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London Mathematical Society Lecture Note Series: 437

Dynamics and Analytic Number Theory

Proceedings of the Durham Easter School 2014

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CAMBRIDGE
UNIVERSITY PRESS

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University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107552371

First published 2016

Printed in the United Kingdom by Clays, St Ives plc

A catalogue record for this publication is available from the British Library.

Library of Congress Cataloging-in-Publication Data
Names: Durham Easter School (2014 : University of Durham) | Badziahin, Dmitry (Dmitry A.), editor. | Gorodnik, Alexander, 1975– editor. | Peyerimhoff, Norbert, 1964– editor.
Title: Dynamics and analytic number theory : proceedings of the Durham Easter School 2014 / edited by Dzmitry Badziahin, University of Durham, Alexander Gorodnik, University of Bristol, Norbert Peyerimhoff, University of Durham.
Description: Cambridge : Cambridge University Press, [2016] | Series: London Mathematical Society lecture note series ; 437 | Includes bibliographical references and index.
Identifiers: LCCN 2016044609 | ISBN 9781107552371 (alk. paper)
Subjects: LCSH: Number theory – Congresses. | Dynamics – Congresses.
Classification: LCC QA241 .D87 2014 | DDC 512.7/3–dc23
LC record available at https://lccn.loc.gov/2016044609

ISBN 978-1-107-55237-1 Paperback

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Preface

This book is devoted to some of the interesting recently discovered interactions between Analytic Number Theory and the Theory of Dynamical Systems. Analytical Number Theory has a very long history. Many people associate its starting point with the work of Dirichet on L-functions in 1837, where he proved his famous result about infinitely many primes in arithmetic progressions. Since then, analytical methods have played a crucial role in proving many important results in Number Theory. For example, the study of the Riemann zeta function allowed to uncover deep information about the distribution of prime numbers. Hardy and Littlewood developed their circle method to establish first explicit general estimates for the Waring problem. Later, Vinogradov used the idea of the circle method to create his own method of exponential sums which allowed him to solve, unconditionally of the Riemann hypothesis, the ternary Goldbach conjecture for all but finitely many natural numbers. Roth also used exponential sums to prove the existence of three-term arithmetic progressions in subsets of positive density. One of the fundamental questions which arise in the investigation of exponential sums, as well as many other problems in Number Theory, is how rational numbers/vectors are distributed and how well real numbers/vectors can be approximated by rationals. Understanding various properties of sets of numbers/vectors that have prescribed approximational properties, such as their size, is the subject of the metric theory of Diophantine approximation, which involves an interesting interplay between Arithmetic and Measure Theory. While these topics are now considered as classical, the behaviour of exponential sums is still not well understood today, and there are still many challenging open problems in Diophantine approximation. On the other hand, in the last decades there have been several important breakthroughs in these areas of Number Theory where progress on long-standing open problems has been achieved by utilising techniques which originated from the Theory of Dynamical Systems. These developments have uncovered many profound and very promising connections between number-theoretic and dynamical objects that are at the forefront of current research. For instance, it turned out that properties of exponential sums are intimately related to the behaviour of orbits of flows on nilmanifolds; the existence of given combinatorial configurations (e.g. arithmetic progressions) in subsets of integers can be established through the study of multiple recurrence properties for dynamical systems; and Diophantine properties of vectors in the Euclidean spaces can be characterised in terms of excursions of orbits of suitable flows in the space of lattices.

The material of this book is based on the Durham Easter School, 'Dynamics and Analytic Number Theory', that was held at the University of Durham in Spring 2014. The intention of this school was to communicate some of these remarkable developments at the interface between Number Theory and Dynamical Systems to young researchers. The Easter School consisted of a series of mini-courses (with two to three lectures each) given by Tim Austin, Manfred Einsiedler, Giovanni Forni, Alex Kontorovich, Sanju Velani and Trevor Wooley, and a talk by Yann Bugeaud presenting a collection of recent results and open problems in Diophantine approximation. The event was very well received by more than 60 participants, many of them PhD students from all around the world. Because of the great interest of young researchers in this topic, we decided to encourage the speakers to write contributions to this Proceedings volume.

One of the typical examples where both classical and dynamical approaches are now actively developing and producing deep results is the theory of Diophantine approximation. One of the classical problems in this area asks how well a given n-dimensional vector $\mathbf{x} \in \mathbb{R}^n$ can be approximated by vectors with rational coefficients. More specifically, one can ask: what is the supremum $\lambda(\mathbf{x})$ of the values λ such that the inequality

$$||q\mathbf{x} - \mathbf{p}||_\infty < Q^{-\lambda} \tag{1}$$

has infinitely many integer solutions $Q \in \mathbb{N}$, $q \in \mathbb{N}$, $\mathbf{p} \in \mathbb{Z}^n$ satisfying $q \leq Q$? This type of problem is referred to as a simultaneous Diophantine approximation. There is also a dual Diophantine approximation problem which asks for the supremum $\omega(\mathbf{x})$ of the values ω such that the inequality

$$|(\mathbf{x}, \mathbf{q}) - p| < Q^{-\omega} \tag{2}$$

has infinitely many solutions $Q \in \mathbb{N}$, $\mathbf{q} \in \mathbb{Z}^n$, $p \in \mathbb{Z}$ with $\mathbf{q} \neq \mathbf{0}$ and $||\mathbf{q}||_\infty \leq Q$. It turns out that there are various relations between the exponents $\lambda(\mathbf{x})$ and $\omega(\mathbf{x})$. Chapter 2 provides an overview of known relations between these and some other similar exponents. It mostly concentrates on the case where $\mathbf{x}$ lies on the so-called Veronese curve which is defined by $\mathbf{x}(t) := (t, t^2, \ldots, t^n)$ with real t. This case is of particular importance for number theorists since it has implications for the question about the distribution of algebraic numbers of bounded degree. For example, condition (2) in this case transforms to $|P(t)| < Q^{-\omega}$ where $P(t)$ is a polynomial with integer coefficients. For large Q this implies that x is very close to the root of P, which is an algebraic number.

Metric theory of Diophantine approximation does not work with particular vectors $\mathbf{x}$. Instead it deals with the sets of all vectors $\mathbf{x}$ satisfying inequalities like (1) or (2) for infinitely many $Q \in \mathbb{N}, q \in \mathbb{N}, \mathbf{p} \in \mathbb{Z}^n$ (respectively, $Q \in \mathbb{N}$, $\mathbf{q} \in \mathbb{Z}^n$, $p \in \mathbb{Z}$, $\mathbf{q} \neq \mathbf{0}$). The central problem is to estimate the measure and the Hausdorff dimension of such sets. This area of Number Theory was founded at the beginning of the twentieth century with Khintchine's work which was later generalised by Groshev. In the most general way they showed that, given a function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, the set of $m \times n$ matrices A which satisfy the inequality

$$||A\mathbf{q} - \mathbf{p}||_\infty < \psi(||\mathbf{q}||_\infty)$$

with $\mathbf{p} \in \mathbb{Z}^n$ and $\mathbf{q} \in \mathbb{Z}^m$, has either zero or full Lebesgue measure. The matrices A satisfying this property are usually called *ψ-well approximable*. Furthermore, with some mild conditions on ψ, the Lebesgue measure of the set of ψ-well approximable matrices is determined by the convergence of a certain series which involves ψ. Later, many other results of this type were established, some of them with help of the classical methods and others by using the ideas from homogeneous dynamics.

Chapter 1 describes several powerful 'classical' techniques used in metric theory of Diophantine approximation, such as the Mass Transference Principle, ubiquitous systems, Cantor sets constructions and winning sets. The Mass Transference Principle allows us to get results about the more sensitive Hausdorff measure and Hausdorff dimension of sets of well approximable matrices or similar objects as soon as results about their Lebesgue measure are known. Ubiquitous systems provide another powerful method originating from works of A. Baker and W. Schmidt. It enables us to obtain the 'full Lebesgue measure'-type results in various analogues of the Khintchine–Groshev theorem. Finally, Chapter 1 introduces the generalised Cantor set construction technique, which helps in investigating badly approximable numbers or vectors. It also relates such sets with so-called winning sets developed by W. Schmidt. The winning sets have several surprising properties. For example, they have the maximal possible Hausdorff dimension and, even though such sets may be null in terms of Lebesgue measure, their countable intersection must also be winning.

Chapter 3 is devoted to the study of exponential sums. Given a real polynomial $P(x) = a_k x^k + \cdots + a_1 x + a_0$, the Weyl sums are defined as

$$W_N := \sum_{n=0}^{N-1} e^{2\pi i P(n)}.$$

The study of Weyl sums has a long history that goes back to foundational works of Hardy, Littlewood, and Weyl. When the coefficients of the polynomial $P(X)$ satisfy a suitable irrationality condition, then it is known that for some $w \in (0, 1)$,

$$W_N = O(N^{1-w}) \quad \text{as } N \to \infty,$$

and improving the value of the exponent in this estimate is a topic of current research. This problem has been approached recently by several very different methods. The method of Wooley is based on refinements of the Vinogradov mean value theorem and a new idea of efficient congruencing, and the method of Flaminio and Forni involves the investigation of asymptotic properties of flows on nilmanifolds using renormalisation techniques. It is quite remarkable that the exponents w obtained by the Flaminio–Forni approach, which is determined by optimal scaling of invariant distributions, essentially coincide with the exponents derived by Wooley using his method of efficient congruencing.

As discussed in Chapter 3, flows on nilmanifolds provide a very convenient tool for investigating the distribution of polynomial sequences modulo one and modelling Weyl sums. We illustrate this by a simple example. Let

$$N := \left\{ [p, q, r] := \begin{pmatrix} 1 & p & r \\ 0 & 1 & q \\ 0 & 0 & 1 \end{pmatrix} : \ p, q, r \in \mathbb{R} \right\}$$

denote the three-dimensional Heisenberg group, and Γ be the subgroup consisting of matrices with integral entries. Then the factor space $M := \Gamma \backslash N$ provides the simplest example of a nilmanifold. Given an upper triangular nilpotent matrix $X = (x_{ij})$, the flow generated by X is defined by

$$\phi_t^X(m) = m \exp(tX) \quad \text{with } m \in M.$$

More explicitly, $\exp(tX) = [x_{12}t, x_{23}t, x_{13}t + x_{12}x_{23}t^2/2]$. The space M contains a two-dimensional subtorus T defined by the condition $q = 0$. If we take $x_{23} = 1$, then the intersection of the orbit $\phi_t^X(\Gamma e)$ with this torus gives the sequence of points $[x_{12}n, 0, x_{13}n + x_{12}n^2/2]$ with $n \in \mathbb{N}$. Hence, choosing suitable matrices X, the flows ϕ_t^X can be used to model values of general quadratic polynomials P modulo one. Moreover, this relation can be made much more precise. In particular, with a suitable choice of a test function F on M and $m \in M$,

$$\sum_{n=0}^{N-1} e^{2\pi i P(n)} = \int_0^N F(\phi_t^X(m))\, dt + O(1).$$

This demonstrates that quadratic Weyl sums are intimately related to averages of one-parameter flows on the Heisenberg manifold. A more elaborate construction discussed in detail in Chapter 3 shows that general Weyl sums can be approximated by integrals along orbits on higher-dimensional nilmanifolds. Chapter 3 discusses asymptotic behaviour of orbits averages on nilmanifolds and related estimates for Weyl sums.

Dynamical systems techniques also provide powerful tools to analyse combinatorial structures of large subsets of integers and of more general groups. This active research field fusing ideas from Ramsey Theory, Additive Combinatorics, and Ergodic Theory is surveyed in Chapter 4. We say that a subset $E \subset \mathbb{Z}$ has positive upper density if

$$\bar{\mathrm{d}}(E) := \limsup_{N-M\to\infty} \frac{|E \cap [M, N]|}{N - M} > 0.$$

Surprisingly, this soft analytic condition on the set E has profound combinatorial consequences, one of the most remarkable of which is the Szemerédi theorem. It states that every subset of positive density contains arbitrarily long arithmetic progressions: namely, configurations of the form $a, a+n, \dots, a+(k-1)n$ with arbitrary large k. It should be noted that the existence of three-term arithmetic progressions had previously been established by Roth using a variant of the circle method, but the case of general progressions required substantial new ideas. Shortly after Szemerédi's work appeared, Furstenberg discovered a very different ingenious approach to this problem that used ergodic-theoretic techniques. He realised that the Szemerédi theorem is equivalent to a new ergodic-theoretic phenomenon called *multiple recurrence*. This unexpected connection is summarised by the Furstenberg correspondence principle which shows that, given a subset $E \subset \mathbb{Z}$, one can construct a probability space (X, μ), a measure-preserving transformation $T : X \to X$, and a measurable subset $A \subset X$ such that $\mu(A) = \bar{\mathrm{d}}(E)$ and

$$\bar{\mathrm{d}}(E \cap (E-n) \cap \cdots \cap (E-(k-1)n)) \geq \mu(A \cap T^{-n}(A) \cap \cdots \cap T^{-(k-1)n}(A)).$$

This allows the proof of Szemerédi's theorem to be reduced to establishing the multiple recurrence property, which shows that if $\mu(A) > 0$ and $k \geq 1$, then there exists $n \geq 1$ such that

$$\mu(A \cap T^{-n}A \cap \cdots \cap T^{-(k-1)n}A) > 0.$$

This result is the crux of Furstenberg's approach, and in order to prove it, a deep structure theorem for general dynamical systems is needed. Furstenberg's work has opened a number of promising vistas for future research and started a new field of Ergodic Theory – ergodic Ramsey theory, which explores the existence of combinatorial structures in large subsets of groups. This is the subject of Chapter 4. In view of the above connection it is of fundamental importance to explore asymptotics of the averages

$$\frac{1}{N}\sum_{n=0}^{N-1}\mu(A\cap T^{-n}(A)\cap\cdots\cap T^{-(k-1)n}(A)),$$

and more generally, the averages

$$\frac{1}{N}\sum_{n=0}^{N-1}(f_1\circ T^n)\cdots(f_{k-1}\circ T^{(k-1)n}) \tag{3}$$

for test functions $f_1,\dots,f_{k-1}\in L^\infty(\mu)$. The existence of limits for these averages was established in the groundbreaking works of Host, Kra, and Ziegler. Chapter 4 explains an elegant argument of Austin which permits the proof of the existence of limits for these multiple averages as well as multiple averages for actions of the group $\mathbb{Z}^d$.

A number of important applications of the Theory of Dynamical Systems to Number Theory involve analysing the distribution of orbits on the space of unimodular lattices in $\mathbb{R}^{d+1}$. This space, which will be denoted by X_{d+1}, consists of discrete cocompact subgroups of $\mathbb{R}^{d+1}$ with covolume one. It can be realised as a homogeneous space

$$\mathsf{X}_{d+1}\simeq \mathrm{SL}_{d+1}(\mathbb{R})/\mathrm{SL}_{d+1}(\mathbb{Z}).$$

which allows us to equip X_{d+1} with coordinate charts and an invariant finite measure. Some of the striking applications of dynamics on the space X_{d+1} to problems of Diophantine approximation are explored in Chapter 5. It was realised by Dani that information about the distribution of suitable orbits on X_{d+1} can be used to investigate the existence of solutions of Diophantine inequalities. In particular, this allows a convenient dynamical characterisation of many Diophantine classes of vectors in $\mathbb{R}^d$ discussed in Chapters 1 and 2 to be obtained, such as, for instance, badly approximable vectors, very well approximable vectors, singular vectors. This connection is explained by the following construction. Given a vector $\mathbf{v}\in\mathbb{R}^d$, we consider the lattice

$$\Lambda_{\mathbf{v}} := \{(q, q\mathbf{v}+\mathbf{p}) : (q,\mathbf{p})\in\mathbb{Z}\times\mathbb{Z}^d\},$$

and a subset of the space X_{d+1} defined as

$$\mathsf{X}_{d+1}(\varepsilon) := \{\Lambda : \Lambda \cap [-\varepsilon, \varepsilon]^{d+1} \neq \{0\}\}.$$

Let $g_Q := \mathrm{diag}(Q^{-d}, Q, \ldots, Q)$. If we establish that the orbit $g_Q\Lambda$ visits the subset $\mathsf{X}_{d+1}(\varepsilon)$, then this will imply that the systems of inequalities

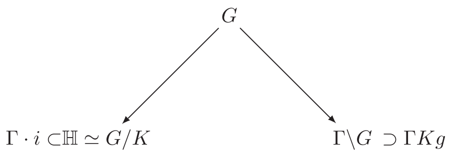
$$|q| \leq \varepsilon Q^d \quad \text{and} \quad \|q\mathbf{v} - \mathbf{p}\|_\infty \leq \frac{\varepsilon}{Q}$$

have a non-trivial integral solution $(q, \mathbf{p}) \in \mathbb{Z} \times \mathbb{Z}^d$. When $\varepsilon \geq 1$, the existence of solutions is a consequence of the classical Dirichlet theorem, but for $\varepsilon < 1$, this is a delicate property which was studied by Davenport and Schmidt. Vectors for which the above system of inequalities has a non-trivial solution for some $\varepsilon \in (0, 1)$ and all sufficiently large Q are called *Dirichlet-improvable*. Chapter 5 explains how to study this property using dynamical systems tools such as the theory of unipotent flow. This approach proved to be very successful. In particular, it was used by Shah to solve the problem posed by Davenport and Schmidt in the 60s. He proved that if $\phi : (0, 1) \to \mathbb{R}^d$ is an analytic curve whose image is not contained in a proper affine subspace, then the vector $\phi(t)$ is not Dirichlet-improvable for almost all t. Chapter 5 explains Shah's proof of this result.

Chapter 5 also discusses how dynamical systems techniques can be used to derive asymptotic counting results. Although this approach is applicable in great generality, its essence can be illustrated by a simple example: counting points in lattice orbits on the hyperbolic upper half-plane $\mathbb{H}$. We recall that the group $G = \mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbb{H}$ by isometries. Given $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ (or, more generally, a discrete subgroup Γ of G with finite covolume), we consider the orbit $\Gamma \cdot i$ in $\mathbb{H}$. We will be interested in asymptotics of the counting function

$$N(R) := |\{\gamma \cdot i : \mathsf{d}_{\mathbb{H}}(\gamma \cdot i, i) < R, \gamma \in \Gamma\}|,$$

where $d_{\mathbb{H}}$ denotes the hyperbolic distance in $\mathbb{H}$. Since $\mathbb{H} \simeq G/K$ with $K = \mathrm{PSO}(2)$, the following diagram:



suggests that the counting function $N(R)$ can be expressed in terms of the space

$$X := \Gamma\backslash G.$$

This idea, which goes back to the work of Duke, Rudnick, and Sarnak, is explained in detail in Chapter 5. Ultimately, one shows that $N(R)$ can be approximated by combinations of averages along orbits $\Gamma K g$ as g varies over some subset of G. This argument reduces the original problem to analysing the distribution of the sets $\Gamma K g$ inside the space X which can be carried out using dynamical systems techniques.

The space X introduced above is of fundamental importance in the Theory of Dynamical Systems and Geometry because it can be identified with the unit tangent bundle of the modular surface $\Gamma\backslash\mathbb{H}$. Of particular interest is the geodesic flow defined on this space, which plays a central role in Chapter 6. This chapter discusses recent striking applications of the sieving theory for thin groups, developed by Bourgain and Kontorovich, to the arithmetics of continued fractions and the distribution of periodic geodesic orbits. It is well known in the theory of hyperbolic dynamical systems that one can construct periodic geodesic orbits with prescribed properties. In particular, a single periodic geodesic orbit may exhibit a very peculiar behaviour. Surprisingly, it turns out that the finite packets of periodic geodesic orbits corresponding to a given fundamental discriminant D become equidistributed as $D \to \infty$. This remarkable result was proved in full generality by Duke, generalising previous works of Linnik and Skubenko. While Duke's proof uses elaborate tools from analytic number theory (in particular, the theory of half-integral modular forms), now there is also a dynamical approach developed by Einsiedler, Lindenstrauss, Michel, and Venkatesh. They raised a question whether there exist infinitely many periodic geodesic orbits corresponding to fundamental discriminants which are contained in a fixed bounded subset of X. Chapter 6 outlines an approach to this problem, which uses that the geodesic flow dynamics is closely related to the symbolic dynamics of the continued fractions expansions. In particular, a quadratic irrational with a periodic continued fraction expansion

$$\alpha = [\overline{a_0, a_1, \ldots, a_\ell}]$$

corresponds to a periodic geodesic orbit. Moreover, the property of having a fundamental discriminant can be characterised in terms of the trace of the matrix

$$M_\alpha := \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix},$$

and the corresponding geodesic orbit lies in a fixed bounded set of X if $a_i \leq A$ for all i for a fixed $A > 0$. Hence, the original question reduces to the investigation of the semigroup

$$\Gamma_A := \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : a \leq A \right\rangle^+ \cap \mathrm{SL}_2(\mathbb{R}),$$

and the trace map $\mathrm{tr} : \Gamma_A \to \mathbb{N}$. The semigroup Γ_A arises naturally in connection with several other deep problems involving periodic geodesic orbits and continued fractions. Chapter 6 outlines a promising approach to the Arithmetic Chaos Conjecture formulated by McMullen, which predicts that there exists a fixed bounded subset of the space X such that, for all real quadratic fields K, the closure of the set of periodic geodesic orbits defined over K and contained in this set has positive entropy. Equivalently, in the language of continued fractions, McMullen's conjecture predicts that for some $A < \infty$, the set

$$\{\alpha = [\overline{a_0, a_1, \ldots, a_\ell}] \in K : \text{ all } a_j \leq A\}$$

has exponential growth as $\ell \to \infty$. Since

$$\alpha \in \mathbb{Q}(\sqrt{\mathrm{tr}(M_\alpha)^2 - 4}),$$

this problem also reduces to the analysis of the map $\mathrm{tr} : \Gamma_A \to \mathbb{N}$. Chapter 6 also discusses progress on the Zaremba conjecture regarding continued fraction expansions of rational fractions. As is explained in Chapter 6, all these problems can be unified by the far-reaching Local-Global Conjectures describing the distribution of solutions of $F(\gamma) = n$, $\gamma \in \Gamma_A$, where F is a suitable polynomial map.

We hope that this book will help to communicate the exciting material written by experts in the field and covering a wide range of different topics which are, nevertheless, in many ways connected to a broad circle of young researchers as well as to other experts working in Number Theory or Dynamical Systems.

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Acknowledgements

We, as part of the organising committee of the Durham Easter School 2014, which followed a one-day LMS Northern Regional Meeting dedicated to the same topic, would like to thank all the people who helped to make this event highly enjoyable and successful. We are also grateful to everybody who made this Proceedings volume possible. Specifically, we would like to acknowledge the help of:

- our Easter School co-organisers, T. Ward, A. Ghosh, and B. Weiss, for their support in this event and the selection of a very impressive list of speakers;
- Sam Harrison and his colleagues from Cambridge University Press for all their encouragement and support in producing this book;
- our distinguished speakers for their interesting talks;
- all participants of the Easter School for their interest and help to create a positive and inspiring atmosphere;
- all contributors to this volume, for their work in producing their chapters within the given timeframe.

Particular thanks are due to Durham University for its great hospitality in hosting the Easter School.

Last, but not least, we would like to stress the fact that the Easter School would not have been possible without the generous funding received, via the University of Bristol, from the ERC grant 239606 and the financial support of the London Mathematical Society.