, that is, in the plane of the Moon's orbit, confirmed his theory that these inequalities were due to the Sun's action.

When we spoke of Newton's results as fragmentary and incomplete, let it not be understood that he gave only very rude approximations to the truth. His results are far more accurate than those arrived at in elementary works of the present day upon the subject.

After Newton, Clairaut (1713—1765) treated the Lunar Theory analytically. He readily found the Variation and many other inequalities, but met with a difficulty in determining the motion of the apse. At first he made its mean motion only about one-half of the observed value, and supposed that this indicated a failure of Newton's law of the inverse square of the distance; but soon he recognized an error, caused by omission of terms which he had imagined would not affect the result. When these were included the calculated amount was nearly doubled.

The first Tables of the Moon which were sufficiently accurate for use in determining longitudes at sea by observation of Lunar Distances were those of Mayer. They obtained a prize offered by our Board of Longitude, and were published in 1770 by Maskelyne, the Astronomer Royal.

The first Theories which gave the Moon's place with an accuracy equal to that of observation were those of Damoiseau and Plana. The former was published in 1827, preceded in 1824 by Tables; the latter was published in 1832.

Hansen's Tables, which are those now used, were constructed from theory and were published in 1857 at the expense of the British Government.]

LECTURE II.

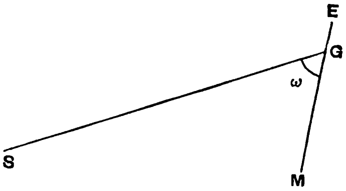
ACCELERATIONS OF THE MOON RELATIVE TO THE EARTH.

WHEN three bodies move under their mutual attraction, their motions are unknown to us except in the cases when they are approximately elliptical; but this restriction includes almost all the most important cases in the Solar System.

If one body of the system is greatly predominant and if the lesser bodies are not close together, the centre of gravity of the greater body may be taken as a common focus around which the others move in approximate ellipses. Or again, if two bodies lie close together, their relative motion may be approximately the same as though they were isolated, although the system contains a third greatly predominant body; for their relative motion is affected by the difference of the attractions of the central body upon them and not by the absolute value of those attractions.

The Sun and Planets are an example of the first kind; the Earth, Moon and Sun of the second. The Earth and Moon describe orbits round the Sun which are approximately ellipses, and the Moon might be regarded as one of the planets; but this point of view would not be a simple one; the disturbing action of the Earth would be too great, though it is never so great as the direct attraction of the Sun, that is to say, never great enough to make the Moon's path convex to the Sun. The more convenient method is to refer the motion of the Moon to the Earth, and counting only the difference of the attractions of the Sun upon the Earth and upon the Moon, to find how this distorts the otherwise elliptical relative orbit. This is the method of the Lunar Theory.

The position of the Sun must be referred to the same origin; but since the Earth describes an ellipse about the Sun which is disturbed by the action of the Moon, if we choose as origin the Earth's centre, we must allow for the disturbance of the Sun's position by the Moon. This correction may be evaded by choosing as origin, not the Earth's centre, but the centre of gravity of the Earth and Moon, with respect to which the Sun describes a curve so closely elliptical that no allowance is required. For, if S, E, M denote respectively the Sun, Earth, and Moon, and G the centre of gravity of E and M, the accelerating forces of S are



$$\text{on } E \quad S/SE^2 \text{ in } ES,$$
$$\text{on } M \quad S/SM^2 \text{ in } MS;$$

and these imply accelerations of G of amount

$$\frac{E}{E+M}\frac{S}{SE^2} \text{ parallel to } ES,$$
$$\frac{M}{E+M}\frac{S}{SM^2} \text{ parallel to } MS;$$

now the accelerations of S are

$$E/SE^2 \text{ in } SE,$$
$$M/SM^2 \text{ in } SM;$$

hence the acceleration of G relative to S is

$$\frac{S+E+M}{E+M}\frac{E}{SE^2} \text{ parallel to } ES,$$
$$\frac{S+E+M}{E+M}\frac{M}{SM^2} \text{ parallel to } MS;$$

or

$$\frac{S+E+M}{E+M}\left(E\,.\,\frac{GE}{SE^3} - M\,.\,\frac{GM}{SM^3}\right) \text{ in } GM,$$
$$\frac{S+E+M}{E+M}\left(E\,.\,\frac{SG}{SE^3} + M\,.\,\frac{SG}{SM^3}\right) \text{ in } GS.$$

Let $EM = r,\ SG = r',\ SGM = \omega$; then

$$GM = \frac{E}{E+M}r, \qquad GE = \frac{M}{E+M}r.$$

Hence

$$\frac{1}{SM^3} = \frac{1}{r'^3}\left[1 + \frac{E}{E+M}\frac{r}{r'}3\cos\omega + \left(\frac{E}{E+M}\frac{r}{r'}\right)^2\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right) + \ldots\ldots\right],$$

$$\frac{1}{SE^3} = \frac{1}{r'^3}\left[1 - \frac{M}{E+M}\frac{r}{r'}3\cos\omega + \left(\frac{M}{E+M}\frac{r}{r'}\right)^2\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right) + \ldots\ldots\right];$$

and the accelerations of G are

$$\frac{S+E+M}{r'^2}\left[-\frac{EM}{(E+M)^2}\frac{r^2}{r'^2}3\cos\omega + \ldots\ldots\right] \text{ in } GM,$$

$$\frac{S+E+M}{r'^2}\left[1 + \frac{EM}{(E+M)^2}\frac{r^2}{r'^2}\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right) + \ldots\ldots\right] \text{ in } GS.$$

Now r/r' is approximately $\frac{1}{400}$; neglecting the square of this quantity, we see that S moves about G in a pure ellipse.

Consider now the accelerations of the Moon relative to the Earth; subtracting the accelerations of the Earth from those of the Moon, we find

$$\frac{E+M}{ME^2} + S\left(\frac{MG}{SM^3}+\frac{EG}{SE^3}\right) \text{ in } MG,$$

$$S\left(\frac{SG}{SM^3}-\frac{SG}{SE^3}\right) \text{ parallel to } GS;$$

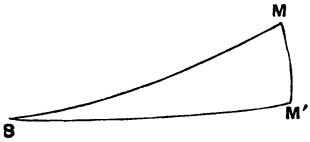
let $E+M=\mu,\ S=m'$; then these become

$$\frac{\mu}{r^2} + \frac{m'r}{r'^3}\left[1 + \frac{E-M}{E+M}\frac{r}{r'}3\cos\omega + \ldots\ldots\right] \text{ in } ME,$$

$$\frac{m'r}{r'^3}\left[3\cos\omega + \frac{E-M}{E+M}\frac{r}{r'}\left(-\frac{3}{2}+\frac{15}{2}\cos^2\omega\right) + \ldots\ldots\right]$$

parallel to GS.

In the accompanying spherical triangle, let G be the centre of the sphere, SM' the ecliptic, and M' the projection of M.



Let $1/u$ be the projection of ME on the plane of the ecliptic;

θ the longitude of the Moon as seen from the Earth,

θ' the longitude of the Sun as seen from G,

s the tangent of the Moon's latitude MM'.

Then

$$SM = \omega, \qquad SM' = \theta - \theta',$$
$$r = (1+s^2)^{\frac{1}{2}} u^{-1}, \qquad \cos\omega = \cos(\theta-\theta')(1+s^2)^{-\frac{1}{2}},$$

and the accelerations of M relative to E are

$$\frac{\mu u^2}{1+s^2} + \frac{m'(1+s^2)^{\frac{1}{2}}}{r'^3 u}\left[1 + \frac{E-M}{E+M}\frac{1}{r'u} 3\cos(\theta-\theta') + \ldots\ldots\right] \text{ in } ME,$$

$$\frac{m'}{r'^3 u}\left[3\cos(\theta-\theta') + \frac{E-M}{E+M}\frac{1}{r'u}\left(-\frac{3}{2}(1+s^2) + \frac{15}{2}\cos^2(\theta-\theta')\right) + \ldots\ldots\right] \text{ parallel to } GS.$$

Call these quantities U and V respectively; then if we resolve parallel to $M'G$, perpendicular to $M'G$ in the plane of the ecliptic, and perpendicular to the plane of the ecliptic, we have the following quantities which we call P, T, S; viz.:—

$$P = U(1+s^2)^{-\frac{1}{2}} - V\cos(\theta-\theta'),$$
$$T = -V\sin(\theta-\theta'),$$
$$S = Us(1+s^2)^{-\frac{1}{2}};$$

and also

$$S - Ps = Vs\cos(\theta-\theta').$$

From these we find

$$P = \frac{\mu u^2}{(1+s^2)^{\frac{3}{2}}} - \frac{m'}{r'^3 u}\left[\frac{1}{2} + \frac{3}{2}\cos 2(\theta-\theta') + \frac{E-M}{E+M}\frac{1}{r'u}\left\{\left(\frac{9}{8} - \frac{3}{2}s^2\right)\cos(\theta-\theta') + \frac{15}{8}\cos 3(\theta-\theta')\right\} + \ldots\right],$$