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PART I

Hodge theory at the boundary

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I.A Period domains and their compactifications

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Classical Period Domains

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We survey the role played by Hermitian symmetric domains in the study of variations of Hodge Structure. These are extended notes based on the lectures given by the first author in Vancouver at the "Advances in Hodge Theory" school (June 2013).

Introduction

There are two classical situations where the period map plays an essential role for the study of moduli spaces, namely the moduli of principally polarized abelian varieties and the moduli of polarized K3 surfaces. What is common for these two situations is the fact that the period domain is in fact a Hermitian symmetric domain. It is well known that the only cases when a period domain is Hermitian symmetric are weight 1 Hodge structures and weight 2 Hodge structures with $h^{2,0} = 1$.

In general, it is difficult to study moduli spaces via period maps. A major difficulty in this direction comes from the Griffiths' transversality relations. Typically, the image Z of the period map in a period domain $\mathbf{D}$ will be a transcendental analytic subvariety of high condimension. The only cases when Z can be described algebraically are when Z is a Hermitian symmetric subdomain of $\mathbf{D}$ with a totally geodesic embedding (and satisfying the horizontality relation). This is closely related to the geometric aspect of the theory of Shimura varieties of Deligne. It is also the case of unconstrained period subdomains in the sense of [GGK12]. We call this case classical, in contrast to the "non-classical" case when the Griffiths' transversality relations are non-trivial.

The purpose of this survey is to review the role of Hermitian symmetric domains in the study of variations of Hodge structure. Let us give a brief

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overview of the content of the paper. In Section 1, we review the basic definitions and properties of Hermitian symmetric domains (Section 1.1) and their classification (Section 1.2) following [Mil04]. The classification is done by reconstructing Hermitian symmetric domains from the associated (semisimple) Shimura data, which are also convenient for the purpose of constructing variations of Hodge structure over Hermitian symmetric domains (Section 1.3). As a digression, we also include the discussion that if the universal family of Hodge structures over a period subdomain satisfies Griffiths transversality then the subdomain must be Hermitian symmetric (i.e. unconstrained $\Rightarrow$ Hermitian symmetric). Section 2 concerns locally symmetric varieties which are quotients of Hermitian symmetric domains. We first review the basic theory of locally symmetric domains and provide some examples of algebraic varieties whose moduli spaces are birational to locally symmetric domains (Section 2.1), and then give a representation theoretic description of variations of Hodge structure on locally symmetric domains (Section 2.2) following [Mil13]. Using the description, we discuss the classification of variations of Hodge structure of abelian variety type and Calabi-Yau type following [Del79] and [FL13] respectively. Baily-Borel and toroidal compactifications of locally symmetric varieties and their Hodge theoretic meanings are reviewed in Section 3.

1 Hermitian Symmetric Domains

In this section, we review the basic concepts and properties related to Hermitian Symmetric domains with an eye towards the theory of Shimura varieties and Hodge theory. The standard (differential geometric) reference for the material in this section is Helgason [Hel78] (see also the recent survey [Viv13]). For the Hodge theoretic point of view, we refer to the original paper of Deligne [Del79] and the surveys of Milne [Mil04] [Mil13].

1.1. Hermitian symmetric spaces and their automorphisms

1.1.1. Hermitian symmetric spaces

We start by recalling the definition of Hermitian symmetric spaces.

Definition 1.1. A *Hermitian manifold* is a pair (M,g) consisting of a complex manifold M together with a Hermitian metric g on M. A Hermitian manifold (M,g) is *symmetric* if additionally

(1) (M,g) is homogeneous, i.e. the holomorphic isometry group $\mathrm{Is}(M,g)$ acts transitively on M;

(2) for any point $p \in M$, there exists an involution s_p (i.e. s_p is a holomorphic isometry and $s_p^2 = \mathrm{Id}$) such that p is an isolated fixed point of s_p (such an involution s_p is called a *symmetry* at p).

A connected symmetric Hermitian manifold is called a *Hermitian symmetric space*. (If there is no ambiguity, we will use M to denote the Hermitian manifold (M, g).)

Note that if (M, g) is homogeneous, it suffices to check Condition (2) at a point (i.e. it suffices to construct a symmetry s_p at some point $p \in M$). Also, the automorphism group $\mathrm{Is}(M, g)$ consists of holomorphic isometries of M:

$$\mathrm{Is}(M, g) = \mathrm{Is}(M^\infty, g) \cap \mathrm{Hol}(M),$$

where M^∞ denotes the underlying C^∞ manifold, $\mathrm{Is}(M^\infty, g)$ is the group of isometries of (M^∞, g) as a Riemannian manifold, and $\mathrm{Hol}(M)$ is the group of automorphisms of M as a complex manifold (i.e. the group of holomorphic automorphisms).

Example 1.2. There are three basic examples of Hermitian symmetric spaces:

(a) the upper half plane $\mathfrak{H}$;
(b) the projective line $\mathbb{P}^1$ (or the Riemann sphere endowed with the restriction of the standard metric on $\mathbb{R}^3$);
(c) any quotient $\mathbb{C}/\Lambda$ of $\mathbb{C}$ by a discrete additive subgroup $\Lambda \subset \mathbb{C}$ (with the natural complex structure and Hermitian metric inherited from $\mathbb{C}$).

To illustrate the definition, we discuss the example of the upper half plane $\mathfrak{H}$. First, it is easy to see that $\mathfrak{H}$, endowed with the metric $\frac{\mathrm{d}z\mathrm{d}\bar{z}}{y^2}$, is a Hermitian manifold. Clearly, $\mathfrak{H}$ is homogeneous with respect to the natural action of $\mathrm{SL}_2(\mathbb{R})$, given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}, \quad \text{for } z \in \mathfrak{H}.$$

In fact, $\mathrm{Is}(\mathfrak{H}) \cong \mathrm{SL}_2(\mathbb{R})/\{\pm I\}$. Finally, the isomorphism $z \mapsto -\frac{1}{z}$ is an involution at the point $i \in \mathfrak{H}$. Since $\mathfrak{H}$ is connected, we conclude that the upper half space $\mathfrak{H}$ is a Hermitian symmetric space.

The three examples above represent the three basic classes of Hermitian symmetric spaces. Specifically, we recall the following:

Definition 1.3. Let M be a Hermitian symmetric space.

(1) M is said to be of *Euclidean type* if it is isomorphic to $\mathbb{C}^n/\Lambda$ for some discrete additive subgroup $\Lambda \subset \mathbb{C}^n$.

(2) M is said to be *irreducible* if it is not of Euclidean type and can not be written as a product of two Hermitian symmetric spaces of lower dimensions.
(3) M is said to be of *compact type* (resp. *noncompact type*) if it is the product of compact (resp. noncompact) irreducible Hermitian symmetric spaces. Moreover, Hermitian symmetric spaces of noncompact type are also called *Hermitian symmetric domains*.

Every Hermitian symmetric space can be decomposed uniquely into a product of Hermitian symmetric spaces of these three types:

Theorem 1.4 (Decomposition Theorem)**.** *Every Hermitian symmetric space M decomposes uniquely as*

$$M = M_0 \times M_- \times M_+,$$

where M_0 is a Euclidean Hermitian symmetric space and M_- (resp. M_+) is a Hermitian symmetric space of compact type (resp. of noncompact type). Moreover, M_- (resp. M_+) is simply connected and decomposes uniquely as a product of compact (resp. noncompact) irreducible Hermitian symmetric spaces.

Proof. See [Hel78, Ch. VIII], especially Proposition 4.4, Theorem 4.6 and Proposition 5.5. □

In this survey, we are mostly interested in Hermitian symmetric domains (or, equivalently, Hermitian symmetric spaces of noncompact type). Note that the terminology is justified by the Harish-Chandra embedding theorem: *every Hermitian symmetric space of noncompact type can be embedded into some $\mathbb{C}^n$ as a bounded domain*. Conversely, every bounded symmetric domain $D \subset \mathbb{C}^n$ has a canonical Hermitian metric (called the Bergman metric) which makes D a Hermitian symmetric domain. For instance, the bounded realization of the upper half plane $\mathfrak{H}$ is the unit ball $\mathcal{B}_1 \subset \mathbb{C}$.

1.1.2. Automorphism groups of Hermitian symmetric domains

Let (D, g) be a Hermitian symmetric domain. Endowed with the compact-open topology, the group $\mathrm{Is}(D^\infty, g)$ of isometries has a natural structure of (real) Lie group. As a closed subgroup of $\mathrm{Is}(D^\infty, g)$, the group $\mathrm{Is}(D, g)$ inherits the structure of a Lie group. Let us denote by $\mathrm{Is}(D, g)^+$ (resp. $\mathrm{Is}(D^\infty, g)^+$, $\mathrm{Hol}(D)^+$) the connected component of $\mathrm{Is}(D, g)$ (resp. $\mathrm{Is}(D^\infty, g)$, $\mathrm{Hol}(D)$) containing the identity.

Proposition 1.5. *Let (D, g) be a Hermitian symmetric domain. The inclusions*

$$\mathrm{Is}(D^\infty, g) \supset \mathrm{Is}(D, g) \subset \mathrm{Hol}(D)$$

induce identities

$$\mathrm{Is}(D^\infty, g)^+ = \mathrm{Is}(D, g)^+ = \mathrm{Hol}(D)^+.$$

Proof. See [Hel78, Lemma 4.3]. □

Since D is homogeneous, one can recover the smooth structure of D as a quotient Lie group of $\mathrm{Is}(D, g)^+$ by the stabilizer of a point. Specifically,

Theorem 1.6. *Notations as above.*

(1) $\mathrm{Is}(D, g)^+$ *is an adjoint (i.e. semisimple with trivial center) Lie group.*
(2) *For any point $p \in D$, the subgroup K_p of* $\mathrm{Is}(D, g)^+$ *fixing p is compact.*
(3) *The map*

$$\mathrm{Is}(D, g)^+/K_p \to D, \quad gK_p \mapsto g \cdot p$$

is an $\mathrm{Is}(D, g)^+$*-equivariant diffeomorphism. In particular,* $\mathrm{Is}(D, g)^+$ *(hence* $\mathrm{Hol}(D)^+$ *and* $\mathrm{Is}(D^\infty, g)^+$*) acts transitively on D.*

Proof. See [Hel78, Ch. IV], especially Theorem 2.5 and Theorem 3.3. □

In particular, every irreducible Hermitian symmetric domain is diffeomorphic to H/K for a unique pair (H, K) (obtained as above) with H a connected noncompact simple adjoint Lie group and K a maximal connected compact Lie group (cf. [Hel78, Ch. VIII, §6]). Conversely, given such a pair (H, K), we obtain a smooth homogenous manifold H/K. The natural question is how to endow H/K with a complex structure and a compatible Hermitian metric so that it is a Hermitian symmetric domain. This can be done in terms of standard Lie theory (see [Viv13, §2.1] and the references therein). However, we shall answer this question from the viewpoint of Shimura data. Specifically, we shall replace the Lie group H by an algebraic group G, replace cosets of K by certain homomorphisms $u : U_1 \to G$ from the circle group U_1 to G, and then answer the question in terms of the pairs (G, u).

To conclude this subsection (and as an initial step to produce a Shimura datum), we discuss how to associate a $\mathbb{R}$-algebraic group G to the real Lie group $\mathrm{Hol}(D)^+$ in such a way that $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$. The superscript $^+$ in $G(\mathbb{R})^+$denotes the neutral connected component relative to the real topology (vs. the Zariski topology). We shall follow [Mil11] for the terminologies on algebraic groups, and also refer the readers to it for the related background materials. For example, we say an algebraic group is *simple* if it is non-commutative and has no proper normal algebraic subgroups, while *almost simple* if it is non-commutative and has no proper normal connected algebraic subgroup (N.B. an almost simple algebraic group can have finite center).

Proposition 1.7. *Let (D,g) be a Hermitian symmetric domain, and let* $\mathfrak{h} = \mathrm{Lie}(\mathrm{Hol}(D)^+)$. *There is a unique connected adjoint real algebraic subgroup G of* $\mathrm{GL}(\mathfrak{h})$ *such that (inside* $\mathrm{GL}(\mathfrak{h})$*)*

$$G(\mathbb{R})^+ = \mathrm{Hol}(D)^+.$$

Moreover, $G(\mathbb{R})^+ = G(\mathbb{R}) \cap \mathrm{Hol}(D)$ *(inside* $\mathrm{GL}(\mathfrak{h})$*); therefore* $G(\mathbb{R})^+$ *is the stabilizer in* $G(\mathbb{R})$ *of D.*

Proof. We sketch the proof of the first statement here, and refer the readers to [Mil04, Prop. 1.7] and references therein for details and the second statement.

Since $\mathrm{Hol}(D)^+$ is adjoint, its adjoint representation on the Lie algebra $\mathfrak{h}$ is faithful, and thus there exists an algebraic group $G \subset \mathrm{GL}(\mathfrak{h})$ such that $\mathrm{Lie}(G) = [\mathfrak{h}, \mathfrak{h}]$ (inside $\mathfrak{gl}(\mathfrak{h})$). Because $\mathrm{Hol}(D)^+$ is semisimple, $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ and so $G(\mathbb{R})^+ = \mathrm{Hol}(D)^+$ (inside $\mathrm{GL}(\mathfrak{h})$). □

1.2. Classification of Hermitian symmetric domains

Consider the circle group $U_1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Motivated by the following fact, one can think of a point of D as a homomorphism $U_1 \to G$.

Theorem 1.8. *Let D be Hermitian symmetric domain. For each $p \in D$, there exists a unique homomorphism* $u_p : U_1 \to \mathrm{Hol}(D)^+$ *such that $u_p(z)$ fixes p and acts on T_pD as multiplication by z.*

Proof. See [Mil04, Thm. 1.9]. □

Remark 1.9. Using the uniqueness of u_p one can easily see that $\mathrm{Hol}(D)^+$ acts on the set of u_p's via conjugation. Clearly, given two different points $p \neq p'$ we choose $f \in \mathrm{Hol}(D)^+$ with $f(p) = p'$, then $f \circ u_p(z) \circ f^{-1}$ $(z \in U_1)$ satisfies the conditions in Theorem 1.8 for p', and thus $u_{p'} = f \circ u_p \circ f^{-1}$.

Example 1.10. Let $p = i \in \mathfrak{H}$. As previously noted, we have $\mathrm{Hol}(\mathfrak{H}) = \mathrm{PSL}_2(\mathbb{R})$. The associated real algebraic group (compare Proposition 1.7) is $(\mathrm{PGL}_2)_{\mathbb{R}}$, and it holds: $\mathrm{PGL}_2(\mathbb{R})^+ = \mathrm{PSL}_2(\mathbb{R})$ (N.B. the group PSL_2 is not an algebraic group). To define $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R})$ we first consider the homomorphism

$$h_i : U_1 \to \mathrm{SL}_2(\mathbb{R}), z = a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

It is easy to verify that $h_i(z)$ fixes i. Since

$$\frac{d}{dw}\left(\frac{aw+b}{-bw+a}\right)\bigg|_{w=i} = \frac{a^2+b^2}{(a-ib)^2} = \frac{z}{\bar{z}} = z^2,$$

$h_i(z)$ acts on the tangent space $T_i\mathfrak{H}$ as multiplication by z^2. Thus, for $z \in U_1$, we choose a square root $\sqrt{z} \in U_1$ and set

$$u_i(z) := h_i(\sqrt{z}).$$

The homomorphism $u_i : U_1 \to \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm I$ is independent of the choice of $\sqrt{z}$ (since $h_i(-1) = -I$). Thus, u_i satisfies the conditions of Theorem 1.8 at the point $i \in \mathfrak{H}$.

Since $G(\mathbb{R})^+ (= \mathrm{Hol}(D)^+)$ acts transitively on D, set-theoretically we can view D as the $G(\mathbb{R})^+$-conjugacy class of $u_p : U_1 \to G(\mathbb{R})$. (Later, we will see that u_p is an algebraic homomorphism). This viewpoint suggests a connection between Hermitian symmetric domains and variations of Hodge structure. Namely, recall that one can view a Hodge structure as a representation of the Deligne torus $\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Then, if we define $h_p : \mathbb{S} \to G$ by $h_p(z) = u_p(z/\bar{z})$, any representation $G \to \mathrm{GL}(V)$ of G (e.g. $\mathrm{Ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$), composed with h_p for all $p \in D$, will produce a variation of Hodge structure on D.

Conversely, given an abstract pair $(G, u : U_1 \to G)$ with G a real adjoint algebraic group and u an algebraic homomorphism it is natural to ask the following questions:

Question 1.11. For a pair (G,u) as above, we let D be the $G(\mathbb{R})^+$-conjugacy class of u. Denote by K_u the subgroup of $G(\mathbb{R})^+$ fixing u. There is a bijection $G(\mathbb{R})^+/K_u \to D$ and so the space D has a natural smooth structure.

(1) Under what conditions can D be given a nice complex structure (or a Hermitian structure)? Under what additional conditions is D a Hermitian symmetric space?
(2) Under what conditions is K_u compact?
(3) Under what conditions is D be a Hermitian symmetric domain (i.e. of noncompact type)?

1.2.1. Representations of U_1

Let T be an algebraic torus defined over a field k, and let K be a Galois extension of k splitting T. The character group $X^*(T)$ is defined by $X^*(T) = \mathrm{Hom}(T_K, \mathbb{G}_m)$. If r is the rank of T, then $X^*(T)$ is a free abelian group of rank r which comes equipped with an action of $\mathrm{Gal}(K/k)$. In general, to give a representation ρ of T on a k-vector space V amounts to giving an $X^*(T)$-grading $V_K = \bigoplus_{\chi \in X^*(T)} V_\chi$ on $V_K := V \otimes_k K$ with the property that

$$\sigma(V_\chi) = V_{\sigma\chi}, \quad \text{all } \sigma \in \mathrm{Gal}(K/k), \quad \chi \in X^*(T).$$

Here V_χ is the K-subspace of V_K on which $T(K)$ acts through χ:

$$V_\chi = \{v \in V_K \mid \rho(t)(v) = \chi(t) \cdot v, \ \forall\, t \in T(K)\}.$$

For instance, we can regard U_1 as a real algebraic torus. As an $\mathbb{R}$-algebraic group, the K-valued points (with K an $\mathbb{R}$-algebra) of U_1 are

$$U_1(K) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_{2\times 2}(K) \;\middle|\; a^2+b^2=1 \right\}.$$

In particular, $U_1(\mathbb{R})$ is the circle group and $U_1(\mathbb{C})$ can be identified with $\mathbb{C}^*$ through

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mapsto a+ib, \;\; \text{conversely } z \mapsto \begin{pmatrix} \frac{1}{2}(z+\frac{1}{z}) & \frac{1}{2i}(z-\frac{1}{z}) \\ -\frac{1}{2i}(z-\frac{1}{z}) & \frac{1}{2}(z+\frac{1}{z}) \end{pmatrix}.$$

Noting that $X^*(U_1) \cong \mathbb{Z}$ and complex conjugation acts on $X^*(U_1)$ as multiplication by -1, we obtain the following proposition.

Proposition 1.12. *Consider a representation ρ of U_1 on a $\mathbb{R}$-vector space V. Then $V_{\mathbb{C}} = \bigoplus_{n\in\mathbb{Z}} V_{\mathbb{C}}^n$ with the property that $\overline{V_{\mathbb{C}}^n} = V_{\mathbb{C}}^{-n}$, where $V_{\mathbb{C}}^n = \{v \in V_{\mathbb{C}} \mid \rho(z)(v) = z^n \cdot v, \quad \forall z \in \mathbb{C}^*\}$. Moreover, if V is irreducible, then it must be isomorphic to one of the following types.*

(a) *$V \cong \mathbb{R}$ with U_1 acting trivially (so $V_{\mathbb{C}} = V_{\mathbb{C}}^0$).*
(b) *$V \cong \mathbb{R}^2$ with $z = x+iy$ acting as $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}^n$ for some $n > 0$ (so $V_{\mathbb{C}} = V_{\mathbb{C}}^n \oplus V_{\mathbb{C}}^{-n}$).*

In particular, every real representation of U_1 is a direct sum of representations of these types.

Remark 1.13. Let V be a $\mathbb{R}$-representation of U_1 and write $V_{\mathbb{C}} = \bigoplus_{n\in\mathbb{Z}} V_{\mathbb{C}}^n$ as above. Because $\overline{V_{\mathbb{C}}^0} = V_{\mathbb{C}}^0$, the weight space $V_{\mathbb{C}}^0$ is defined over $\mathbb{R}$; in other words, it is the complexification of the real subspace V^0 of V defined by $V \cap V_{\mathbb{C}}^0$: $V^0 \otimes_{\mathbb{R}} \mathbb{C} = V_{\mathbb{C}}^0$. The natural homomorphism $V/V^0 \to V_{\mathbb{C}}/\bigoplus_{n\le 0} V_{\mathbb{C}}^n \cong \bigoplus_{n>0} V_{\mathbb{C}}^n$ is a $\mathbb{R}$-linear isomorphism.

The representations of U_1 have the same description no matter if we regard it as a Lie group or an algebraic group, and so every homomorphism $U_1 \to \mathrm{GL}(V)$ of Lie groups is algebraic. In particular, the homomorphism $u_p : U_1 \to \mathrm{Hol}(D)^+ \cong G(\mathbb{R})^+$ is algebraic for any $p \in D$. Let K_p be the subgroup of $G(\mathbb{R})^+$ fixing p. By Theorem 1.8, $u_p(z)$ acts on the $\mathbb{R}$-vector space

$$\mathrm{Lie}(G)/\mathrm{Lie}(K_p) \cong T_pD$$

as multiplication by z, and it acts on $\mathrm{Lie}(K_p)$ trivially. Suppose $T_pD \cong \mathbb{C}^k$ and identify it with $\mathbb{R}^{2k}$ by $(a_1+ib_1,\dots,a_k+ib_k) \mapsto (a_1,b_1,\dots,a_k,b_k)$, then it is easy to write down the matrix of multiplication by $z = x+iy$ and conclude that