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Metric Diophantine Approximation: Aspects of Recent Work

Victor Beresnevich[1], *Felipe Ramírez*[2] *and Sanju Velani*[1]

Abstract

In these notes, we begin by recalling aspects of the classical theory of metric Diophantine approximation, such as theorems of Khintchine, Jarník, Duffin–Schaeffer and Gallagher. We then describe recent strengthening of various classical statements as well as recent developments in the area of Diophantine approximation on manifolds. The latter includes the well approximable, the badly approximable and the inhomogeneous aspects.

1.1 Background: Dirichlet and Bad

1.1.1 Dirichlet's Theorem and Two Important Consequences

Diophantine approximation is a branch of number theory that can loosely be described as a quantitative analysis of the density of the rationals $\mathbb{Q}$ in the reals $\mathbb{R}$. Recall that to say that $\mathbb{Q}$ is dense in $\mathbb{R}$ is to say that:

> for any real number x and $\epsilon > 0$ there exists a rational number p/q $(q > 0)$ such that $|x - p/q| < \epsilon$.

In other words, any real number can be approximated by a rational number with any assigned degree of accuracy. But how 'rapidly' can we approximate a given $x \in \mathbb{R}$?

> Given $x \in \mathbb{R}$ and $q \in \mathbb{N}$, how small can we make ϵ? Trivially, we can take any $\epsilon > 1/2q$. Can we do better than $1/2q$?

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The following rational numbers all lie within $1/(\text{denominator})^2$ of the circle constant $\pi = 3.141\ldots$:

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \frac{355}{113}, \frac{103993}{33102}. \tag{1.1}$$

This shows that, at least sometimes, the answer to the last question is 'yes'. A more complete answer is given by *Dirichlet's theorem*, which is itself a simple consequence of the following powerful fact.

Pigeonhole Principle *If n objects are placed in m boxes and $n > m$, then some box will contain at least two objects.*

Theorem 1.1.1 (Dirichlet, 1842) *For any $x \in \mathbb{R}$ and $N \in \mathbb{N}$, there exist $p, q \in \mathbb{Z}$ such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{qN} \qquad \textit{and} \qquad 1 \le q \le N. \tag{1.2}$$

The proof can be found in most elementary number theory books. However, given the important consequences of the theorem and its various hybrids, we have decided to include the proof.

Proof As usual, let $[x] := \max\{n \in \mathbb{Z} : n \le x\}$ denote the integer part of the real number x and let $\{x\} = x - [x]$ denote the fractional part of x. Note that for any $x \in \mathbb{R}$ we have that $0 \le \{x\} < 1$.

Consider the $N + 1$ numbers

$$\{0x\}, \{x\}, \{2x\}, \ldots, \{Nx\} \tag{1.3}$$

in the unit interval $[0, 1)$. Divide $[0, 1)$ into N equal semi-open subintervals as follows:

$$[0, 1) = \bigcup_{u=0}^{N-1} I_u \quad \text{where} \quad I_u := \left[\frac{u}{N}, \frac{u+1}{N} \right), \quad u = 0, 1, \ldots, N-1. \tag{1.4}$$

Since the $N + 1$ points (1.3) are situated in the N subintervals (1.4), the Pigeonhole principle guarantees that some subinterval contains at least two points, say $\{q_2 x\}, \{q_1 x\} \in I_u$, where $0 \le u \le N - 1$ and $q_1, q_2 \in \mathbb{Z}$ with $0 \le q_1 < q_2 \le N$. Since the length of I_u is N^{-1} and I_u is semi-open we have that

$$|\{q_2 x\} - \{q_1 x\}| < \frac{1}{N}. \tag{1.5}$$

We have that $q_i x = p_i + \{q_i x\}$ where $p_i = [q_i x] \in \mathbb{Z}$ for $i = 1, 2$. Returning to (1.5) we get

$$|\{q_2x\} - \{q_1x\}| = |q_2x - p_2 - (q_1x - p_1)| = |(q_2 - q_1)x - (p_2 - p_1)|. \quad (1.6)$$

Now define $q = q_2 - q_1 \in \mathbb{Z}$ and $p = p_2 - p_1 \in \mathbb{Z}$. Since $0 \le q_1, q_2 \le N$ and $q_1 < q_2$ we have that $1 \le q \le N$. By (1.5) and (1.6), we get

$$|qx - p| < \frac{1}{N}$$

whence (1.2) readily follows. □

The following statement is an important consequence of Dirichlet's theorem.

Theorem 1.1.2 (Dirichlet, 1842) *Let $x \in \mathbb{R}\setminus\mathbb{Q}$. Then there exist infinitely many integers q, p such that* $\gcd(p, q) = 1$, $q > 0$ *and*

$$\left|x - \frac{p}{q}\right| < \frac{1}{q^2}. \quad (1.7)$$

Remark 1.1.3 Theorem 1.1.2 is true for all $x \in \mathbb{R}$ if we remove the condition that p and q are coprime; that is, if we allow approximations by non-reduced rational fractions.

Proof Observe that Theorem 1.1.1 is valid with $\gcd(p, q) = 1$. Otherwise $p/q = p'/q'$ with $\gcd(p', q') = 1$ and $0 < q' < q \le N$ and $|x - p/q| = |x - p'/q'| < 1/(qN) < 1/(q'N)$.

Suppose x is irrational and that there are only finitely many rationals

$$\frac{p_1}{q_1}, \frac{p_2}{q_2}, \dots, \frac{p_n}{q_n},$$

where $\gcd(p_i, q_i) = 1$, $q_i > 0$ and

$$\left|x - \frac{p_i}{q_i}\right| < \frac{1}{q_i^2}$$

for all $i = 1, 2, \dots, n$. Since x is irrational, $x - \frac{p_i}{q_i} \neq 0$ for $i = 1, \dots, n$. Then there exists $N \in \mathbb{N}$ such that

$$\left|x - \frac{p_i}{q_i}\right| > \frac{1}{N} \qquad \text{for all } 1 \le i \le n.$$

By Theorem 1.1.1, there exists a reduced fraction $\frac{p}{q}$ such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{qN} \le \frac{1}{N} \qquad (1 \le q \le N).$$

Therefore, $\frac{p}{q} \neq \frac{p_i}{q_i}$ for any i but satisfies (1.7). A contradiction. □

Theorem 1.1.2 tells us in particular that the list (1.1) of good rational approximations to π is not just a fluke. This list can be extended to an infinite sequence, and furthermore, such a sequence of good approximations exists for *every* irrational number. (See §1.1.2.)

Another important consequence of Theorem 1.1.1 is Theorem 1.1.4. Unlike Theorem 1.1.2, its significance is not so immediately clear. However, it will become apparent during the course of these notes that it is the key to the two fundamental theorems of classical metric Diophantine approximation: namely, the theorems of Khintchine and Jarník.

First, some notational matters. Unless stated otherwise, given a set $X \subset \mathbb{R}$, we will denote by $m(X)$ the one-dimensional Lebesgue measure of X. And we will use $B(x, r)$ to denote $(x - r, x + r) \subset \mathbb{R}$, the ball around $x \in \mathbb{R}$ of radius $r > 0$.

Theorem 1.1.4 *Let $[a, b] \subset \mathbb{R}$ be an interval and $k \geq 6$ be an integer. Then*

$$m\left([a,b] \cap \bigcup_{k^{n-1}<q\leq k^n} \bigcup_{p\in\mathbb{Z}} B\left(\tfrac{p}{q}, \tfrac{k}{k^{2n}}\right)\right) \geq \tfrac{1}{2}(b-a)$$

for all sufficiently large $n \in \mathbb{N}$.

Proof By Dirichlet's theorem, for any $x \in I := [a, b]$ there are coprime integers p, q with $1 \leq q \leq k^n$ satisfying $|x - p/q| < (qk^n)^{-1}$. We therefore have that

$$\begin{aligned} m(I) &= m\left(I \cap \bigcup_{q\leq k^n} \bigcup_{p\in\mathbb{Z}} B\Big(\frac{p}{q}, \frac{1}{qk^n}\Big)\right) \\ &\leq m\left(I \cap \bigcup_{q\leq k^{n-1}} \bigcup_{p\in\mathbb{Z}} B\Big(\frac{p}{q}, \frac{1}{qk^n}\Big)\right) + m\left(I \cap \bigcup_{k^{n-1}<q\leq k^n} \bigcup_{p\in\mathbb{Z}} B\Big(\frac{p}{q}, \frac{k}{k^{2n}}\Big)\right). \end{aligned}$$

Also, notice that

$$\begin{aligned} m\left(I \cap \bigcup_{q\leq k^{n-1}} \bigcup_{p\in\mathbb{Z}} B\Big(\frac{p}{q}, \frac{1}{qk^n}\Big)\right) &= m\left(I \cap \bigcup_{q\leq k^{n-1}} \bigcup_{p=aq-1}^{bq+1} B\Big(\frac{p}{q}, \frac{1}{qk^n}\Big)\right) \\ &\leq 2 \sum_{q\leq k^{n-1}} \frac{1}{qk^n}\Big(m(I)q + 3\Big) \leq \frac{3}{k} m(I) \end{aligned}$$

for large n. It follows that for $k \geq 6$,

$$m\left(I \cap \bigcup_{k^{n-1}<q\leq k^n} \bigcup_{p\in\mathbb{Z}} B\left(\tfrac{p}{q}, \tfrac{k}{k^{2n}}\right)\right) \geq m(I) - \tfrac{3}{k}\, m(I) \geq \tfrac{1}{2}\, m(I)$$

for large n. □

1.1.2 Basics of Continued Fractions

From Dirichlet's theorem we know that for any real number x there are infinitely many 'good' rational approximates p/q; but how can we find them? The theory of continued fractions provides a simple mechanism for generating them. We collect some basic facts about continued fractions in this section. For proofs and a more comprehensive account, see, for example, [57, 66, 80].

Let x be an irrational number and let $[a_0; a_1, a_2, a_3, \ldots]$ denote its continued fraction expansion. Denote its nth convergent by

$$\frac{p_n}{q_n} := [a_0; a_1, a_2, a_3, \ldots, a_n].$$

Recall that the convergents can be obtained by the following recursion

$$\begin{aligned} p_0 &= a_0, & q_0 &= 1, \\ p_1 &= a_1 a_0 + 1, & q_1 &= a_1, \\ p_k &= a_k p_{k-1} + p_{k-2}, & q_k &= a_k q_{k-1} + q_{k-2} & \text{for } k \geq 2, \end{aligned}$$

and that they satisfy the inequalities

$$\frac{1}{q_n(q_{n+1}+q_n)} \leq \left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}. \tag{1.8}$$

From this it is clear that the convergents provide explicit solutions to the inequality in Theorem 1.1.2 (Dirichlet); that is,

$$\left| x - \frac{p_n}{q_n} \right| \leq \frac{1}{q_n^2} \qquad \forall n \in \mathbb{N}.$$

In fact, it turns out that for irrational x the convergents are *best approximates* in the sense that if $1 \leq q < q_n$, then any rational $\frac{p}{q}$ satisfies

$$\left| x - \frac{p_n}{q_n} \right| < \left| x - \frac{p}{q} \right|.$$

Regarding $\pi = 3.141\ldots$, the rationals (1.1) are the first five convergents.

1.1.3 Competing with Dirichlet and Losing Badly

We have presented Dirichlet's theorem as an answer to whether the trivial inequality $|x - p/q| \leq 1/2q$ can be beaten. Naturally, one may also ask if we can do any better than Dirichlet's theorem. Let us formulate this a little more precisely. For $x \in \mathbb{R}$, let

$$\|x\| := \min\{|x - m| : m \in \mathbb{Z}\}$$

denote the distance from x to the nearest integer. Dirichlet's theorem (Theorem 1.1.2) can be restated as follows: *for any $x \in \mathbb{R}$, there exist infinitely many integers $q > 0$ such that*

$$q\,\|qx\| \leq 1. \tag{1.9}$$

Can we replace right-hand side of (1.9) by arbitrary $\epsilon > 0$? In other words, is it true that $\liminf_{q\to\infty} q\|qx\| = 0$ for every x? One might notice that (1.8) implies that there certainly do exist x for which this is true. (One can write down a continued fraction whose partial quotients grow as fast as one pleases.) Still, the answer to the question is 'no'. It was proved by Hurwitz (1891) that, for every $x \in \mathbb{R}$, we have $q\,\|qx\| < \epsilon = 1/\sqrt{5}$ for infinitely many $q > 0$, and that this is the best possible answer in the sense that the statement becomes false if $\epsilon < 1/\sqrt{5}$.

The fact that $1/\sqrt{5}$ is the best possible answer is relatively easy to see. Assume that it can be replaced by

$$\frac{1}{\sqrt{5}+\epsilon} \qquad (\epsilon > 0,\ \text{arbitrary}).$$

Consider the Golden Ratio $x_1 = \frac{\sqrt{5}+1}{2}$, root of the polynomial

$$f(t) = t^2 - t - 1 = (t - x_1)(t - x_2),$$

where $x_2 = \frac{1-\sqrt{5}}{2}$. Assume there exists a sequence of rationals $\frac{p_i}{q_i}$ satisfying

$$\left|x_1 - \frac{p_i}{q_i}\right| < \frac{1}{(\sqrt{5}+\epsilon)q_i^2}.$$

Then, for sufficiently large values of i, the right-hand side of the above inequality is less than ϵ and so

$$\left|x_2 - \frac{p_i}{q_i}\right| \leq |x_2 - x_1| + \left|x_1 - \frac{p_i}{q_i}\right| < \sqrt{5} + \epsilon.$$

It follows that

$$0 \neq \left|f\left(\frac{p_i}{q_i}\right)\right| < \frac{1}{(\sqrt{5}+\epsilon)q_i^2}\cdot(\sqrt{5}+\epsilon)$$

$$\implies \left|q_i^2 f\left(\frac{p_i}{q_i}\right)\right| < 1.$$

However, the left-hand side is a strictly positive integer. This is a contradiction, for there are no integers in $(0, 1)$ – an extremely useful fact.

The above argument shows that if $x = \frac{\sqrt{5}+1}{2}$ then there are at most finitely many rationals p/q such that

$$\left|x - \frac{p}{q}\right| < \frac{1}{(\sqrt{5}+\epsilon)q^2}.$$

Therefore, there exists a constant $c(x) > 0$ such that

$$\left|x - \frac{p}{q}\right| > \frac{c(x)}{q^2} \qquad \forall\ p/q \in \mathbb{Q}.$$

All of this shows that there exist numbers for which we cannot improve Dirichlet's theorem arbitrarily. These are called *badly approximable numbers* and are defined by

$$\begin{aligned}\mathbf{Bad} &:= \{x \in \mathbb{R} : \inf_{q\in\mathbb{N}} q\|qx\| > 0\}\\ &= \{x \in \mathbb{R} : c(x) := \liminf_{q\to\infty} q\|qx\| > 0\}.\end{aligned}$$

Note that if x is badly approximable then for the associated badly approximable constant $c(x)$ we have that

$$0 < c(x) \le \frac{1}{\sqrt{5}}.$$

Clearly, **Bad** $\neq \varnothing$ since the Golden Ratio is badly approximable. Indeed, if $x \in$ **Bad** then $tx \in$ **Bad** for any $t \in \mathbb{Z}\setminus\{0\}$ and so **Bad** is at least countable.

Bad has a beautiful characterisation via continued fractions.

Theorem 1.1.5 *Let $x = [a_0; a_1, a_2, a_3, \ldots]$ be irrational. Then*

$$x \in \mathbf{Bad} \iff \exists\, M = M(x) \ge 1 \textit{ such that } a_i \le M \ \ \forall i.$$

That is, **Bad** *consists exactly of the real numbers whose continued fractions have bounded partial quotients.*

Proof It follows from (1.8) that

$$\frac{1}{q_n^2(a_{n+1}+2)} \le \left|x - \frac{p_n}{q_n}\right| < \frac{1}{a_{n+1}q_n^2}, \tag{1.10}$$

and from this it immediately follows that if $x \in$ **Bad**, then

$$a_n \le \max\{|a_o|, 1/c(x)\}.$$

Conversely, suppose the partial quotients of x are bounded, and take any $q \in \mathbb{N}$. Then there is $n \geq 1$ such that $q_{n-1} \leq q < q_n$. On using the fact that convergents are best approximates, it follows that

$$\left|x - \frac{p}{q}\right| \geq \left|x - \frac{p_n}{q_n}\right| \geq \frac{1}{q_n^2(M+2)} = \frac{1}{q^2(M+2)} \frac{q^2}{q_n^2}.$$

It is easily seen that

$$\frac{q}{q_n} \geq \frac{q_{n-1}}{q_n} \geq \frac{1}{M+1},$$

which proves that

$$c(x) \geq \frac{1}{(M+2)(M+1)^2} > 0,$$

hence $x \in$ **Bad**. □

Recall that a continued fraction is said to be *periodic* if it is of the form $x = [a_0; \dots, a_n, \overline{a_{n+1}, \dots, a_{n+m}}]$. Also, recall that an irrational number α is called a *quadratic irrational* if α is a solution to a quadratic equation with integer coefficients:

$$ax^2 + bx + c = 0 \qquad (a, b, c \in \mathbb{Z}, a \neq 0).$$

It is a well-known fact that an irrational number x has periodic continued fraction expansion if and only if x is a quadratic irrational. This and Theorem 1.1.5 imply the following corollary.

Corollary 1.1.6 *Every quadratic irrational is badly approximable.*

The simplest instance of this is the Golden Ratio, a root of $x^2 - x - 1$, whose continued fraction is

$$\frac{\sqrt{5}+1}{2} = [1; 1, 1, 1, \dots] := [\,\overline{1}\,],$$

with partial quotients clearly bounded.

Indeed, much is known about the badly approximable numbers, yet several simple questions remain unanswered. For example:

Folklore Conjecture *The only algebraic irrationals that are in* **Bad** *are the quadratic irrationals.*

Remark 1.1.7 Though this conjecture is widely believed to be true, there is no direct evidence for it. That is, there is no single algebraic irrational of degree greater than two whose membership (or non-membership) in **Bad** has been verified.

A particular goal of these notes is to investigate the 'size' of **Bad**. We will show:

(a) $m(\mathbf{Bad}) = 0$,
(b) $\dim \mathbf{Bad} = 1$,

where dim refers to the Hausdorff dimension (see §1.3.1). In other words, we will see that **Bad** is a small set in that it has measure zero in $\mathbb{R}$, but it is a large set in that it has the same (Hausdorff) dimension as $\mathbb{R}$.

Let us now return to Dirichlet's theorem (Theorem 1.1.2). Every $x \in \mathbb{R}$ can be approximated by rationals p/q with 'rate of approximation' given by q^{-2} – the right-hand side of inequality (1.7) determines the 'rate' or 'error' of approximation by rationals. The above discussion shows that this rate of approximation cannot be improved by an arbitrary constant for every real number – **Bad** is non-empty. On the other hand, we have stated above that **Bad** is a zero-measure set, meaning that the set of points for which we *can* improve Dirichlet's theorem by an arbitrary constant is full. In fact, we will see that if we exclude a set of real numbers of measure zero, then from a measure theoretic point of view the rate of approximation can be improved not just by an arbitrary constant but by a logarithm (see Remark 1.2.8).

1.2 Metric Diophantine Approximation: The Classical Lebesgue Theory

In the previous section, we have been dealing with variations of Dirichlet's theorem in which the right-hand side or rate of approximation is of the form ϵq^{-2}. It is natural to broaden the discussion to general approximating functions. More precisely, for a function $\psi : \mathbb{N} \to \mathbb{R}^+ = [0, \infty)$, a real number x is said to be *ψ-approximable* if there are infinitely many $q \in \mathbb{N}$ such that

$$\|qx\| < \psi(q). \tag{1.11}$$

The function ψ governs the 'rate' at which the rationals approximate the reals and will be referred to as an *approximating function*.

One can readily verify that the set of ψ-approximable numbers is invariant under translations by integer vectors. Therefore, without any loss of generality, and to ease the 'metrical' discussion which follows, we shall restrict our attention to ψ-approximable numbers in the unit interval $\mathrm{I} := [0, 1)$. The set of such numbers is clearly a subset of I and will be denoted by $W(\psi)$; i.e.

$$W(\psi) := \{x \in \mathrm{I}\colon \|qx\| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}.$$

Notice that in this notation we have that

$$\text{Dirichlet's theorem (Theorem 1.1.2)} \quad \Longrightarrow \quad W(\psi) = \mathrm{I} \text{ if } \ \psi(q) = q^{-1}.$$

Yet, the existence of badly approximable numbers implies that there exist approximating functions ψ for which $W(\psi) \neq I$. Furthermore, the fact that $m(\mathbf{Bad}) = 0$ implies that we can have $W(\psi) \neq I$ while $m(W(\psi)) = 1$.

A key aspect of the classical theory of Diophantine approximation is to determine the 'size' of $W(\psi)$ in terms of:

(a) Lebesgue measure;
(b) Hausdorff dimension; and
(c) Hausdorff measure.

From a measure theoretic point of view, as we move from (a) to (c) in the above list, the notion of size becomes subtler. In this section we investigate the 'size' of $W(\psi)$ in terms of one-dimensional Lebesgue measure m.

We start with the important observation that $W(\psi)$ is a lim sup set of balls. For a fixed $q \in \mathbb{N}$, let

$$\begin{aligned} A_q(\psi) &:= \{x \in \mathrm{I} : \|qx\| < \psi(q)\} \\ &:= \bigcup_{p=0}^{q} B\Big(\frac{p}{q}, \frac{\psi(q)}{q}\Big) \cap \mathrm{I}. \end{aligned} \tag{1.12}$$

Note that

$$m\big(A_q(\psi)\big) \leqslant 2\psi(q) \tag{1.13}$$

with equality when $\psi(q) < 1/2$ since then the intervals in (1.12) are disjoint.

The set $W(\psi)$ is simply the set of real numbers in I which lie in infinitely many sets $A_q(\psi)$ with $q = 1, 2, \ldots$ i.e.

$$W(\psi) = \limsup_{q\to\infty} A_q(\psi) := \bigcap_{t=1}^{\infty}\bigcup_{q=t}^{\infty} A_q(\psi)$$

is a lim sup set. Now notice that for each $t \in \mathbb{N}$

$$W(\psi) \subset \bigcup_{q=t}^{\infty} A_q(\psi),$$

i.e. for each t, the collection of balls $B(p/q, \psi(q)/q)$ associated with the sets $A_q(\psi) : q = t, t+1, \ldots$ form a cover for $W(\psi)$. Thus, it follows via (1.13) that