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Recent Advances in Hodge Theory
Period Domains, Algebraic Cycles, and Arithmetic

Edited by

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Preface

The summer school and conference on “Recent advances in Hodge theory: Period domains, algebraic cycles, and arithmetic” was held from June 10-20, 2013 at the Pacific Institute of Mathematical Sciences, on the beautiful campus of the University of British Columbia in Vancouver, Canada. It was organized by Matt Kerr (Washington University in St. Louis), James D. Lewis (University of Alberta), and Gregory Pearlstein (Texas A&M University).

The conference received generous financial and administrative assistance from PIMS, and we are grateful to Ruth Situma and the rest of the PIMS staff for their hard work. In addition to PIMS, it was supported by the NSF (Conference Grant DMS-1259024), NSA, and Foundation Compositio Mathematica. We are grateful to all these sponsor organizations, whose support allowed for a large number of junior participants, as well as a significant number of participants from outside North America. This led to a vibrant and memorable event.

This proceedings volume for the 2013 Hodge theory conference is edited by Matt Kerr and Gregory Pearlstein. All papers in this volume have been rigorously peer-reviewed, in several instances by more than one referee. The editors wish to express their appreciation to all the referees for their efforts, especially given the short time-frame. In addition, we want to thank Sam Harrison and the staff at Cambridge University Press for their assistance in preparing the volume.

A couple of the articles contained herein congratulate our co-organizer James Lewis on his 60th birthday, which was celebrated on the last day of the conference. We wanted to add our congratulations and thanks, both for his work as co-organizer and for his friendship, support, and inspiring mathematical presence over the years.

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James D. Lewis

Introduction

In its simplest form, Hodge theory is the study of *periods* – integrals of algebraic differential forms which arise in the study of complex geometry, number theory and physics. Its difficulty and richness arise in part from the non-algebraicity of these integrals. According to the beautiful conjectures of Hodge, Bloch and Beilinson, what algebraic structure they do have should be explained by “algebraic cycles.” There has been much recent progress on these conjectures and on classifying spaces for periods, as well as their asymptotics and arithmetic.

The main goal of the Vancouver conference was to bring together a diverse community of world-leading mathematicians – in Hodge theory, arithmetic geometry, algebraic cycles, complex geometry, and representation theory – around the common theme of *period maps* (considered broadly). With an intensive summer school followed by 32 research talks, it attracted over 85 participants from the US, Canada, Mexico, Europe and Japan, and sparked several new collaborations. With this book, we hope to draw an even larger audience into this area, and to cement the impact of the conference. In particular, this volume includes careful write-ups of expository talks by Wushi Goldring, Radu Laza and Richard Hain, as well as papers presenting key recent developments in each of several focus areas. We hope that it is useful for graduate students and seasoned researchers alike.

Overview of this volume

The birth of modern Hodge theory began with the work of P. Griffiths, who devised an extension of Lefschetz’s original proof of the Hodge conjecture using normal functions and variations of Hodge structure. Despite the fact that this program did not lead to the desired fruition in higher codimension, it has had a lasting impact on the subject. Alongside the study of *algebraic cycles*, the last half-century has seen the development of rich theories of *Hodge theory at*

the boundary and symmetries of Hodge structures. In particular, by the recent work of Griffiths and others, the Hodge Conjecture itself can now be stated in terms of the asymptotic behavior of normal functions; while Mumford-Tate (symmetry) groups of Hodge structures have led to proofs of the Hodge and Beilinson-Hodge conjectures in special cases.

Algebraic cycles (and normal functions), Mumford-Tate groups (and representation theory), and the Hodge-theoretic boundary (degenerations, compactifications) were major themes of the conference, and remain so for this volume, with sections organized around asymptotics and cycles, and Mumford-Tate groups appearing throughout in papers by Green-Griffiths, Laza-Zheng, Kaplan, Arapura, and Abdulali. Another closely related theme of the volume is the *arithmetic of periods*, which is older than Hodge theory proper – dating back to Euler’s work on relations between multiple zeta values, and later work on modular forms and L-functions. Again, Hodge theory at the boundary plays a critical role, as can be seen in Hain’s contribution.

We now turn to an outline of the volume’s contents.

(I) Hodge theory at the boundary ***Period domains and their compactifications***

The volume begins with a detailed version of R. Laza’s expository lectures on *Classical period domains* (i.e., the unconstrained Hermitian symmetric case of Mumford-Tate domains), written with Z. Zhang. This article reviews the classification of Hermitian symmetric domains and the associated locally symmetric (i.e. connected Shimura) varieties, and treats the representation-theoretic description of variations of Hodge structure over the latter, including the classification of VHS of abelian variety type and Calabi-Yau type. It concludes with a discussion of the Hodge-theoretic meaning of the boundary components in the Baily-Borel and toroidal compactifications.

This is followed by two articles studying the behavior of invariant metrics on boundary components of M-T domains, in the classical (i.e. mixed Shimura variety) resp. period domain cases. The first article, by Burgos, Kramer, and Kühn, considers the behavior at infinity of the (metrized) Jacobi line bundle on the universal elliptic curve $E^0(N)$. The authors show that its “naive” extension as a \mathbb{Q} -line bundle encounters singularities at the double points of Néron N -gons, which are directly connected to the “height jumping” phenomenon studied by Brosnan, Hain, and Pearlstein. To obtain an extension functorial with respect to pullback, they instead consider a limit over all possible toroidal compactifications, and show that Chern-Weil theory is compatible with intersections on the resulting object.

The second article is by A. Kaplan. By the work of Griffiths, Schmid, and others, the classifying spaces of pure Hodge structures are homogeneous

spaces which admit an invariant hermitian metric with negative holomorphic sectional curvature along horizontal directions. In contrast, due to the presence of a unipotent radical, there is no such statement in general for classifying spaces of graded-polarized mixed Hodge structures. Nonetheless, Kaplan shows that at each point F of such a classifying space \mathcal{M} , there exists a pair of subgroups $G_{\mathbb{R}}$ and e^{Λ} such that (i) $\mathcal{M} = G_{\mathbb{R}}e^{\Lambda}F$ and (ii) there is a natural Hermitian structure on \mathcal{M} such that for any $g \in G_{\mathbb{R}} \cup e^{\Lambda}$, left translation by g is an isometry from F to gF .

Period maps and algebraic geometry

Classically, the differential of the period mapping sends the tangent space of the Kuranishi space of a smooth projective variety to the tangent space of the period domain. This section begins with an article by Green and Griffiths on *deformation theory and limiting mixed Hodge structures* (LMHS), which took shape out of the authors' desire to extend this differential "to the boundary," by considering first-order deformations of a singular variety. Making use of the work of Friedman, Fujisawa, and Steenbrink, they treat both the map from semistable smoothing deformations (in several variables) to LMHS, and from non-smoothing deformations to the tangent space of the boundary component classifying LMHS.

The relation between mirror symmetry and variational Hodge theory comes into view in the next two papers. In a series of joint works with Kato and Nakayama, S. Usui has defined logarithmic partial compactifications of classifying spaces for mixed Hodge structures. The purpose of his contribution to this volume is to study open mirror symmetry by combining this framework with Iritani's $\hat{\Gamma}$ -integral structure and recent work of Morrison and Walcher. In particular, the truncated normal function arising from the domainwall tension is shown to extend to a log normal function, i.e. a section of the log Néron model.

The second article, by 5 authors (Clingher, Doran, Harder, Novoseltsev, and Thompson), is motivated by a 2006 paper of Doran and Morgan which classified the Calabi-Yau VHS with $h^{2,1} = 1$ over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ (and monodromies of prescribed forms) into fourteen cases, finding CY-threefold realizations for all but the "14th case." This article is a study of two families of $K3$ -fibred $h^{2,1} = 3$ CY 3-folds, and the main result is that a 1-parameter singular subfamily of one of them (which is in fact the mirror of a bidegree $(2, 12)$ complete intersection in $\mathbb{W}\mathbb{P}(1, 1, 1, 1, 4, 6)$) realizes the 14th case VHS. The authors also obtain a counterexample to a conjecture of Morrison, by showing that this 1-parameter family has no CY resolution, while its mirror admits a CY smoothing.

(II) Algebraic cycles and normal functions

Higher Chow cycles were introduced by S. Bloch to “geometrize” higher algebraic K -theory, and these days are also (in the smooth case) known as motivic cohomology classes. This section begins with two papers on these “generalized” algebraic cycles. In his article, Asakura constructs real-regulator indecomposable K_1 -cycles (more precisely, $CH^2(X, 1)$) on families of elliptic surfaces with “colliding semistable singular fibers.” This gives a simple, general approach to constructions in the literature that have required difficult computations. The article of de Jeu, Lewis and Patel reports new evidence for the “relative Beilinson-Hodge conjecture” formulated in work of Lewis with S.-J. Kang, especially in the case of $CH^r(X, 1)$. Unlike the original B-H conjecture, the relative version is expected to be valid even when the varieties are not defined over a number field.

At the heart of current thought on the Hodge conjecture in the last decade, two intertwined programs have emerged. The approach referred to above, proposed by Griffiths and Green (and influenced by work of Clemens and Thomas) reduces the conjecture to the existence of singularities for certain several-variable admissible normal functions obtained from Hodge classes. While this criterion pertains *a priori* to degenerations of normal functions, a result of Schnell reveals the importance of estimates on the *dimension* of its zero loci, which have recently been proven to be algebraic by three different groups of researchers. Another approach, championed by Voisin (building on work of Cattani, Deligne and Kaplan), is to break the Hodge conjecture into two pieces: first, to show that the locus of Hodge classes in a variation is defined over a number field; and second, to prove the Hodge conjecture on arithmetic varieties. Key to this approach is showing that a given family of Hodge classes is *absolute*, extending Deligne’s theorem for abelian varieties.

The papers of M. Saito and Saito-Schnell in this volume contain an important advance in this direction. Saito’s paper considers an admissible normal function ν of an admissible variation of MHS of strictly negative weights, and an irreducible component Z of its zero locus (known to be algebraic by work of Brosnan-Pearlstein, Schnell, and Kato-Nakayama-Usui). The results of his paper imply that if Z contains a k -rational point ($k \subset \mathbb{C}$ of finite transcendence degree), then Z is defined over k ; here ν need *not* come from an algebraic cycle. The joint paper with Schnell proves the comparable result for the Hodge loci of a VHS of weight zero, which is closely related to the theory of absolute Hodge classes when $k = \bar{\mathbb{Q}}$.

The remaining papers in this section concern two of the basic conjectures on (usual) algebraic cycles: the generalized Hodge conjecture (GHC) and Bloch conjecture. An observation of Grothendieck is that, for a variety X , one can deduce the GHC from the Hodge conjecture for certain products $X \times Y_i$. (The

idea is that all Hodge substructures of $H^*(X)$ must be Tate twists of Hodge substructures with $V^{n,0} \neq 0$ that are contained in H^* of some Y_i .) Abdulali's article collects together several known cases of the GHC for abelian varieties, many due to him, which can be proved in this way (and variants thereof). The paper that follows it, by Pedrini and Weibel, provides many examples of surfaces of general type with $p_g(S) = 0$ for which Bloch's conjecture holds, for all values of K_S^2 less than 9. The authors' method, which proceeds by using involutions to show the vanishing of the transcendental motive $t_2(S)$, leads to a more uniform treatment of known examples and produces several new ones.

(III) Arithmetic of periods

Motives, Galois representations, and automorphic forms

Another writeup of an expository minicourse from the summer school begins this section: an introduction to the Langlands program by W. Goldring, focusing on the history and current status of results associating Galois representations to automorphic representations. This story begins with modular forms on the upper half plane, specifically Hecke eigenforms of weight at least 2, whose realization in the étale cohomology of the modular curve $X_0(N)$ produces a 2-dimensional representation of $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. The article then proceeds to a thorough explanation of Langlands's parametrization of the (infinite-dimensional) archimedean components of automorphic representations via L -packets, with particular attention to discrete series representations and limits thereof, and concludes with clear statements of the conjectures and recent results on the Langlands correspondence itself.

Turning to the contribution of S. Patrikis, a classical construction in complex algebraic geometry (due to Kuga and Satake) associates to any $K3$ surface an abelian variety, with a precise relationship between their Hodge structures. A major innovation of Patrikis is his recasting of the Kuga-Satake construction in terms of the existence of a geometric lift of a motivic Galois representation, and his proposal that this is merely the first instance of a general phenomenon. In this paper he demonstrates, using Katz's theory of middle convolutions, the existence of "generalized Kuga-Satake lifts" of motives whose ℓ -adic realizations are isomorphic to some fiber of an SO_5 or SO_6 -cohomologically rigid local system. This gives the first non-trivial examples of his proposed lift beyond the Tannakian subcategory of motives generated by abelian varieties and Artin motives.

The subsection concludes with an article of Yoshida on the *fundamental periods of a motive M* over \mathbb{Q} . Recall that the Betti-de Rham comparison isomorphism produces a period matrix, whose determinant was refined by Deligne to (conjecturally) describe the critical values of the L-function $L(M, s)$. The fundamental periods are these Deligne periods together with further period

invariants, previously introduced by the author to study the change of Deligne periods under linear-algebra operations. In this paper, he gives applications to Siegel modular forms, Harris's quadratic periods, a formula of Blasius (for periods of a tensor product of motives), and also generalizes fundamental periods to motives over number fields.

Modular forms and iterated integrals

Arapura's note points out a simple construction of a *motive* with H^k having any prescribed list of Hodge numbers. (This is distinct from Schreider's results on projective varieties with prescribed Hodge numbers, and Abdulali's theorem that every CM Hodge structure is motivic.) He also makes the interesting conjecture that all Hodge structures of rank 2 and odd level that arise from a motive over \mathbb{Q} , in fact arise from cusp forms in the natural way (via higher Kuga varieties), and offers some supporting evidence.

The volume concludes with a substantial article by Hain which builds on his expository and research talks from the conference. This is a study of the *completion of a modular group* with respect to its inclusion into $SL_2(\mathbb{Q})$, which is an object which unifies the theory of iterated integrals of modular forms (as in the work of Manin) with that of admissible variations of MHS over the modular curve. One Hodge-theoretic application is the explicit construction of normal functions associated to Hecke eigenforms. Another focus of the article is on relations in the "maximal Eisenstein quotient" of the completion of $SL_2(\mathbb{Z})$. Using a computation of Terasoma, Hain proves that Pollack's quadratic relations in the derivation Lie algebra of a rank two free Lie algebra lift to this quotient, and are thus motivic.

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