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# Inequalities for Graph Eigenvalues

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# Contents

# Preface

This book has been written to be of use to mathematicians working in algebraic (or more precisely, spectral) graph theory. It also contains material that may be of interest to graduate students dealing with the same subject area. It is primarily a theoretical book with an indication of possible applications, and so it can be used by computer scientists, chemists, physicists, biologists, electrical engineers, and other scientists who are using the theory of graph spectra in their work.

The rapid development of the theory of graph spectra has caused the appearance of various inequalities involving spectral invariants of a graph. The main purpose of this book is to expose those results along with their proofs, discussions, comparisons, examples, and exercises. We also indicate some conjectures and open problems that might provide initiatives for further research.

The book is written to be as self-contained as possible, but we assume familiarity with linear algebra, graph theory, and particularly with the basic concepts of the theory of graph spectra. For those who need some additional material, we recommend the books [58, 98, 102, 170].

The graphs considered here are finite, simple (so without loops or multiple edges), and undirected, and the spectra considered in the largest part of the book are those of the adjacency matrix, Laplacian matrix, and signless Laplacian matrix of a graph. Although the results may be exposed in different ways, say from simple to more complicated, or in parts by following their historical appearance, here we follow the concept of *from general to specific*, that is, whenever possible, we give a general result, idea or method, and then its consequences or particular cases. This concept is applied in many places, see for example Theorem 2.2 and its consequences, the whole of Subsection 2.1.2 or Theorem 2.19 and its consequences.

We briefly outline the content of the book. In Chapter 1 we fix the terminology and notation, introduce the matrices associated with a graph, give the necessary results, select possible applications, and give more details about the content. In this respect, the last section of this chapter can be considered as an extension of this Preface. In Chapters 2–4 we consider inequalities that include the largest, the least, and the second largest eigenvalue of the adjacency matrix of a graph, respectively. The last section of Chapter 4 contains the lists of graphs obtained, together with some additional data. The remaining, less investigated, eigenvalues of the adjacency matrix are considered in Chapter 5. Chapters 6 and 7 deal with the inequalities for single eigenvalues of the Laplacian and signless Laplacian matrix. The inequalities that include multiple eigenvalues of any of three spectra considered before are singled out in Chapter 8. In Chapter 9 we consider the normalized Laplacian matrix, the Seidel matrix, and the distance matrix of a graph.

Each of Chapters 2–9 contains theoretical results, comments (including additional explanations, similar results or possible applications), comparisons of inequalities obtained, and numerical or other examples. Each of these chapters ends with exercises and notes. The exercises contain selected problems or a small number of the previous results whose proofs were omitted. The notes contain brief surveys of unmentioned results and directions to the corresponding literature.

Spectral inequalities occupy a central place in this book. Mostly, they are lower or upper bounds for selected eigenvalues. Apart from these, we consider some results written rather in the form of an inequality that bounds some structural invariant in terms of graph eigenvalues (and possibly some other quantities) or, as we have already said, inequalities that include more than one eigenvalue. All inequalities exposed are listed at the end of the book.

In an informal sense, *extremal graph theory* deals with the problem of determining extremal graphs for a given graph invariant in a set of graphs with prescribed properties. In the context of the theory of graph spectra, the invariant in question is a fixed eigenvalue of a matrix associated with a graph or a spectral invariant based on a number of graph eigenvalues (like the graph energy). Extremal graphs for a given spectral invariant in various sets of graphs are widely considered.

The terminology and notation are mainly taken from [98, 102], and they can also be found in similar literature. However, since there is some overlap in the wider notation used, we have made some small adjustments for this book only.

The author is grateful to Dragoš Cvetković and Vladimir Nikiforov, who read the manuscript and gave valuable suggestions. In addition, these colleagues – together with Kinkar Chandra Das, Martin Hasler, and Slobodan K. Simić – gave permission to use some of their proofs with no significant change. Finally, Sarah Lewis helped with correcting language and technical errors, which is much appreciated.