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Classical solutions to the two-dimensional Euler equations and elliptic boundary value problems, an overview

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Abstract

Consider the classical initial, boundary-value problem for the 2D Euler equations, which describes the motion of an ideal, incompressible, fluid in a impermeable vessel. In the early eighties we introduced and studied a Banach space, denoted $C_*(\overline{\Omega})$, which enjoys the following property: if the curl of the initial velocity belongs to $C_*(\overline{\Omega})$, and the curl of the external forces is integrable in time with values in the above space $C_*(\overline{\Omega})$, then all derivatives appearing in the differential equations and in the boundary conditions are continuous in space-time, up to the boundary (we call these solutions *classical solutions*). At that time this conclusion was know if $C_*(\overline{\Omega})$ is replaced by a Hölder space $C^{0,\lambda}(\overline{\Omega})$. In the proof of the above result we appealed to a $C^2(\overline{\Omega})$ regularity result for solutions to the Poisson equation, vanishing on the boundary and with external forces in $C_*(\overline{\Omega})$. Actually, at that time, we have proved this regularity result for solutions to more general second-order linear elliptic boundary-value problems. However the proof remained unpublished. Recently, we have published an adaptation of the proof to solutions of the Stokes system. We recall these results in Section 1.1 below. On the other hand, attempts to prove the above regularity results for data in functional spaces properly containing $C_*(\overline{\Omega})$, have also been done. Bellow we prove some partial results in this direction. This possibly unfinished picture leads to interesting open problems.

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1.1 The Euler and Stokes equations with data in $C_*(\overline{\Omega})$.

In these notes we want to give an overview on some results, both old and new. Some are old, but remained unpublished for a long time. The starting point will be Beirão da Veiga (1981, 1982, 1984).

We start by introducing some notation. Ω is an open, bounded, connected set in $\mathbb{R}^n$, $n \geq 2$, locally situated on one side of its boundary Γ. We assume that Γ is of class $C^{2,\lambda}(\overline{\Omega})$, for some positive λ. By $C(\overline{\Omega})$ we denote the Banach space of all real, continuous functions in $\overline{\Omega}$ with the norm

$$\|f\| \equiv \sup_{x \in \overline{\Omega}} |f(x)|.$$

In the sequel we use the notation

$$\|\nabla u\| = \sum_{i=1}^{n} \|\partial_i u\|, \quad \|\nabla^2 u\| = \sum_{i,j=1}^{n} \|\partial_{ij} u\|,$$

and appeal to the canonical spaces $C^1(\overline{\Omega})$ and $C^2(\overline{\Omega})$, with the norms

$$\|u\|_1 \equiv \|u\| + \|\nabla u\|, \quad \|u\|_2 \equiv \|u\| + \|\nabla^2 u\|$$

respectively. Further, for each $\lambda \in (0,1]$, we define the semi-norm

$$[f]_{0,\lambda} \equiv \sup_{x,y \in \Omega; x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\lambda}, \tag{1.1}$$

and the Hölder space $C^{0,\lambda}(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_{0,\lambda} < \infty\}$, with the norm

$$\|f\|_{0,\lambda} = \|f\| + [f]_{0,\lambda}.$$

In particular, $C^{0,1}(\overline{\Omega})$ is the space of Lipschitz continuous functions in $\overline{\Omega}$. By $C^\infty(\overline{\Omega})$ we denote the set of all restrictions to $\overline{\Omega}$ of infinitely differentiable functions in $\mathbb{R}^n$. We will use boldface notation to denote vectors, vector spaces, and so on. We denote the components of a generic vector $\boldsymbol{u}$ by u_i, and similarly for tensors. Norms in functional spaces whose elements are vector fields are defined in the usual way, by appealing to the corresponding norms of the components.

In considering the two-dimensional Euler equations we will introduce the following well-known simplification. For a scalar function $u(x)$ (identified here with the third component of a vector field, normal to the plane of motion) we define the vector field $\mathrm{Rot}\, u = (\partial_2 u, -\partial_1 u)$. For a vector field $\boldsymbol{v} = (v_1, v_2)$ we define the scalar field $\mathrm{rot}\boldsymbol{v} = \partial_1 \boldsymbol{v}_2 - \partial_2 \boldsymbol{v}_1$ (the normal component of the curl). One has $-\Delta = \mathrm{rotRot}$. Note that

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Rot$u$ is the rotation of the gradient ∇u by $\pi/2$ in the counter-clockwise direction.

Next we describe the motivation and origin of this research. We follow Beirão da Veiga (1981, 1982, 1984) which were essentially written during a visiting professorship to the Mathematics Research Center and the Mathematics Department in Wisconsin-Madison, in the semester October 1981-March 1982. In the above references we consider the initial boundary value problem for the two dimensional Euler equations

$$\begin{cases} \partial_t \boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{v} - \nabla \pi \quad \text{in} \quad Q \equiv \mathbb{R} \times \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 \quad \text{in} \quad Q, \\ \boldsymbol{v}_0 \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \mathbb{R} \times \Gamma, \\ \boldsymbol{v}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{1.2}$$

At that time our main interest was to determine minimal conditions on the data which imply that the global, unique, solutions to the above problem are *classical*. This means here that all derivatives appearing in the equations are continuous, up to the boundary, in the space-time cylinder. The main result on this problem was stated and proved in the preprint by Beirão da Veiga (1982), see the theorem 1.9 below. Exactly the same work was published in Beirão da Veiga (1984), to which we will refer in the sequel. To explain, in the simplest way, the main lines followed in our study, assume for now that no external forces are present, and that Ω is simply connected. In Beirão da Veiga (1984) we started by considering the Banach space

$$\boldsymbol{E}(\overline{\Omega}) \equiv \{\boldsymbol{v} \in \mathbf{C}(\overline{\Omega}) : \operatorname{div} \boldsymbol{v} = 0 \text{in } \Omega; \operatorname{rot} \boldsymbol{v} \in C(\overline{\Omega}); \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}, \tag{1.3}$$

endowed with the norm (in the simply connected case)

$$|||\boldsymbol{v}||| = \|\operatorname{rot} \boldsymbol{v}\|, \tag{1.4}$$

and show the global boundedness, strong-continuous dependence on the data, and other basic properties with respect to data in the above space $\boldsymbol{E}(\overline{\Omega})$ (see the theorems 1.1, 1.2, and 1.3, in the above reference). These preliminary results were obtained by improving techniques already used by other authors; see for instance Kato (1967), and Schaeffer (1937). However these results do not imply that solutions are classical under the given assumption on the initial data, since

$$\operatorname{rot} \boldsymbol{v}_0 \in C(\overline{\Omega})$$

leads to rot$\boldsymbol{v}(t, \cdot) \in C(\overline{\Omega})$, but this last property does not imply $\nabla \boldsymbol{v}(t, \cdot) \in$

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$\mathbf{C}(\overline{\Omega})$. This gap is strictly related to a corresponding gap for solutions to elliptic equations, namely, the solution $\boldsymbol{v}$ to the system (see equation (1.3) in Beirão da Veiga, 1982)

$$\begin{cases} \mathrm{rot}\boldsymbol{v} = f & \text{in} \quad \Omega, \\ \mathrm{div}\boldsymbol{v} = 0 & \text{in} \quad \Omega, \\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.5}$$

does not necessarily belong to $\mathbf{C}^1(\overline{\Omega})$, whenever $f \in C(\overline{\Omega})$. On the other hand, at that time, it was already well known that if f belongs to a Hölder space $C^{0,\lambda}(\overline{\Omega})$, then $\boldsymbol{v} \in \mathbf{C}^{1,\lambda}(\overline{\Omega})$. This result, together with a clever use of Lagrangian coordinates, makes it possible to prove that solutions to the system (1.2) are classical under the hypothesis

$$\mathrm{rot}\boldsymbol{v}_0 \in C^{0,\lambda}(\overline{\Omega}).$$

This was a well known result at that time, see Bardos (1972), Judovich (1963), Kato (1967), and Schaeffer (1937).

Having the above picture in mind, it seemed natural to start our approach to the Euler equations by studying the system (1.5). We wanted to single out a Banach spaces $C_*(\overline{\Omega})$, strictly contained in the Hölder spaces $C^{0,\lambda}(\overline{\Omega})$, such that solutions $\boldsymbol{v}$ to the first order system (1.5) are classical under the assumption $f \in C_*(\overline{\Omega})$. On the other hand, a classical argument shows that the solution $\boldsymbol{v}$ to the system (1.5) can be obtained by setting $\boldsymbol{v} = -\mathrm{Rot}u$, where u solves the problem

$$\begin{cases} -\Delta u = f & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \Gamma. \end{cases} \tag{1.6}$$

It follows that solutions $\boldsymbol{v}$ to system (1.5) belong to $\mathbf{C}^1(\overline{\Omega})$ if the solutions $\boldsymbol{u}$ to the system (1.6) belong to $\mathbf{C}^2(\overline{\Omega})$. This situation led us to look for a Banach space $C_*(\overline{\Omega})$, for which the following result holds.

Theorem 1.1.1. *Let $f \in C_*(\overline{\Omega})$ and let u be the solution to problem* (1.6)*. Then $u \in C^2(\overline{\Omega})$, moreover, $\|u\|_2 \leq c_0 \|f\|_*$.*

The above theorem was stated in Beirão da Veiga (1984) as Theorem 4.5. For convenience, the space $C_*(\overline{\Omega})$ will be defined at the end of this section.

Having obtained the above result, we succeeded in proving that the solutions to the Euler equations (1.2) are classical under the assumption

$$\mathrm{rot}\boldsymbol{v}_0 \in C_*(\overline{\Omega}).$$

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This is the main result in Beirão da Veiga (1984). More precisely, we proved the following statement.

Theorem 1.1.2. *Let* $\mathrm{rot}\,\boldsymbol{u}_0 \in C_*(\overline{\Omega})$ *and* $\mathrm{rot}\,\boldsymbol{v} \in L^1(\mathbb{R}^+; C_*(\overline{\Omega}))$. *Then, the global solution* $\boldsymbol{v}$ *to problem* (1.2) *is continuous in time with values in* $\mathbf{C}^1(\overline{\Omega})$, *that is*

$$\boldsymbol{v} \in C(\mathbb{R}^+; \mathbf{C}^1(\overline{\Omega})). \tag{1.7}$$

Furthermore, the estimate

$$\|\boldsymbol{v}(t)\|_{\mathbf{C}^1(\overline{\Omega})} \leq c e^{c_1 B_t t}\{\|\mathrm{rot}\,\boldsymbol{v}_0\|_{C_*(\overline{\Omega})} + \|\mathrm{rot}\,\boldsymbol{v}\|_{L^1(0,t;C_*(\overline{\Omega}))}\} \tag{1.8}$$

holds for all $t \in \mathbb{R}^+$, *where*

$$B_t = \|\mathrm{rot}\,\boldsymbol{v}_0\| + \|\mathrm{rot}\,\boldsymbol{v}\|_{L^1(0,t;C(\overline{\Omega}))}. \tag{1.9}$$

Moreover, $\partial_t \boldsymbol{v}$ *and* $\nabla\pi$ *are continuous in* $\overline{Q}$ *if both terms* $\boldsymbol{v}_0$ *and* ∇F, *in the canonical Helmholtz decomposition* $\boldsymbol{v} = \boldsymbol{v}_0 + \nabla F$ *separately satisfy this same continuity property. Then all derivatives that appear in equations* (1.2) *are continuous in* $\overline{Q}$, *that is, we have a classical solution.*

The conclusion of the theorem is false in general for data $\boldsymbol{v}_0 \in \mathbf{C}^1(\overline{\Omega})$, or $\boldsymbol{v} \in L^1(\mathbb{R}^+; \mathbf{C}^1(\overline{\Omega}))$.

If Ω is not simply connected the results still apply, as remarked in Beirão da Veiga (1984), by appealing to well known devices. See, for instance, the appendix 1 in the above reference.

Concerning the 2D Euler equations, we also refer the reader to Koch (2002). In this interesting work the author considers not only the 2D Euler equations but also many other central problems. However, the claims and proofs that followed to treat the particular two-dimensional problem considered in reference Beirão da Veiga (1984) are not very dissimilar to those previously showed by us in this last reference. Related results can also be found in reference Vishik (1998).

In Beirão da Veiga (1984) it was remarked that Theorem 1.1.1 could also be extended to solutions to more general linear elliptic boundary value problems. In fact, in Beirão da Veiga (1981) we proved the following regularity result.

Theorem 1.1.3. *For every* $f \in C_*(\overline{\Omega})$ *the solution* u *to the problem*

$$\begin{cases} \mathcal{L}u = f & \text{in} \quad \Omega, \\ \mathcal{B}u = 0 & \text{on} \quad \Gamma, \end{cases} \tag{1.10}$$

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belongs to $C^2(\overline{\Omega})$*. Moreover, there is a constant* c_0 *such that the estimate*

$$\|u\|_2 \leq c_0 \|f\|_*, \quad \forall f \in C_*(\overline{\Omega}). \tag{1.11}$$

holds.

In the above theorem $\mathcal{L}$ is a second order partial differential elliptic operator with smooth coefficients, and $\mathcal{B}$ is a linear differential operator, of order less or equal to one, acting on the boundary Γ. In Beirão da Veiga (1981) we assumed that $\mathcal{L}$, $\mathcal{B}$, and Ω are such that, for each $f \in C(\overline{\Omega})$, problem (1.10) has a unique solution $u \in C^1(\overline{\Omega})$, given by

$$u(x) = \int_\Omega g(x,y) f(y)\, \mathrm{d}y, \tag{1.12}$$

where g is the Green function associated with he above boundary value problem. Our hypotheses on $\mathcal{L}$, $\mathcal{B}$, and Ω are given by the following two requirements:

– For each $f \in C(\overline{\Omega})$ the solution u of problem (1.10) is unique, belongs to $C^1(\overline{\Omega})$, and is is given by (1.12). Furthermore, if $f \in C^\infty(\overline{\Omega})$ then $u \in C^2(\overline{\Omega})$.

– The above Green's function $g(x,y)$ satisfies the estimates

$$\left|\frac{\partial g}{\partial x_i}\right| \leq \frac{k}{|x-y|^{n-1}}, \quad \left|\frac{\partial^2 g}{\partial x_i \partial x_j}\right| \leq \frac{k}{|x-y|^n}, \tag{1.13}$$

where $i, j = 1, ..., n$.

The above estimates for Green's functions have been well known for a large class of problems for a long time. Classical works are due to, for instance, Levi (1908, 1909), Hadamard (1914), Lichtenstein (1918), Eidus (1958), Levy (1920), and many other authors. We refer in particular to (Miranda, 1955, Chap. III, Sections 21, 22, and 23), and references therein (in particular, to Giraud's references). For much more general results on Green functions see Solonnikov (1970, 1971).

It is worth noting that the proof of Theorem 1.1.3 may be extended to a larger class of problems, like non-homogeneous boundary-value problems, elliptic systems, and in particular the Stokes system, higher order problems, etc. The main point is that solutions u are given by expressions like (1.12), where the Green's functions g satisfy suitable estimates, which extend that shown in equation (1.13). Recently, we have adapted the unpublished proof of theorem 1.1.3 to show a similar regularity result for solutions to the Stokes system (1.10). Actually, in Beirão da Veiga (2014) we prove the following result.

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Theorem 1.1.4. *For every $\boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega})$ the solution $(\boldsymbol{u}, p)$ of the Stokes system*

$$\begin{cases} -\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} & in\,\Omega, \\ \nabla \cdot \boldsymbol{u} = 0 & in\,\Omega, \\ \boldsymbol{u} = 0 & on\,\Gamma, \end{cases} \tag{1.14}$$

belongs to $\boldsymbol{C}^2(\overline{\Omega}) \times C^1(\overline{\Omega})$. Moreover, there exists a constant c_0, depending only on Ω, such that the estimate

$$\|\boldsymbol{u}\|_2 + \|\nabla p\| \leq c_0 \|\boldsymbol{f}\|_*, \quad \forall \boldsymbol{f} \in \boldsymbol{C}_*(\overline{\Omega}), \tag{1.15}$$

holds.

In the final part of the section we define the Banach space $C_*(\overline{\Omega})$.
If $f \in C(\overline{\Omega})$ set, for each $r > 0$,

$$\omega_f(r) \equiv \sup_{x,y\in\Omega;0<|x-y|\leq r} |f(x) - f(y)|, \tag{1.16}$$

and define the semi-norm

$$[f]_* = [f]_{*,\delta} \equiv \int_0^\delta \omega_f(r) \frac{\mathrm{d}r}{r}. \tag{1.17}$$

If $0 < \delta < R$, one has

$$[f]_{*,\delta} \leq [f]_{*,R} \leq [f]_{*,\delta} + 2\Big(\log \frac{R}{\delta}\Big)\|f\|. \tag{1.18}$$

It follows that norms (obtained by the addition of $\|f\|$, see (1.20) below), are equivalent.

In the literature, the condition

$$\int_0^\delta \omega_f(r) \frac{\mathrm{d}r}{r} < +\infty$$

is called *Dini's continuity condition*, see Gilbarg & Trudinger (1977), equation (4.47). In Gilbarg & Trudinger (1977), problem 4.2, it is remarked that if f satisfies Dini's condition in $\mathbb{R}^n$, then its Newtonian potential is a C^2 solution of Poisson's equation $\Delta u = f$ in $\mathbb{R}^n$.

Definition 1.1.5.

$$C_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : [f]_* < \infty\}. \tag{1.19}$$

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As claimed in Beirão da Veiga (1984), $C_*(\overline{\Omega})$ endowed with the norm

$$\|f\|_* \equiv [f]_* + \|f\|, \tag{1.20}$$

is a Banach space, compactly embedded in $C(\overline{\Omega})$. Furthermore, $C^\infty(\overline{\Omega})$ is dense in $C_*(\overline{\Omega})$. We have appealed to these properties in reference Beirão da Veiga (1984), however the complete proofs were written only in an unpublished manuscript Beirão da Veiga (1981). For the complete proofs see the recent publication Beirão da Veiga (2014).

In Beirão da Veiga (1981) we introduced a functional space $B_*(\overline{\Omega})$, which strictly contains $C_*(\overline{\Omega})$, for which we have proven that the second order derivatives of the solutions to the system (1.10) are bounded in Ω for all $f \in B_*(\overline{\Omega})$. However, we did not succeed in proving, or disproving, the full result, namely, the continuity up to the boundary of the second order derivatives. This led us to leave unpublished the statements concerning the space $B_*(\overline{\Omega})$. In the next sections we show some of these results and proofs, and related open problems. Some results are proved below for data in a larger space $D_*(\overline{\Omega}) \supset B_*(\overline{\Omega})$.

As remarked in Beirão da Veiga (2014), another significant candidate could be obtained by replacing in the definition of $C_*(\overline{\Omega})$ given in (1.17) by the quantity $\omega_f(x;r)$ by

$$\widetilde{\omega}_f(x;r) = \sup_{x\in\Omega} \left| f(x) - |\Omega(x;r)|^{-1} \int_{\Omega(x;r)} f(y)\,\mathrm{d}y \right|. \tag{1.21}$$

1.2 The functional spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$.

In this section we define the spaces $B_*(\overline{\Omega})$ and $D_*(\overline{\Omega})$. We start with $B_*(\overline{\Omega})$. Set

$$\omega_f(x;r) = \sup_{y\in\Omega(x;r)} |f(x) - f(y)|, \tag{1.22}$$

and define, for each $x \in \overline{\Omega}$, the "point-wise" semi-norms

$$p_x(f) \equiv \int_0^\delta \omega_f(x;r)\frac{\mathrm{d}r}{r}, \tag{1.23}$$

and also the "global" semi-norm

$$\langle f\rangle_* = \sup_{x\in\overline{\Omega}} \int_0^\delta \omega_f(x;r)\frac{\mathrm{d}r}{r} = \sup_{x\in\overline{\Omega}} p_x(f). \tag{1.24}$$

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Note that

$$[f]_* = \int_0^\delta \sup_{x \in \overline{\Omega}} \omega_f(x;r) \frac{\mathrm{d}r}{r}. \tag{1.25}$$

Definition 1.2.1.

$$B_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : \langle f \rangle_* < +\infty\}. \tag{1.26}$$

The space $B_*(\overline{\Omega})$ endowed with

$$\|f\|^* \equiv \|f\| + \langle f \rangle_*, \tag{1.27}$$

is a normed linear space. Clearly $\langle f \rangle_* \leq [f]_*$. Further, in Beirão da Veiga (1981), we proved that the embedding $B_*(\overline{\Omega}) \subset C_*(\overline{\Omega})$ is strict, by constructing an oscillating function which belongs to $B_*(\overline{\Omega})$ but not to $C_*(\overline{\Omega})$; for the counterexample we take $\overline{\Omega} = [0,1]$. We show this construction in Section 1.7 below.

Next we define $D_*(\overline{\Omega})$. Set

$$S(x;r) = \{y \in \Omega : |x - y| = r\},$$

and define

$$\mu_f(x;r) = \sup_{y \in S(x;r)} |f(x) - f(y)|, \tag{1.28}$$

for each fixed $x \in \overline{\Omega}$ and $r > 0$. Further, fix a real positive δ, and define the semi-norms

$$q_x(f) \equiv \int_0^\delta \mu_f(x;r) \frac{\mathrm{d}r}{r}, \tag{1.29}$$

for each $x \in \overline{\Omega}$. As in (1.18), the particular positive value δ is not significant here. Note that the continuity of f at single point x follows necessarily from the finiteness of the integral in equation (1.29). To avoid unnecessary complications, we assume in the sequel that $f \in C(\overline{\Omega})$. Next define the semi-norm

$$(f)_* = \sup_{x \in \overline{\Omega}} \int_0^\delta \omega_f(x;r) \frac{\mathrm{d}r}{r} = \sup_{x \in \overline{\Omega}} q_x(f). \tag{1.30}$$

It is worth noting that all the semi-norms introduced above enjoy property (1.18).

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Definition 1.2.2.

$$D_*(\overline{\Omega}) \equiv \{f \in C(\overline{\Omega}) : (f)_* < +\infty\}. \tag{1.31}$$

The linear space $D_*(\overline{\Omega})$ endowed with

$$|||f|||^* \equiv \|f\| + (f)_*, \tag{1.32}$$

is a normed linear space. Obviously, $B_*(\overline{\Omega}) \subset D_*(\overline{\Omega})$. Finally, note that (1.18) holds for the above two functional spaces, with the obvious modifications.

1.3 Results and open problems.

Theorem 1.3.1. *Let $f \in D_*(\overline{\Omega})$, and let u be the solution to problem* (1.10). *Then the first order derivatives of the solution u are Lipschitz continuous in $\overline{\Omega}$. Furthermore, the estimate*

$$\|\nabla^2 u\|_{L^\infty(\Omega)} \leq c_0 \; |||f|||^* \tag{1.33}$$

holds.

The proof of this result is an extension of the unpublished proof given in Beirão da Veiga (1981) for data $f \in B_*(\overline{\Omega})$. The proof will be shown in Section 1.4.

It remains an open problem whether the Theorems 1.1.3 and 1.1.4 hold with $C_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$ or by $D_*(\overline{\Omega})$. Let us discuss this point. Below we prove the following *conditional* result.

Theorem 1.3.2. *Let u be the solution of problem* (1.10) *with a given data $f \in D_*(\overline{\Omega})$. Assume that there is a sequence of data $f_m \in D_*(\overline{\Omega})$, convergent to f in $D_*(\overline{\Omega})$, such that the solutions u^m of problem* (1.10) *with data f_m belong to $C^2(\overline{\Omega})$. Then $u \in C^2(\overline{\Omega})$, and moreover*

$$\|\nabla^2 u\| \leq c_0 |||f|||^*. \tag{1.34}$$

Theorem 1.3.2 will be proven in Section 1.5. It is worth noting that, since $B_*(\overline{\Omega}) \subset D_*(\overline{\Omega})$, the above two theorems hold with $D_*(\overline{\Omega})$ replaced by $B_*(\overline{\Omega})$, and $|||f|||^*$ replaced by $\|f\|^*$.

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