Collected Papers of
SRINIVASA RAMANUJAN

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Edited by

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and
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Cambridge
AT THE UNIVERSITY PRESS
1927

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107536517

First published 1927
First paperback edition 2015

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-53651-7 Paperback

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CORRIGENDA

P. 37, line 14: *for* $\frac{1 \cdot 2}{3^3} k^2$ *read* $\frac{1 \cdot 2}{3^2} k^2$.

P. 45, equation (11): insert dots after the right-hand member.

P. 57, line 11: *for* $\Pi \{1 + x^2 (a+n)^2\}$ *read* $\Pi \{1 + x^2/(a+n)^2\}$.

Cambridge University Press
978-1-107-53651-7 - Collected Paper of Srinivasa Ramanujan
Edited by G. H. Hardy, P. V. Seshu Aiyar and B. M. Wilson
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PREFACE

THIS volume contains everything published by Ramanujan except a few solutions of questions by other mathematicians printed in the *Journal of the Indian Mathematical Society*, and a certain amount of additional matter. Its publication has been made possible by the liberality of the University of Madras, the Royal Society, and Trinity College, Cambridge, each of which bodies has guaranteed a proportion of the expense of printing.

The editorial comments in Appendix I do not profess to be in any way systematic or exhaustive. We have merely put down such comments and references to the literature as occurred to us or were suggested to us by other mathematicians. In particular we are indebted to Prof. L. J. Mordell for a number of valuable suggestions.

We have also printed in Appendix II those parts of Ramanujan's letters from India which have not been printed before. It may seem that it would have been more natural to incorporate these in their proper places in the second *Notice*, but to do this would have expanded it unduly and destroyed its proportion, and the letters consist so largely of an enumeration of isolated theorems that they hardly suffer by division.

There is still a large mass of unpublished material. None of the contents of Ramanujan's notebooks has been printed, unless incorporated in later papers, except that one chapter, on generalised hypergeometric series, was analysed by Hardy* in the *Proceedings of the Cambridge Philosophical Society*. This chapter is sufficient to show that, while the notebooks are naturally unequal in quality, they contain much which should certainly be published. It would be a very formidable task to work through them systematically, select particular passages, and edit these with adequate comment, and it is impossible to print the notebooks as they stand without further monetary assistance. The singular quality of Ramanujan's work, and the romance which surrounds his career, encourage us to hope that this volume may enjoy sufficient success to make possible the publication of another.

* G. H. Hardy, "A chapter from Ramanujan's notebook", *Proc. Camb. Phil. Soc.*, XXI (1923), pp. 492–503.

SRINIVASA RAMANUJAN (1887—1920)

By P. V. Seshu Aiyar and R. Ramachandra Rao

Srinivasa Ramanujan Aiyangar, the remarkable mathematical genius who is the subject of this biographical sketch, was a member of a Brahmin family in somewhat poor circumstances in the Tanjore District of the Madras Presidency. There is nothing specially noteworthy about his ancestry to account for his great gifts. His father and paternal grandfather were *gumastas* (petty accountants) to cloth merchants in Kumbakonam, an important town in the Tanjore District. His mother, a woman of strong common-sense who still survives to mourn the loss of her distinguished son, was the daughter of a Brahmin petty official who held the position of *amin* (bailiff) in the Munsiff's court at Erode in the neighbouring district of Coimbatore. For some time after her marriage she had no children, but her father prayed to the famous goddess Namagiri, in the neighbouring town of Namakkal, to bless his daughter with children. Shortly afterwards, her eldest child, the mathematician Ramanujan, was born on the ninth day of Margasirsha in the Samvath Sarvajit, answering to the English date of 22nd December 1887.

Ramanujan was born in Erode, in the house of his maternal grandfather, to which in accordance with custom his mother had gone for the birth of her first child. In 1892, when in his fifth year, he was, as is usual with Brahmin boys, sent to a *pial* school, i.e. an indigenous elementary school conducted on very simple lines. Two years later he was admitted into the Town High School at Kumbakonam, in which he spent the rest of his school career.

During the first ten years of his life the only indication that he gave of special ability was that in 1897 he stood first amongst the successful candidates of the Tanjore District in the Primary Examination. This success secured for him the concession of being permitted to pay half-fees in his school.

Even in these early days he was remarkably quiet and meditative. It is remembered that he used to ask questions about the distances of the stars. As he held a high place in his class his class fellows used often to go to his house, but as he knew that his parents did not care for him to go out he used only to talk to them from a window which overlooked the street.

While he was in the second form he had, it appears, a great curiosity to know the "highest truth" in Mathematics, and asked some of his friends in the higher classes about it. It seems that some mentioned the Theorem of Pythagoras as the highest truth, and that some others gave the highest place to "Stocks and Shares". While in the third form, when his teacher was explaining to the class that any quantity divided by itself was equal to unity, he is said to have stood up and asked if zero divided by zero also was equal to unity. It was at about this time that he mastered the properties of the three progressions. While in the fourth form, he took to the study of Trigonometry. He is said to have borrowed a copy of the second part of Loney's *Trigonometry* from a student of the B.A. class, who was his neighbour. This student was struck with wonder to learn that this young lad of the fourth form had not only finished reading the book but could do every problem in it without any aid whatever; and not infrequently this B.A. student used to go to Ramanujan for the solution of difficult problems. While in the fifth form, he obtained unaided Euler's Theorems for the sine and the cosine and, when he found out later that the theorems had been already proved, he kept the paper containing the results secreted in the roofing of his house.

It was in 1903, while he was in the sixth form, on a momentous day for Ramanujan, that a friend of his secured for him the loan of a copy of Carr's *Synopsis of Pure Mathematics* from the library of the local Government College. Through the new world thus opened to him, Ramanujan went ranging with delight. It was this book that awakened his genius. He set himself to establish the formulæ given therein. As he was without the aid of other books, each solution was a piece of research so far as he was concerned. He first devised some methods for constructing magic squares. Then, he branched off to Geometry, where he took up the squaring of the circle and succeeded so far as to get a result for the length of the equatorial circumference of the earth which differed from the true length only by a few feet. Finding the scope of geometry limited, he turned his attention to Algebra and obtained several new series. Ramanujan used to say that the goddess of Namakkal inspired him with the formulæ in dreams. It is a remarkable fact that frequently, on rising from bed, he would note down results and rapidly verify them, though he was not always able to supply a rigorous proof. These results were embodied in a notebook which he afterwards used to show to mathematicians interested in his work.

In December 1903 he passed the Matriculation Examination of the University of Madras, and in the January of the succeeding year he joined the Junior First in Arts class of the Government College, Kumbakonam, and won the Subrahmanyam scholarship, which is generally awarded for proficiency in English and Mathematics. By this time, he was so much absorbed in the study of Mathematics that in all lecture hours—whether devoted to English, History or Physiology—he used to engage himself in some mathematical investigation, unmindful of what was happening in the class. This excessive devotion to Mathematics and his consequent neglect of the other subjects resulted in his failure to secure promotion to the senior class and in the consequent discontinuance of the scholarship. Partly owing to disappointment and partly owing to the influence of a friend, he ran away northwards into the Telugu country, but returned to Kumbakonam after some wandering and rejoined the college. As owing to his absence he failed to make sufficient attendances to obtain his term certificate in 1905, he entered Pachaiyappa's College, Madras, in 1906, but falling ill returned to Kumbakonam. He appeared as a private student for the F.A. Examination of December 1907 and failed. Afterwards he had no very definite occupation till 1909, but continued working at Mathematics in his own way and jotting down his results in another notebook.

In the summer of 1909 he married and wanted to settle down in life. Belonging to a poor and humble family, with an unfortunate college career, and without influence, he was hard put to it to secure some means of livelihood. In the hope of finding some employment he went, in 1910, to Tirukoilur, a small sub-division town in the South Arcot District, to see Mr V. Ramaswami Aiyar, M.A., the founder of the Indian Mathematical Society, who was then Deputy Collector of that place, and asked him for a clerical post in a municipal or *taluq* office of his division. This gentleman, being himself a mathematician of no mean order, and finding that the results contained in Ramanujan's notebook were remarkable, thought rightly that this unusual genius would be wasted if consigned to the dull routine of a *taluq* office, and helped Ramanujan on to Madras with a letter of introduction to Mr P. V. Seshu Aiyar, now Principal of Government College, Kumbakonam. Mr Seshu Aiyar had already known Ramanujan while the latter was at Kumbakonam, as he was the mathematical lecturer there while Ramanujan was in the F.A. class. Through him Ramanujan secured for a few months an acting post in the Madras Accountant-General's office and, when this arrangement ceased, he lived for a few months earning what little he could by giving private tuition. Not satisfied with such make-shift arrangements, Mr Seshu Aiyar sent him with a note of recommendation to Diwan Bahadur R. Ramachandra Rao, who was then Collector at Nellore, a small town 80 miles north of Madras, and who had already been introduced to Ramanujan and seen his notebook. His first interview with Ramanujan in December 1910 is better described in his own words:

"Several years ago, a nephew of mine perfectly innocent of mathematical knowledge said to me, 'Uncle, I have a visitor who talks of mathematics; I do not understand him; can you see if there is anything in his talk?' And in the plenitude of my mathematical wisdom, I condescended to permit Ramanujan to walk into my presence. A short uncouth figure, stout, unshaved, not overclean, with one conspicuous feature—shining eyes—walked in with a frayed notebook under his arm. He was miserably poor. He had run away from Kumbakonam to get leisure in Madras to pursue his studies. He never craved for any distinction. He wanted leisure; in other words, that simple food should be provided for him without exertion on his part and that he should be allowed to dream on.

"He opened his book and began to explain some of his discoveries. I saw quite at once that there was something out of the way; but my knowledge did not permit me to judge whether he talked sense or nonsense. Suspending judgment, I asked him to come over again, and he did. And then he had gauged my ignorance and shewed me some of his simpler results. These transcended existing books and I had no doubt that he was a remarkable man. Then, step by step, he led me to elliptic integrals and hypergeometric series and at last his theory of divergent series not yet announced to the world converted me. I asked him what he wanted. He said he wanted a pittance to live on so that he might pursue his researches."

Mr Ramachandra Rao sent him back to Madras, saying that it was cruel to make an intellectual giant like Ramanujan rot in a *mofussil* station like Nellore, and recommended that he should stay at Madras, undertaking to pay his expenses for a time. After a while, other attempts to obtain for him a scholarship having failed and Ramanujan being unwilling to be a burden on anybody for any length of time, he took up a small appointment on Rs 30 *per mensem* in the Madras Port Trust office, on the 9th February 1912.

He did not slacken his work at Mathematics in the meantime. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by Mr Seshu Aiyar and published in the February number of Volume III (1911). His first long article was on "Some Properties of Bernoulli's Numbers" and was published in the December number of the same volume. In 1912 he contributed two more notes to the fourth volume of the same Journal, and also several questions for solution.

By this time, Mr Ramachandra Rao had induced Mr Griffith of the Madras Engineering College to take an interest in Ramanujan, and Mr Griffith spoke to Sir Francis Spring, the Chairman of the Madras Port Trust, in which Ramanujan was then employed; and from that time onwards it became easy to secure recognition of his work. Fortunately also the then manager of the Port Trust office was Mr S. Narayana Aiyar, M.A., a very keen and devoted student of Mathematics. He gave every encouragement to Ramanujan and very frequently worked with him during this period.

On the suggestion of Mr Seshu Aiyar and others, Ramanujan began a correspondence with Mr G. H. Hardy, then Fellow of Trinity College, Cambridge, on the 16th January 1913. In that letter he wrote:

"I had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics....I have made a special investigation of divergent series....Very recently I came across a tract published by you, styled *Orders of Infinity*, in page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value, I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get; but I have indicated the lines on which I proceed. Being inexperienced, I would very highly value any advice you give me...."

The papers enclosed contained the enunciations of a hundred or more mathematical theorems.

In his second letter of date 27th February 1913, he wrote:

"...I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed....To preserve my brains, I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from the Government...."

But in the meantime Mr Hardy had written to the Secretary for Indian Students in London, saying that Ramanujan might prove to be a mathematician of the very highest class, and asking him to enquire whether some means could not be found for getting him a Cambridge education. This question was transmitted to the Secretary of the Students' Advisory Committee in Madras, who, in his turn, asked Ramanujan if he would go to England. But since his caste prejudices were very strong, he definitely declined to go. Upon the receipt of this unfavourable reply, the Secretary wrote, early in March 1913, to the Registrar of the University of Madras, explaining the circumstances of the case.

By this time Ramanujan's case had been brought to the notice of the University of Madras in another way. Early in February, Dr G. T. Walker, F.R.S., Director-General of Observatories, Simla, and formerly Fellow of Trinity College, Cambridge, happened to visit Madras on one of his official tours; and Sir Francis Spring took this opportunity to bring some of Ramanujan's work to Dr Walker's notice. As a result, Dr Walker addressed, on the 26th February 1913, the following letter to the Registrar of the University of Madras:

"...I have the honour to draw your attention to the case of S. Ramanujan, a clerk in the Accounts Department of the Madras Port Trust. I have not seen him, but was yesterday shewn some of his work in the presence of Sir Francis Spring. He is, I am told, 22 years of age and the character of the work that I saw impressed me as comparable in originality with that of a mathematical fellow in a Cambridge college....It was perfectly clear to me that the University would be justified in enabling S. Ramanujan for a few years at least to spend the whole of his time on Mathematics, without any anxiety as to his livelihood...."

As a result of this momentous letter and on the recommendation of the Board of Studies in Mathematics, the University granted to Ramanujan, with the previous approval of Government, a special scholarship of Rs 75 *per mensem* tenable for two years. The Syndicate took a special interest in getting this scholarship sanctioned, as may be seen from the following extract from the letter of the Registrar to the Government in this connection:

"The regulations of the University do not at present provide for such a special scholarship. But the Syndicate assumes that Section XV of the Act of Incorporation and Section 3 of the Indian Universities Act, 1904, allow of the grant of such a scholarship, subject to the express consent of the Governor of Fort St George in Council."

He was accordingly relieved from his clerical post in the Madras Port Trust office on the 1st of May 1913, and from that time he became and remained for the rest of his life a professional mathematician.

In accordance with the conditions of award of the scholarship, he submitted to the Board of Studies in Mathematics three quarterly reports on his researches on the 5th August 1913, 7th November 1913 and 9th March 1914 respectively.

But Mr Hardy was very much disappointed at Ramanujan's refusal to go to Cambridge. He had been at frequent intervals writing persuasive letters pointing out the advantages of a short stay in Cambridge, and when, early in 1914, the University of Madras invited Mr E. H. Neville, M.A., Fellow of Trinity College, Cambridge, to deliver a course of lectures at Madras, Mr Hardy used this opportunity and entrusted to Mr Neville the mission of persuading Ramanujan to give up his caste prejudices and come to Cambridge. In the meantime, many Indian friends also had been influencing him and, by the time Mr Neville approached him, Ramanujan himself had almost made up his mind; but his chief difficulty was to obtain his mother's consent. This consent was at last got very easily in an unexpected manner. For one morning his mother announced that she had had a dream on the previous night, in which she saw her son seated in a big hall amidst a group of Europeans, and that the goddess Namagiri had commanded her not to stand in the way of her son fulfilling his life's purpose. This was a very agreeable surprise to all concerned.

As soon as Ramanujan's consent was obtained, Mr Neville sent a memorandum to the authorities of the University of Madras on 28th January 1914. The memorandum ran as follows:

> "The discovery of the genius of S. Ramanujan of Madras promises to be the most interesting event of our time in the mathematical world....The importance of securing to Ramanujan a training in the refinements of modern methods and a contact with men who know what ranges of ideas have been explored and what have not cannot be overestimated....
>
> "I see no reason to doubt that Ramanujan himself will respond fully to the stimulus which contact with western mathematicians of the highest class will afford him. In that case his name will become one of the greatest in the history of mathematics and the University and the City of Madras will be proud to have assisted in his passage from obscurity to fame."

The next day, Mr R. Littlehailes, M.A., who was then Professor of Mathematics in the Presidency College, Madras, and now is the Director of Public Instruction, Madras, wrote another long letter to the Registrar of the University and made definite proposals regarding the scholarship to be granted.

The authorities of the University readily seized the opportunity and within a week decided, with the approval of the Government of Madras, to grant Ramanujan a scholarship of £250 a year, tenable in England for a period of two years, with free passage and a reasonable sum for outfit. This scholarship was subsequently extended up to 1st April 1919. Having arranged that the University should forward Rs 60 *per mensem* out of his scholarship to his mother at Kumbakonam, Ramanujan sailed for England on the 17th March

1914. He reached Cambridge in April and was admitted into Trinity College, which supplemented his scholarship by the award of an exhibition of £60.

He was now for the first time in his life in a really comfortable position and could devote himself to his researches without anxiety. Mr Hardy and Mr Littlewood helped him in publishing his papers in the English periodicals and under their guidance he developed rapidly.

On the 11th November 1915, Mr Hardy wrote to the Registrar of the Madras University:

> "Ramanujan has been much handicapped by the war. Mr Littlewood, who would naturally have shared his teaching with me, has been away, and one teacher is not enough for so fertile a pupil......He is beyond question the best Indian mathematician of modern times...He will always be rather eccentric in his choice of subjects and methods of dealing with them......But of his extraordinary gifts there can be no question; in some ways he is the most remarkable mathematician I have ever known."

Mr Hardy's official report of date 16th June 1916 to the University of Madras was also in terms of very high praise*. Ramanujan had already published about a dozen papers in European journals. Everything went on well till the spring of 1917.

About May 1917, Mr Hardy wrote that it was suspected that Ramanujan had contracted an incurable disease. Since sea voyages were then risky on account of submarines and since the war had depleted India of good medical men, it was decided that he should stay in England for some time more. Hence he went into a nursing home at Cambridge in the early summer, and he was never out of bed for any length of time again. He was in sanatoria at Wells, at Matlock and in London, and it was not until the autumn of 1918 that he shewed any decided symptom of improvement.

On the 28th February 1918, he was elected a Fellow of the Royal Society. He was the first Indian on whom this high honour was conferred, and his election at the early age of thirty, and on the first occasion that his name was proposed, is a remarkable tribute to his distinguished genius. Stimulated perhaps by this election, he resumed active work, in spite of his ill-health, and some of his most beautiful theorems were discovered about this time. On the 13th October of the same year, he was elected a Fellow of Trinity College, Cambridge—a prize fellowship worth about £250 a year for six years, with no duties or conditions. In announcing this election, Mr Hardy wrote to the Registrar of the University of Madras, "He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is", and urged the authorities of the University to make permanent provision for him in a way which could leave him free for research. This time also the University of Madras rose to the occasion and, in recognition of Ramanujan's services to the science of Mathematics, it granted him an allowance of £250 a year for

* Cf. *Journal of the Indian Mathematical Society*, 9 (1917), pp. 30–45.

five years from 1st April 1919, the date of the expiry of his scholarship, together with the actual expenses incurred by him in returning from England to India and on such passages from India to Europe and back as the Syndicate might approve of during the five years. At the suggestion of Mr Littlehailes the University of Madras also contemplated creating a University Professorship of Mathematics and offering it to him.

By this time his health shewed some signs of improvement. Although he shewed a tubercular tendency, the doctors said that he had never been gravely affected. Since the climate of England was suspected of retarding his recovery, it was decided to send him back to India. Accordingly, he left England on 27th February 1919, landed in Bombay on 27th March and arrived at Madras on the 2nd April.

When he returned he was in a precarious state of health. His friends grew very anxious. The best medical attendance was arranged for. He stayed three months in Madras and then spent two months in Kodumudi, a village on the banks of the Cauvery, not far from the place of his birth. He was a difficult patient, always inclined to revolt against medical treatment, and after a time he declined to be treated further. On the 3rd September he went to Kumbakonam, and since it was reported by many medical friends that he was getting worse, he was with great difficulty induced to come to Madras for treatment in January 1920 and was put under the best available medical care. Several philanthropic gentlemen assisted him during this period, notably Mr S. Srinivasa Aiyangar, who found all his expenses, and Rao Bahadur T. Numberumal Chetty, who gave his house free. The members of the Syndicate of the University of Madras also made a contribution towards his expenses in their individual capacity. But all this was of no avail. He died on the 26th April 1920, at Chetput, a suburb of Madras. He had no children but was survived by his parents and his wife.

We must refer to Mr Hardy's notice for an account of his mathematical work, but we add a few words about his appearance and personality. Before his illness he was inclined to stoutness; he was of moderate height (5 feet 5 inches); and had a big head with a large forehead and long wavy dark hair. His most remarkable feature was his sharp and bright dark eyes. A fairly faithful representation of him adorns the walls of the Madras University Library. On his return from England, he was very thin and emaciated and had grown very pale. He looked as if racked with pain. But his intellect was undimmed, and till about four days before he died he was engaged in work. All his work on "mock theta functions", of which only rough indications survive, was done on his death bed.

Ramanujan had definite religious views. He had a special veneration for the Namakkal goddess. Fond of the *Puranas*, he used to attend popular lectures on the Great Epics of Ramayana and Nahabharata, and to enter into discussions with learned pundits. He believed in the existence of a Supreme

Being and in the attainment of Godhood by men by proper methods of service and realisation of oneness with the Deity. He had settled convictions about the problem of life and after, and even the certain approach of death did not unsettle his faculties or spirits.

In manners he was very simple and he had absolutely no conceit. In a letter of date 26th November 1918, i.e. after Ramanujan had been honoured by being elected a Fellow of the Royal Society and a Fellow of Trinity, Mr Hardy wrote: "His natural simplicity has never been affected in the least by his success; indeed all that is wanted is to get him to realise that he really is a success." He was much distressed, when he had so little money for his own expenses, about his inability to help his poor parents; and when he received his scholarship, his first act was to devote a part of it to them. Ramanujan's simplicity and largeness of heart are further revealed in the following letter that he sent to the Registrar of the University of Madras:

2 Colinette Road, Putney, S.W. 15.
11*th January* 1919.

To The Registrar,
University of Madras.

Sir,

I beg to acknowledge the receipt of your letter of 9th December 1918, and gratefully accept the very generous help which the University offers me.

I feel, however, that, after my return to India, which I expect to happen as soon as arrangements can be made, the total amount of money to which I shall be entitled will be much more than I shall require. I should hope that, after my expenses in England have been paid, £50 a year will be paid to my parents and that the surplus, after my necessary expenses are met, should be used for some educational purpose, such in particular as the reduction of school-fees for poor boys and orphans and provision of books in schools. No doubt it will be possible to make an arrangement about this after my return.

I feel very sorry that, as I have not been well, I have not been able to do so much mathematics during the last two years as before. I hope that I shall soon be able to do more and will certainly do my best to deserve the help that has been given me.

I beg to remain, Sir,
Your most obedient servant,
S. Ramanujan.

SRINIVASA RAMANUJAN (1887—1920)

By G. H. Hardy*

I

Srinivasa Ramanujan, who died at Kumbakonam on April 26th, 1920, had been a member of the Society since 1917. He was not a man who talked much about himself, and until recently I knew very little of his early life. Two notices, by P. V. Seshu Aiyar and R. Ramachandra Rao, two of the most devoted of Ramanujan's Indian friends, have been published recently in the *Journal of the Indian Mathematical Society*; and Sir Francis Spring has very kindly placed at my disposal an article which appeared in the *Madras Times* of April 5th, 1919. From these sources of information I can now supply a good many details with which I was previously unacquainted. Ramanujan (Srinivasa Iyengar Ramanuja Iyengar, to give him for once his proper name) was born on December 22nd, 1887, at Erode in southern India. His father was an accountant (*gumasta*) to a cloth merchant at Kumbakonam, while his maternal grandfather had served as *amin* in the *Munsiff's* (or local judge's) Court at Erode. He first went to school at five, and was transferred before he was seven to the Town High School at Kumbakonam, where he held a "free scholarship", and where his extraordinary powers appear to have been recognised immediately. "He used", so writes an old schoolfellow to Mr Seshu Aiyar, "to borrow Carr's *Synopsis of Pure Mathematics* from the College library, and delight in verifying some of the formulæ given there....He used to entertain his friends with his theorems and formulæ, even in those early days.... He had an extraordinary memory and could easily repeat the complete lists of Sanscrit roots (*atmanepada* and *parasmepada*); he could give the values of $\sqrt{2}$, π, e, ... to any number of decimal places....In manners, he was simplicity itself...."

He passed his matriculation examination to the Government College at Kumbakonam in 1904; and secured the "Junior Subrahmanyam Scholarship". Owing to weakness in English, he failed in his next examination and lost his scholarship; and left Kumbakonam, first for Vizagapatam and then for Madras. Here he presented himself for the "First Examination in Arts' in December 1906, but failed and never tried again. For the next few years he continued his independent work in mathematics, "jotting down his results in two good-sized notebooks": I have one of these notebooks in my possession still. In 1909 he married, and it became necessary for him to find some permanent employment. I quote Mr Seshu Aiyar:

To this end, he went to Tirukoilur, a small sub-division town in South Arcot District, to see Mr V. Ramaswami Aiyar, the founder of the Indian Mathematical Society, but

* Obituary notice in the *Proceedings of the London Mathematical Society* (2), XIX (1921), pp. xl—lviii. The same notice was printed, with slight changes, in the *Proceedings of the Royal Society* (A), XCIX (1921), pp. xiii—xxix.

Mr Aiyar, seeing his wonderful gifts, persuaded him to go to Madras. It was then after some four years' interval that Ramanujan met me at Madras, with his two good-sized notebooks referred to above. I sent Ramanujan with a note of recommendation to that true lover of Mathematics, Diwan Bahadur R. Ramachandra Rao, who was then District Collector at Nellore, a small town some eighty miles north of Madras. Mr Rao sent him back to me, saying it was cruel to make an intellectual giant like Ramanujan rot at a *mofussil* station like Nellore, and recommended his stay at Madras, generously undertaking to pay Ramanujan's expenses for a time. This was in December 1910. After a while, other attempts to obtain for him a scholarship having failed, and Ramanujan himself being unwilling to be a burden on anybody for any length of time, he decided to take up a small appointment under the Madras Port Trust in 1912.

But he never slackened his work at Mathematics. His earliest contribution to the *Journal of the Indian Mathematical Society* was in the form of questions communicated by me in Vol. III (1911). His first long article on "Some Properties of Bernoulli's Numbers" was published in the December number of the same volume. Ramanujan's methods were so terse and novel and his presentation was so lacking in clearness and precision, that the ordinary reader, unaccustomed to such intellectual gymnastics, could hardly follow him. This particular article was returned more than once by the Editor before it took a form suitable for publication. It was during this period that he came to me one day with some theorems on Prime Numbers, and when I referred him to Hardy's Tract on *Orders of Infinity*, he observed that Hardy had said on p. 36 of his Tract "the exact order of $\rho(x)$ [defined by the equation

$$\rho(x) = \pi(x) - \int_2^x \frac{dt}{\log t},$$

where $\pi(x)$ denotes the number of primes less than x], has not yet been determined", and that he himself had discovered a result which gave the order of $\rho(x)$. On this I suggested that he might communicate his result to Mr Hardy, together with some more of his results.

This passage brings me to the beginning of my own acquaintance with Ramanujan. But before I say anything about the letters which I received from him, and which resulted ultimately in his journey to England, I must add a little more about his Indian career. Dr G. T. Walker, F.R.S., Head of the Meteorological Department, and formerly Fellow and Mathematical Lecturer of Trinity College, Cambridge, visited Madras for some official purpose some time in 1912; and Sir Francis Spring, K.C.I.E., the Chairman of the Madras Port Authority, called his attention to Ramanujan's work. Dr Walker was far too good a mathematician not to recognise its quality, little as it had in common with his own. He brought Ramanujan's case to the notice of the Government and the University of Madras. A research studentship, "Rs. 75 *per mensem* for a period of two years", was awarded him; and he became, and remained for the rest of his life, a professional mathematician.

II

Ramanujan wrote to me first on January 16th, 1913, and at fairly regular intervals until he sailed for England in 1914. I do not believe that his letters were entirely his own. His knowledge of English, at that stage of his life, could scarcely have been sufficient, and there is an occasional phrase which is hardly characteristic. Indeed I seem to remember his telling me that his friends had given him some assistance. However, it was the mathematics that mattered, and that was very emphatically his.

MADRAS, 16*th January* 1913.

DEAR SIR,

I beg to introduce myself to you as a clerk in the Accounts Department of the Port Trust Office at Madras on a salary of only £20 per annum. I am now about 23 years of age. I have had no University education but I have undergone the ordinary school course. After leaving school I have been employing the spare time at my disposal to work at Mathematics. I have not trodden through the conventional regular course which is followed in a University course, but I am striking out a new path for myself. I have made a special investigation of divergent series in general and the results I get are termed by the local mathematicians as "startling".

Just as in elementary mathematics you give a meaning to a^n when n is negative and fractional to conform to the law which holds when n is a positive integer, similarly the whole of my investigations proceed on giving a meaning to Eulerian Second Integral for all values of n. My friends who have gone through the regular course of University education tell me that $\int_0^\infty x^{n-1} e^{-x}\,dx = \Gamma(n)$ is true only when n is positive. They say that this integral relation is not true when n is negative. Supposing this is true only for positive values of n and also supposing the definition $n\Gamma(n) = \Gamma(n+1)$ to be universally true, I have given meanings to these integrals and under the conditions I state the integral is true for all values of n negative and fractional. My whole investigations are based upon this and I have been developing this to a remarkable extent so much so that the local mathematicians are not able to understand me in my higher flights.

Very recently I came across a tract published by you styled *Orders of Infinity* in page 36 of which I find a statement that no definite expression has been as yet found for the number of prime numbers less than any given number. I have found an expression which very nearly approximates to the real result, the error being negligible. I would request you to go through the enclosed papers. Being poor, if you are convinced that there is anything of value I would like to have my theorems published. I have not given the actual investigations nor the expressions that I get but I have indicated the lines on which I proceed. Being inexperienced I would very highly value any advice you give me. Requesting to be excused for the trouble I give you.

I remain, Dear Sir, Yours truly,

S. RAMANUJAN.

P.S. My address is S. Ramanujan, Clerk Accounts Department, Port Trust, Madras, India.

I quote now from the "papers enclosed", and from later letters*:

In page 36 it is stated that "the number of prime numbers less than

$$x = \int_2^x \frac{dt}{\log t} + \rho(x)$$

where the precise order of $\rho(x)$ has not been determined...."

I have observed that $\rho(e^{2\pi x})$ is of such a nature that its value is very small when x lies between 0 and 3 (its value is less than a few hundreds when $x=3$) and rapidly increases when x is greater than 3....

The difference between the number of prime numbers of the form $4n-1$ and which are less than x and those of the form $4n+1$ less than x is infinite when x becomes infinite....

* [See Appendix II for parts of the letters not printed here.]

The following are a few examples from my theorems:

(1) The numbers of the form 2^p3^q less than $n = \frac{1}{2}\frac{\log(2n)\log(3n)}{\log 2 \log 3}$ where p and q may have any positive integral value including 0.

(2) Let us take all numbers containing an odd number of dissimilar prime divisors, viz.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 30, 31, 37, 41, 42, 43, 47, ...

(*a*) The number of such numbers less than $n = \frac{3n}{\pi^2}$.

(*b*) $\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots+\frac{1}{30^2}+\frac{1}{31^2}+\ldots=\frac{9}{2\pi^2}$.

(*c*) $\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\ldots=\frac{15}{2\pi^4}$.

(3) Let us take the number of divisors of natural numbers, viz.

1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, ... (1 having 1 divisor, 2 having 2, 3 having 2, 4 having 3, 5 having 2, ...).

The sum of such numbers to n terms

$$= n(2\gamma - 1 + \log n) + \tfrac{1}{2} \text{ of the number of divisors of } n$$

where $\gamma = \cdot 5772156649\ldots$, the Eulerian Constant.

(4) 1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, ... are numbers which are either themselves squares or which can be expressed as the sum of two squares.

The number of such numbers greater than A and less than B

$$= K\int_A^B \frac{dx}{\sqrt{\log x}} + \theta(x)^* \quad \text{where} \quad K = \cdot 764\ldots$$

and $\theta(x)$ is very small when compared with the previous integral. K and $\theta(x)$ have been exactly found though complicated....

Ramanujan's theory of primes was vitiated by his ignorance of the theory of functions of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeros. His methods of proof depended upon a wholesale use of divergent series. He disregarded entirely all the difficulties which are involved in the interchange of double limit operations; he did not distinguish, for example, between the sum of a series Σa_n and the value of the Abelian limit

$$\lim_{x\to 1} \Sigma a_n x^n,$$

or that of any other limit which might be used for similar purposes by a modern analyst. There are regions of mathematics in which the precepts of modern rigour may be disregarded with comparative safety, but the Analytic Theory of Numbers is not one of them, and Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong. That his proofs should have been invalid was only to be expected. But the mistakes went deeper than that, and many of the actual results were false. He had obtained the dominant terms of the classical formulæ, although by invalid methods; but none of them is such a close approximation as he supposed.

* This should presumably be $\theta(B)$.

This may be said to have been Ramanujan's one great failure. And yet I am not sure that, in some ways, his failure was not more wonderful than any of his triumphs. Consider, for example, problem (4). The dominant term, viz. $KB(\log B)^{-\frac{1}{2}}$, in Ramanujan's notation, was first obtained by Landau in 1908. Ramanujan had none of Landau's weapons at his command; he had never seen a French or German book; his knowledge even of English was insufficient to enable him to qualify for a degree. It is sufficiently marvellous that he should have even dreamt of problems such as these, problems which it has taken the finest mathematicians in Europe a hundred years to solve, and of which the solution is incomplete to the present day.

...IV. Theorems on integrals. The following are a few examples:

(1) $$\int_0^\infty \frac{1+\left(\frac{x}{b+1}\right)^2}{1+\left(\frac{x}{a}\right)^2} \cdot \frac{1+\left(\frac{x}{b+2}\right)^2}{1+\left(\frac{x}{a+1}\right)^2} \cdots dx = \frac{\sqrt{\pi}}{2} \frac{\Gamma(a+\frac{1}{2})}{\Gamma(a)} \frac{\Gamma(b+1)}{\Gamma(b+\frac{1}{2})} \frac{\Gamma(b-a+\frac{1}{2})}{\Gamma(b-a+1)}.$$

......

(3) If $$\int_0^\infty \frac{\cos nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n),$$

then $$\int_0^\infty \frac{\sin nx}{e^{2\pi\sqrt{x}}-1}\,dx = \phi(n) - \frac{1}{2n} + \phi\left(\frac{\pi^2}{n}\right)\sqrt{\frac{2\pi^3}{n^3}}.$$

$\phi(n)$ is a complicated function. The following are certain special values:

$$\phi(0) = \frac{1}{12}; \quad \phi\left(\frac{\pi}{2}\right) = \frac{1}{4\pi}; \quad \phi(\pi) = \frac{2-\sqrt{2}}{8}; \quad \phi(2\pi) = \frac{1}{16};$$

$$\phi\left(\frac{2\pi}{5}\right) = \frac{8-8\sqrt{5}}{16}; \quad \phi\left(\frac{\pi}{5}\right) = \frac{6+\sqrt{5}}{4} - \frac{5\sqrt{10}}{8}; \quad \phi(\infty) = 0;$$

$$\phi\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \sqrt{3}\left(\frac{3}{16} - \frac{1}{8\pi}\right).$$

(4) $$\int_0^\infty \frac{dx}{(1+x^2)(1+r^2x^2)(1+r^4x^2)\cdots} = \frac{\pi}{2(1+r+r^3+r^6+r^{10}+\ldots)}$$

where 1, 3, 6, 10... are sums of natural numbers.

(5) $$\int_0^\infty \frac{\sin 2nx}{x(\cosh \pi x + \cos \pi x)}\,dx = \frac{\pi}{4} - 2\left(\frac{e^{-n}\cos n}{\cosh\frac{\pi}{2}} - \frac{e^{-3n}\cos 3n}{3\cosh\frac{3\pi}{2}} \cdots\right)$$

......

V. Theorems on summation of series*; e.g.

(1) $$\frac{1}{1^3}\cdot\frac{1}{2} + \frac{1}{2^3}\cdot\frac{1}{2^2} + \frac{1}{3^3}\cdot\frac{1}{2^3} + \frac{1}{4^3}\cdot\frac{1}{2^4} + \ldots = \frac{1}{6}(\log 2)^3 - \frac{\pi^2}{12}\log 2 + \left(\frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \ldots\right).$$

(2) $$1 + 9\cdot(\tfrac{1}{4})^4 + 17\cdot\left(\frac{1\cdot5}{4\cdot8}\right)^4 + 25\cdot\left(\frac{1\cdot5\cdot9}{4\cdot8\cdot12}\right)^4 + \ldots = \frac{2\sqrt{2}}{\sqrt{\pi}\,\{\Gamma(\frac{3}{4})\}^2}.$$

* There is always more in one of Ramanujan's formulæ than meets the eye, as anyone who sets to work to verify those which look the easiest will soon discover. In some the interest lies very deep, in others comparatively near the surface; but there is not one which is not curious and entertaining.

(3) $1-5\left(\frac{1}{2}\right)^3+9\left(\frac{1\,.\,3}{2\,.\,4}\right)^3-\ldots=\frac{2}{\pi}.$

(4) $\frac{1^{13}}{e^{2\pi}-1}+\frac{2^{13}}{e^{4\pi}-1}+\frac{3^{13}}{e^{6\pi}-1}+\ldots=\frac{1}{24}.$

(5) $\frac{\coth\pi}{1^7}+\frac{\coth 2\pi}{2^7}+\frac{\coth 3\pi}{3^7}+\ldots=\frac{19\pi^7}{56700}.$

(6) $\frac{1}{1^5\cosh\frac{\pi}{2}}-\frac{1}{3^5\cosh\frac{3\pi}{2}}+\frac{1}{5^5\cosh\frac{5\pi}{2}}-\ldots=\frac{\pi^5}{768}.$

......

VI. Theorems on transformation of series and integrals, e.g.

(1) $\pi\left(\frac{1}{2}-\frac{1}{\sqrt{1}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{5}}-\frac{1}{\sqrt{5}+\sqrt{7}}+\ \ldots\right)=\frac{1}{1\sqrt{1}}-\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}-\ldots.$

......

(3) $$1-\frac{x^2\,3\,!}{(1\,!\,2\,!)^3}+\frac{x^4\,6\,!}{(2\,!\,4\,!)^3}-\frac{x^6\,9\,!}{(3\,!\,6\,!)^3}+\ldots$$
$$=\left\{1+\frac{x}{(1\,!)^3}+\frac{x^2}{(2\,!)^3}+\ \ldots\right\}\left\{1-\frac{x}{(1\,!)^3}+\frac{x^2}{(2\,!)^3}-\ \ldots\right\}.$$

......

(6) If $\alpha\beta=\pi^2$, then
$$\frac{1}{\sqrt[4]{\alpha}}\left\{1+4\alpha\int_0^\infty\frac{xe^{-\alpha x^2}}{e^{2\pi x}-1}\,dx\right\}=\frac{1}{\sqrt[4]{\beta}}\left\{1+4\beta\int_0^\infty\frac{xe^{-\beta x^2}}{e^{2\pi x}-1}\,dx\right\}.$$

(7) $n\left(e^{-n^2}-\frac{e^{-\frac{1}{3}n^2}}{3\sqrt{3}}+\frac{e^{-\frac{1}{5}n^2}}{5\sqrt{5}}-\ \ldots\right)=\sqrt{\pi}\,(e^{-n\sqrt{\pi}}\sin n\sqrt{\pi}-e^{-n\sqrt{3\pi}}\sin n\sqrt{3\pi}+\ldots).$

(8) If n is any positive integer excluding 0,
$$\frac{1^{4n}}{(e^{\pi}-e^{-\pi})^2}+\frac{2^{4n}}{(e^{2\pi}-e^{-2\pi})^2}+\ldots=\frac{n}{\pi}\left\{\frac{B_{4n}}{8n}+\frac{1^{4n-1}}{e^{2\pi}-1}+\frac{2^{4n-1}}{e^{4\pi}-1}+\ldots\right\}$$
where $B_2=\frac{1}{6}$, $B_4=\frac{1}{30}$, ...

VII. Theorems on approximate integration and summation of series.

......

(2) $1+\frac{x}{1\,!}+\frac{x^2}{2\,!}+\frac{x^3}{3\,!}+\ldots+\frac{x^x}{x\,!}\,\theta=\frac{e^x}{2}$

where $\theta=\frac{1}{3}+\frac{4}{135\,(x+k)}$ where k lies between $\frac{8}{45}$ and $\frac{2}{21}$.

(3) $1+\left(\frac{x}{1\,!}\right)^5+\left(\frac{x^2}{2\,!}\right)^5+\left(\frac{x^3}{3\,!}\right)^5+\ldots=\frac{\sqrt{5}}{4\pi^2}\cdot\frac{e^{5x}}{5x^2-x+\theta}$

where θ vanishes when $x=\infty$.

(4) $$\frac{1^2}{e^x-1}+\frac{2^2}{e^{2x}-1}+\frac{3^2}{e^{3x}-1}+\frac{4^2}{e^{4x}-1}+\ldots=\frac{2}{x^3}\left(\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\ \ldots\right)-\frac{1}{12x}-\frac{x}{1440}+\frac{x^3}{181440}$$
$$+\frac{x^5}{7257600}+\frac{x^7}{159667200}+\ldots \text{ when } x \text{ is small.}$$

(*Note*: x may be given values from 0 to 2.)

(5) $\frac{1}{1001}+\frac{1}{1002^2}+\frac{3}{1003^3}+\frac{4^2}{1004^4}+\frac{5^3}{1005^5}+\ldots=\frac{1}{1000}-10^{-440}\times 1{\cdot}0125$ nearly.

(6) $\int_0^a e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2} - \frac{e^{-a^2}}{2a+}\,\frac{1}{a+}\,\frac{2}{2a+}\,\frac{3}{a+}\,\frac{4}{2a+}\dots$.

(7) The coefficient of x^n in $\frac{1}{1-2x+2x^4-2x^9+2x^{16}-\dots}$

$$= \text{the nearest integer to } \frac{1}{4n}\left\{\cosh(\pi\sqrt{n}) - \frac{\sinh(\pi\sqrt{n})}{\pi\sqrt{n}}\right\}^*.$$

......

IX. Theorems on continued fractions, a few examples are:

(1) $\frac{4}{x+}\,\frac{1^2}{2x+}\,\frac{3^2}{2x+}\,\frac{5^2}{2x+}\,\frac{7^2}{2x+}\dots = \left\{\frac{\Gamma\left(\frac{x+1}{4}\right)}{\Gamma\left(\frac{x+3}{4}\right)}\right\}^2.$

......

(4) If
$$u = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+}\,\frac{x^{20}}{1+}\dots$$
and
$$v = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+}\dots,$$
then
$$v^5 = u\,.\,\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}.$$

(5) $\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+}\dots = \left(\sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$

(6) $\frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1+}\dots = \left(\sqrt{\frac{5-\sqrt{5}}{3}} - \frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$

(7) $\frac{1}{1+}\,\frac{e^{-\pi\sqrt{n}}}{1+}\,\frac{e^{-2\pi\sqrt{n}}}{1+}\,\frac{e^{-3\pi\sqrt{n}}}{1+}\dots$ can be exactly found if n be any positive rational quantity....

27 *February* 1913.

...I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed....I find in many a place in your letter rigorous proofs are required and you ask me to communicate the methods of proof.... I told him† that the sum of an infinite number of terms of the series $1+2+3+4+\dots = -\frac{1}{12}$ under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal....What I tell you is this. Verify the results I give and if they agree with your results...you should at least grant that there may be some truths in my fundamental basis....

To preserve my brains I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government....

1. The number of prime numbers less than $e^a = \int_0^\infty \frac{a^x\,dx}{xS_{x+1}\Gamma(x+1)}$,

where
$$S_{x+1} = \frac{1}{1^{x+1}} + \frac{1}{2^{x+1}} + \dots.$$

2. The number of prime numbers less than n
$$= \frac{2}{\pi}\left\{\frac{2}{B_2}\left(\frac{\log n}{2\pi}\right) + \frac{4}{3B_4}\left(\frac{\log n}{2\pi}\right)^3 + \frac{6}{5B_6}\left(\frac{\log n}{2\pi}\right)^5 + \dots\right\},$$
where $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, ..., the Bernoullian numbers....

* This is quite untrue. But the formula is extremely interesting for a variety of reasons.
† Referring to a previous correspondence.

The order of $\theta(x)$ which you asked in your letter is $\sqrt{\left(\frac{x}{\log x}\right)}$.

......

(1) If $$F(x)=\frac{1}{1}+\frac{x}{1}+\frac{x^2}{1}+\frac{x^3}{1}+\frac{x^4}{1}+\frac{x^5}{1}+\dots,$$

then $$\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\alpha}{5}}F(e^{-2\alpha})\right\}\left\{\frac{\sqrt{5}+1}{2}+e^{-\frac{2\beta}{5}}F(e^{-2\beta})\right\}=\frac{5+\sqrt{5}}{2}$$

with the conditions $\alpha\beta=\pi^2\dots$,

e.g. $$\frac{1}{1}+\frac{e^{-2\pi\sqrt{5}}}{1}+\frac{e^{-4\pi\sqrt{5}}}{1}+\dots=e^{\frac{2\pi}{\sqrt{5}}}\left\{\frac{\sqrt{5}}{1+\sqrt[5]{5^{\frac{3}{4}}\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{5}{2}}-1}}-\frac{\sqrt{5}+1}{2}\right\}.$$

The above theorem is a particular case of a theorem on the continued fraction
$$\frac{1}{1}+\frac{ax}{1}+\frac{ax^2}{1}+\frac{ax^3}{1}+\frac{ax^4}{1}+\frac{ax^5}{1}+\dots,$$
which is a particular case of the continued fraction
$$\frac{1}{1}+\frac{ax}{1+bx}+\frac{ax^2}{1+bx^2}+\frac{ax^3}{1+bx^3}+\dots,$$
which is a particular case of a general theorem on continued fractions.

(2) (i) $$4\int_0^\infty \frac{xe^{-x\sqrt{5}}}{\cosh x}dx=\frac{1}{1}+\frac{1^2}{1}+\frac{1^2}{1}+\frac{2^2}{1}+\frac{2^2}{1}+\frac{3^2}{1}+\frac{3^2}{1}+\dots.$$

(ii) $$4\int_0^\infty \frac{x^2e^{-x\sqrt{3}}}{\cosh x}dx=\frac{1}{1}+\frac{1^3}{1}+\frac{1^3}{3}+\frac{2^3}{1}+\frac{2^3}{5}+\frac{3^3}{1}+\frac{3^3}{7}+\dots.$$

(3) $$1-5.\left(\frac{1}{2}\right)^5+9.\left(\frac{1.3}{2.4}\right)^5-13.\left(\frac{1.3.5}{2.4.6}\right)^5+\dots=\frac{2}{\{\Gamma(\frac{3}{4})\}^4}.$$

......

(6) If $$v=\frac{x}{1}+\frac{x^3+x^6}{1}+\frac{x^6+x^{12}}{1}+\frac{x^9+x^{18}}{1}+\dots,$$

then (i) $$x\left(1+\frac{1}{v}\right)=\frac{1+x+x^3+x^6+x^{10}+\dots}{1+x^9+x^{27}+x^{54}+x^{90}+\dots},$$

(ii) $$x^3\left(1+\frac{1}{v^3}\right)=\left(\frac{1+x+x^3+x^6+x^{10}+\dots}{1+x^3+x^9+x^{18}+x^{30}+\dots}\right)^4.$$

(7) If n is any odd integer,
$$\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}}-\frac{1}{3\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)}+\frac{1}{5\left(\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2n}\right)}-\dots=\frac{\pi}{8}.$$

......

(10) If $$F(\alpha,\beta,\gamma,\delta,\epsilon)=1+\frac{\alpha}{1!}.\frac{\beta}{\delta}.\frac{\gamma}{\epsilon}+\frac{\alpha(\alpha+1)}{2!}.\frac{\beta(\beta+1)}{\delta(\delta+1)}.\frac{\gamma(\gamma+1)}{\epsilon(\epsilon+1)}+\dots,$$

then $$F(\alpha,\beta,\gamma,\delta,\epsilon)=\frac{\Gamma(\delta)\Gamma(\delta-\alpha-\beta)}{\Gamma(\delta-\alpha)\Gamma(\delta-\beta)}.F(\alpha,\beta,\epsilon-\gamma,\alpha+\beta-\delta+1,\epsilon)$$
$$+\frac{\Gamma(\delta)\Gamma(\epsilon)\Gamma(\alpha+\beta-\delta)\Gamma(\delta+\epsilon-\alpha-\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\epsilon-\gamma)\Gamma(\delta+\epsilon-\alpha-\beta)}$$
$$\times F(\delta-\alpha,\delta-\beta,\delta+\epsilon-\alpha-\beta-\gamma,\delta-\alpha-\beta+1,\delta+\epsilon-\alpha-\beta).$$

(13) $\dfrac{a}{1+n+}\,\dfrac{a^2}{3+n+}\,\dfrac{(2a)^2}{5+n+}\,\dfrac{(3a)^2}{7+n+}\dots$

$$=2a\int_0^1 z^{\frac{n}{\sqrt{(1+a^2)}}}\frac{dz}{\{\sqrt{(1+a^2)}+1\}+z^2\{\sqrt{(1+a^2)}-1\}}.$$

(14) If $$F(\alpha,\beta)=\alpha+\frac{(1+\beta)^2+k}{2\alpha}\,+\,\frac{(3+\beta)^2+k}{2\alpha}\,+\,\frac{(5+\beta)^2+k}{2\alpha}\,+\dots,$$

then $$F(\alpha,\beta)=F(\beta,\alpha).$$

(15) If $$F(\alpha,\beta)=\frac{\alpha}{n+}\,\frac{\beta^2}{n+}\,\frac{(2\alpha)^2}{n}\,+\,\frac{(3\beta)^2}{n}\,+\dots,$$

then $$F(\alpha,\beta)+F(\beta,\alpha)=2F\{\tfrac{1}{2}(\alpha+\beta),\ \sqrt{(\alpha\beta)}\}.$$

......

(17) If $F(k)=1+\left(\frac{1}{2}\right)^2k+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^2+\dots$ and $F(1-k)=\sqrt{(210)}\,F(k)$,

then

$$k=(\sqrt{2}-1)^4(2-\sqrt{3})^2(\sqrt{7}-\sqrt{6})^4(8-3\sqrt{7})^2(\sqrt{10}-3)^4(4-\sqrt{15})^4(\sqrt{15}-\sqrt{14})^2(6-\sqrt{35})^2.$$

......

(20) If $$F(\alpha)=\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{\{1-(1-\alpha)\sin^2\phi\}}}\Big/\int_0^{\frac{1}{2}\pi}\frac{d\phi}{\sqrt{\{1-\alpha\sin^2\phi\}}}$$

and $$F(\alpha)=3F(\beta)=5F(\gamma)=15F(\delta),$$

then (i) $[(\alpha\delta)^{\frac{1}{8}}+\{(1-\alpha)(1-\delta)\}^{\frac{1}{8}}]\,[(\beta\gamma)^{\frac{1}{8}}+\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}}]=1.$

......

(v) $$(\alpha\beta\gamma\delta)^{\frac{1}{8}}+\{(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{8}}$$
$$+\{16\alpha\beta\gamma\delta(1-\alpha)(1-\beta)(1-\gamma)(1-\delta)\}^{\frac{1}{24}}=1.$$

......

(21) If $$F(\alpha)=3F(\beta)=13F(\gamma)=39F(\delta)$$

or $$F(\alpha)=5F(\beta)=11F(\gamma)=55F(\delta)$$

or $$F(\alpha)=7F(\beta)=\ 9F(\gamma)=63F(\delta),$$

then $$\frac{\{(1-\alpha)(1-\delta)\}^{\frac{1}{8}}-(\alpha\delta)^{\frac{1}{8}}}{\{(1-\beta)(1-\gamma)\}^{\frac{1}{8}}-(\beta\gamma)^{\frac{1}{8}}}=\frac{1+\{(1-\alpha)(1-\delta)\}^{\frac{1}{4}}+(\alpha\delta)^{\frac{1}{4}}}{1+\{(1-\beta)(1-\gamma)\}^{\frac{1}{4}}+(\beta\gamma)^{\frac{1}{4}}}.$$

......

(23) $(1+e^{-\pi\sqrt{1353}})(1+e^{-3\pi\sqrt{1353}})(1+e^{-5\pi\sqrt{1353}})\dots$

$$=\sqrt[4]{2}\,e^{-\frac{1}{24}\pi\sqrt{1353}}\times\sqrt{\left\{\sqrt{\left(\frac{569+99\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{561+99\sqrt{33}}{8}\right)}\right\}}$$
$$\times\sqrt{\left\{\sqrt{\left(\frac{25+3\sqrt{33}}{8}\right)}+\sqrt{\left(\frac{17+3\sqrt{33}}{8}\right)}\right\}}\times\sqrt[4]{\left(\frac{123+11}{\sqrt{2}}\right)}$$
$$\times\sqrt[8]{(10+3\sqrt{11})}\times\sqrt[8]{(26+15\sqrt{3})}\times\sqrt[12]{\left(\frac{6817+321\sqrt{451}}{\sqrt{2}}\right)}.$$

......

17 *April* 1913.

...I am a little pained to see what you have written*....I am not in the least apprehensive of my method being utilized by others. On the contrary my method has been in my possession for the last eight years and I have not found anyone to appreciate the method. As I wrote in my last letter I have found a sympathetic friend in you and

* Ramanujan might very reasonably have been reluctant to give away his secrets to an English mathematician, and I had tried to reassure him on this point as well as I could.

I am willing to place unreservedly in your hands what little I have. It was on account of the novelty of the method I have used that I am a little diffident even now to communicate my own way of arriving at the expressions I have already given....

...I am glad to inform you that the local University has been pleased to grant me a scholarship of £60 per annum for two years and this was at the instance of Dr Walker, F.R.S., Head of the Meteorological Department in India, to whom my thanks are due.... I request you to convey my thanks also to Mr Littlewood, Dr Barnes, Mr Berry and others who take an interest in me....

III

It is unnecessary to repeat the story of how Ramanujan was brought to England. There were serious difficulties; and the credit for overcoming them is due primarily to Prof. E. H. Neville, in whose company Ramanujan arrived in April 1914. He had a scholarship from Madras of £250, of which £50 was allotted to the support of his family in India, and an exhibition of £60 from Trinity. For a man of his almost ludicrously simple tastes, this was an ample income; and he was able to save a good deal of money which was badly wanted later. He had no duties and could do as he pleased; he wished indeed to qualify for a Cambridge degree as a research student, but this was a formality. He was now, for the first time in his life, in a really comfortable position, and could devote himself to his researches without anxiety.

There was one great puzzle. What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity. Here was a man who could work out modular equations, and theorems of complex multiplication, to orders unheard of, whose mastery of continued fractions was, on the formal side at any rate, beyond that of any mathematician in the world, who had found for himself the functional equation of the Zeta-function, and the dominant terms of many of the most famous problems in the analytic theory of numbers; and he had never heard of a doubly periodic function or of Cauchy's theorem, and had indeed but the vaguest idea of what a function of a complex variable was. His ideas as to what constituted a mathematical proof were of the most shadowy description. All his results, new or old, right or wrong, had been arrived at by a process of mingled argument, intuition, and induction, of which he was entirely unable to give any coherent account.

It was impossible to ask such a man to submit to systematic instruction, to try to learn mathematics from the beginning once more. I was afraid too that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand there were things of which it was impossible that he should remain in ignorance. Some of his results were wrong, and in particular those which concerned the distribution of primes, to which he attached the greatest importance. It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to try to teach him, and in a measure I succeeded, though obviously I learnt from him much more