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SOME PROPERTIES OF BERNOULLI'S NUMBERS

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1. Let the well-known expansion of $x \cot x$ (*vide* Edwards' *Differential Calculus*, § 149) be written in the form

$$x \cot x = 1 - \frac{B_2}{2\,!}(2x)^2 - \frac{B_4}{4\,!}(2x)^4 - \frac{B_6}{6\,!}(2x)^6 - \dots, \qquad \dots\dots\dots\dots(1)$$

from which we infer that B_0 may be supposed to be -1.

Now
$$\cot x = \frac{\cos x}{\sin x} = \frac{1 - \dfrac{x^2}{2\,!} + \dfrac{x^4}{4\,!} - \dfrac{x^6}{6\,!} + \dots}{x - \dfrac{x^3}{3\,!} + \dfrac{x^5}{5\,!} - \dfrac{x^7}{7\,!} + \dots}$$

$$= \frac{\sin 2x}{1 - \cos 2x} = \frac{\dfrac{2x}{1\,!} - \dfrac{(2x)^3}{3\,!} + \dfrac{(2x)^5}{5\,!} - \dots}{\dfrac{(2x)^2}{2\,!} - \dfrac{(2x)^4}{4\,!} + \dfrac{(2x)^6}{6\,!} - \dots}$$

$$= \frac{1 + \cos 2x}{\sin 2x} = \frac{2 - \dfrac{(2x)^2}{2\,!} + \dfrac{(2x)^4}{4\,!} - \dfrac{(2x)^6}{6\,!} + \dots}{2x - \dfrac{(2x)^3}{3\,!} + \dfrac{(2x)^5}{5\,!} - \dfrac{(2x)^7}{7\,!} + \dots}.$$

Multiplying both sides in each of the above three relations by the denominator of the right-hand side and equating the coefficients of x^n on both sides, we can write the results thus:

$$c_1 \frac{B_{n-1}}{2} - c_3 \frac{B_{n-3}}{2^3} + c_5 \frac{B_{n-5}}{2^5} - \dots + \frac{(-1)^{\frac{1}{2}(n-1)}}{2^n} B_0 + \frac{n}{2^n}(-1)^{\frac{1}{2}(n-1)} = 0, \dots\dots(2)$$

where n is any odd integer;

$$c_2 B_{n-2} - c_4 B_{n-4} + c_6 B_{n-6} - \dots + (-1)^{\frac{1}{2}(n-2)} B_0 + \frac{n}{2}(-1)^{\frac{1}{2}(n-2)} = 0, \dots\dots(3)$$

where n is any even integer;

$$c_1 B_{n-1} - c_3 B_{n-3} + c_5 B_{n-5} - \dots + (-1)^{\frac{1}{2}(n-1)} B_0 + \frac{n}{2}(-1)^{\frac{1}{2}(n-1)} = 0, \dots\dots(4)$$

where n is any odd integer greater than unity.

From any one of (2), (3), (4) we can calculate the B's. But as n becomes greater and greater the calculation will get tedious. So we shall try to find simpler methods.

2. We know $$(x \cot x)^2 = -x^2 \left(1 + \frac{d \cot x}{dx}\right).$$

Using (1) and equating the coefficients of x^n on both sides, and simplifying, we have

$$\tfrac{1}{2}(n+1) B_n = c_2 B_2 B_{n-2} + c_4 B_4 B_{n-4} + c_6 B_6 B_{n-6} + \dots,$$

the last term being $c_{\frac{1}{2}n-1} B_{\frac{1}{2}n-1} B_{\frac{1}{2}n+1}$ or $\frac{1}{2} c_{\frac{1}{2}n} (B_{\frac{1}{2}n})^2$ according as $\frac{1}{2}n$ is odd or even. ....(5)

A similar result can be obtained by equating the coefficients of x^n in the identity

$$\frac{d \tan x}{dx} = 1 + \tan^2 x.$$

3. Again

$$-\tfrac{1}{2}x(\cot \tfrac{1}{2}x + \coth \tfrac{1}{2}x) = -\tfrac{1}{2}x(\cot \tfrac{1}{2}x + i \cot \tfrac{1}{2}ix)$$
$$= 2\left\{B_0 + B_4 \frac{x^4}{4!} + B_8 \frac{x^8}{8!} + \dots\right\},$$

by using (1). The expression may also be written

$$-\tfrac{1}{2}x \frac{(\cos \frac{1}{2}x \sin \frac{1}{2}ix + i \sin \frac{1}{2}x \cos \frac{1}{2}ix)}{\sin \frac{1}{2}x \sin \frac{1}{2}ix}$$
$$= -\tfrac{1}{2}x \frac{(1+i)\sin \frac{1}{2}x(1+i) - (1-i)\sin \frac{1}{2}x(1-i)}{\cos \frac{1}{2}x(1-i) - \cos \frac{1}{2}x(1+i)}$$
$$= -x \frac{\dfrac{x}{1!} - \dfrac{x^5}{2^2.5!} + \dfrac{x^9}{2^4.9!} - \dots}{\dfrac{x^2}{2!} - \dfrac{x^6}{2^2.6!} + \dfrac{x^{10}}{2^4.10!} - \dots},$$

by expanding the numerator and the denominator, and simplifying by De Moivre's theorem.

Hence $$2\left(B_0 + B_4 \frac{x^4}{4!} + B_8 \frac{x^8}{8!} + \dots\right) = -x \frac{\dfrac{x}{1!} - \dfrac{x^5}{2^2.5!} + \dots}{\dfrac{x^2}{2!} - \dfrac{x^6}{2^2.6!} + \dots}. \quad \text{.........(6)}$$

Similarly

$$-\tfrac{1}{2}x(\cot \tfrac{1}{2}x - \coth \tfrac{1}{2}x) = 2\left(B_2 \frac{x^2}{2!} + B_6 \frac{x^6}{6!} + B_{10} \frac{x^{10}}{10!} + \dots\right)$$
$$= x \frac{\frac{1}{2}(1-i)\sin \frac{1}{2}x(1+i) - \frac{1}{2}(1+i)\sin \frac{1}{2}x(1-i)}{\cos \frac{1}{2}x(1+i) - \cos \frac{1}{2}x(1-i)}$$
$$= x \frac{\dfrac{x^3}{2.3!} - \dfrac{x^7}{2^3.7!} + \dfrac{x^{11}}{2^5.11!} + \dots}{\dfrac{x^2}{2!} - \dfrac{x^6}{2^2.6!} + \dfrac{x^{10}}{2^4.10!} - \dots}. \quad \text{........................(7)}$$

Proceeding as in § 1 we have, if n is an even integer greater than 2,

$$c_2 \frac{B_{n-2}}{2} - c_6 \frac{B_{n-6}}{2^3} + c_{10} \frac{B_{n-10}}{2^5} - \ldots + \frac{n}{2^{\frac{1}{2}(n+2)}}(-1)^{\frac{1}{4}n} \text{ or } \frac{n}{2^{\frac{1}{2}(n+2)}}(-1)^{\frac{1}{4}(n-2)} = 0, \quad (8)$$

according as n or $n-2$ is a multiple of 4.

Analogous results can be obtained from $\tan \frac{1}{2}x \pm \tanh \frac{1}{2}x$.

In (2), (3) and (4) there are $\frac{1}{2}n$ terms, while in (5) and (8) there are $\frac{1}{4}n$ or $\frac{1}{4}(n-2)$ terms. Thus B_n can be found from only half of the previous B's.

4. A still simpler method can be deduced from the following identities.

If 1, ω, ω^2 be the three cube roots of unity, then

$$4 \sin x \sin x\omega \sin x\omega^2 = -(\sin 2x + \sin 2x\omega + \sin 2x\omega^2),$$

as may easily be verified.

By logarithmic differentiation, we have

$$\cot x + \omega \cot x\omega + \omega^2 \cot x\omega^2 = 2 \frac{\cos 2x + \omega \cos 2x\omega + \omega^2 \cos 2x\omega^2}{\sin 2x + \sin 2x\omega + \sin 2x\omega^2}.$$

Writing $\frac{1}{2}x$ for x,

$$-\tfrac{1}{2}x(\cot \tfrac{1}{2}x + \omega \cot \tfrac{1}{2}x\omega + \omega^2 \cot \tfrac{1}{2}x\omega^2) = -x \frac{\cos x + \omega \cos x\omega + \omega^2 \cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2},$$

and, proceeding as in § 3, we get

$$3\left(B_0 + B_6 \frac{x^6}{6!} + B_{12}\frac{x^{12}}{12!} + \ldots\right) = -x \frac{\dfrac{x^2}{2!} - \dfrac{x^8}{8!} + \dfrac{x^{14}}{14!} - \ldots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \ldots}. \quad \ldots\ldots(9)$$

Again

$$\cot \tfrac{1}{2}x\omega - \cot \tfrac{1}{2}x\omega^2 = \frac{\cos x\omega^2 - \cos x\omega}{2 \sin \frac{1}{2}x \sin \frac{1}{2}x\omega \sin \frac{1}{2}x\omega^2} = \frac{2(\cos x\omega - \cos x\omega^2)}{\sin x + \sin x\omega + \sin x\omega^2}.$$

Multiplying both sides by $-\frac{1}{2}x(\omega^2 - \omega)$ and adding to the corresponding sides of the previous result, we have

$$-\tfrac{1}{2}x(\cot \tfrac{1}{2}x + \omega^2 \cot \tfrac{1}{2}x\omega + \omega \cot \tfrac{1}{2}x\omega^2) = -x \frac{\cos x + \omega^2 \cos x\omega + \omega \cos x\omega^2}{\sin x + \sin x\omega + \sin x\omega^2}.$$

Hence, as before,

$$3\left(B_2 \frac{x^2}{2!} + B_8 \frac{x^8}{8!} + B_{14}\frac{x^{14}}{14!} + \ldots\right) = x \frac{\dfrac{x^4}{4!} - \dfrac{x^{10}}{10!} + \dfrac{x^{16}}{16!} - \ldots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \ldots}. \quad \ldots\ldots(10)$$

Similarly

$$-x(\cot \tfrac{1}{2}x + \cot \tfrac{1}{2}x\omega + \cot \tfrac{1}{2}x\omega^2) = x \frac{\cos x + \cos x\omega + \cos x\omega^2 - 3}{\sin x + \sin x\omega + \sin x\omega^2},$$

and therefore

$$6\left(B_4 \frac{x^4}{4!} + B_{10} \frac{x^{10}}{10!} + B_{16}\frac{x^{16}}{16!} + \ldots\right) = x \frac{\dfrac{x^6}{6!} - \dfrac{x^{12}}{12!} + \dfrac{x^{18}}{18!} - \ldots}{\dfrac{x^3}{3!} - \dfrac{x^9}{9!} + \dfrac{x^{15}}{15!} - \ldots}. \quad \ldots(11)$$

Multiplying up and equating coefficients in (9), (10) and (11) as usual, we have,

$$c_3 B_{n-3} - c_9 B_{n-9} + c_{15} B_{n-15} - \dots = 0, \quad \text{.................(12)}$$

the last term being $\frac{1}{6}n(-1)^{\frac{1}{6}(n-1)}$, $\frac{1}{3}n(-1)^{\frac{1}{6}(n+1)}$, or $\frac{1}{3}n(-1)^{\frac{1}{6}(n-3)}$.

Again, dividing both sides in (10) by x and differentiating, we have

$$3\left(B_2\frac{1}{2!} + 7B_8\frac{x^6}{8!} + 13B_{14}\frac{x^{12}}{14!} + \dots\right) = \frac{d}{dx}\left(\frac{\frac{x^4}{4!} - \frac{x^{10}}{10!} + \frac{x^{16}}{16!} - \dots}{\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} - \dots}\right)$$

$$= 1 - \frac{\frac{x^2}{2!} - \frac{x^8}{8!} + \frac{x^{14}}{14!} - \dots}{\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} - \dots} \cdot \frac{\frac{x^4}{4!} - \frac{x^{10}}{10!} + \frac{x^{16}}{16!} - \dots}{\frac{x^3}{3!} - \frac{x^9}{9!} + \frac{x^{15}}{15!} - \dots}.$$

Hence by (9) and (10),

$$3\left(B_2\frac{x^2}{2!} + 7B_8\frac{x^8}{8!} + 13B_{14}\frac{x^{14}}{14!} + \dots\right)$$

$$= x^2 + 9\left(B_0 + B_6\frac{x^6}{6!} + B_{12}\frac{x^{12}}{12!} + \dots\right)\left(B_2\frac{x^2}{2!} + B_8\frac{x^8}{8!} + B_{14}\frac{x^{14}}{14!} + \dots\right).$$

Equating the coefficients of x^n we have, if $n > 2$ and $n - 2$ is a multiple of 6,

$$\tfrac{1}{3}(n+2)B_n = c_6 B_{n-6} B_6 + c_{12} B_{n-12} B_{12} + c_{18} B_{n-18} B_{18} + \dots . \quad \text{......(13)}$$

From (12) the B's can be calculated very quickly and (13) may prove useful in checking the calculations. The number of terms is one-third of that in (4); thus B_{24} is found from B_{18}, B_{12} and B_6.

5. We shall see later on how the B's can be obtained from their properties only. But to know these properties, it will be convenient to calculate a few B's by substituting 3, 5, 7, 9, ..., for n in succession in (12). Thus

$$B_0 = -1;\quad B_2 = \frac{1}{6};\quad B_4 = \frac{1}{30};\quad B_6 = \frac{1}{42};\quad B_8 - \tfrac{1}{3}B_2 = -\frac{1}{45};$$

$$B_{10} - \frac{5}{2}B_4 = -\frac{1}{132};\quad B_{12} - 11B_6 = -\frac{4}{455};\quad B_{14} - \frac{143}{4}B_8 + \frac{B_2}{5} = \frac{1}{120};$$

$$B_{16} - \frac{286}{3}B_{10} + 4B_4 = \frac{1}{306};\quad B_{18} - 221B_{12} + \frac{204}{5}B_6 = \frac{3}{665};$$

$$B_{20} - \frac{3230}{7}B_{14} + \frac{1938}{7}B_8 - \frac{B_2}{7} = -\frac{1}{231};$$

$$B_{22} - \frac{3553}{4}B_{16} + \frac{7106}{5}B_{10} - \frac{11}{2}B_4 = -\frac{1}{552};$$

and so on. Hence we have finally the following values:

$$B_2=\frac{1}{6};\ B_4=\frac{1}{30};\ B_6=\frac{1}{42};\ B_8=\frac{1}{30};\ B_{10}=\frac{5}{66};\ B_{12}=\frac{691}{2730};$$

$$B_{14}=\frac{7}{6};\ B_{16}=\frac{3617}{510};\ B_{18}=\frac{43867}{798};\ B_{20}=\frac{174611}{330};\ B_{22}=\frac{854513}{138};$$

$$B_{24}=\frac{236364091}{2730};\ B_{26}=\frac{8553103}{6};\ B_{28}=\frac{23749461029}{870};$$

$$B_{30}=\frac{8615841276005}{14322};\ B_{32}=\frac{7709321041217}{510};\ B_{34}=\frac{2577687858367}{6};$$

$$B_{36}=\frac{26315271553053477373}{1919190};\ B_{38}=\frac{2929993913841559}{6};$$

$$B_{40}=\frac{261082718496449122051}{13530};\ \ldots,\ B_\infty=\infty.$$

6. It will be observed* that, if n is even but not equal to zero,

(i) B_n is a fraction and the numerator of B_n/n in its lowest terms is a prime number, ...(14)

(ii) the denominator of B_n contains each of the factors 2 and 3 once and only once, ...(15)

(iii) $2^n(2^n-1)B_n/n$ is an integer and consequently $2(2^n-1)B_n$ is an *odd* integer. ....(16)

From (16) it can easily be shewn that the denominator of $2(2^n-1)B_n/n$ in its lowest terms is the greatest power of 2 which divides n; and consequently, if n is not a multiple of 4, then $4(2^n-1)B_n/n$ is an odd integer. ....(17)

It follows from (14) that the numerator of B_n in its lowest terms is divisible by the greatest measure of n prime to the denominator, and the quotient is a prime number. ....(18)

Examples: (*a*) 2 and 3 are the only prime factors of 12, 24 and 36, and they are found in the denominators of B_{12}, B_{24} and B_{36} and their numerators are prime numbers.

(*b*) 11 is not found in the denominator of B_{22}, and hence its numerator is divisible by 11; similarly, the numerators of B_{26}, B_{34}, B_{38} are divisible by 13, 17, 19, respectively and the quotients in all cases are prime numbers.

(*c*) 5 is found in the denominator of B_{20} and not in that of B_{30}, and consequently the numerator of B_{30} is divisible by 5 while that of B_{20} is a prime number. Thus we may say that if a prime number appearing in n is not found in the denominator it will appear in the numerator, and *vice versa*.

* See § 12 below.

7. Next, let us consider the denominators.

All the denominators are divisible by 6; those of $B_4, B_8, B_{12}, \ldots$ by 5; those of B_6, B_{12}, $B_{18}, \ldots$ by 7; those of B_{10}, B_{20}, $B_{30}, \ldots$ by 11; but those of B_8, B_{16}, $B_{24}, \ldots$ are *not* divisible by 9; and those of B_{14}, $B_{28}, \ldots$ are *not* divisible by 15. Hence we may infer that:

the denominator of B_n is the continued product of prime numbers which are the next numbers (in the natural order) to the factors of n (including unity and the number itself). ....(19)

As an example take the denominator of B_{24}. Write all the factors of 24, viz. 1, 2, 3, 4, 6, 8, 12, 24. The next numbers to these are 2, 3, 4, 5, 7, 9, 13, 25. Strike out the *composite* numbers and we have the prime numbers 2, 3, 5, 7, 13. And the denominator of B_{24} is the product of 2, 3, 5, 7, 13, i.e. 2730.

It is unnecessary to write the *odd* factors of n except unity, as the next numbers to these are even and hence composite.

The following are some further examples:

	Even factors of n and unity	Denominator of B_n
B_2	... 1, 2	2 . 3=6
B_6	... 1, 2, 6	2 . 3 . 7=42
B_{12}	... 1, 2, 4, 6, 12	2 . 3 . 5 . 7 . 13=2730
B_{20}	... 1, 2, 4, 10, 20	2 . 3 . 5 . 11=330
B_{30}	... 1, 2, 6, 10, 30	2 . 3 . 7 . 11 . 31=14322
B_{42}	... 1, 2, 6, 14, 42	2 . 3 . 7 . 43=1806
B_{56}	... 1, 2, 4, 8, 14, 28, 56	2 . 3 . 5 . 29=870
B_{72}	... 1, 2, 4, 6, 8, 12, 18, 24, 36, 72 ...	2 . 3 . 5 . 7 . 13 . 19 . 37 . 73=140100870
B_{90}	... 1, 2, 6, 10, 18, 30, 90	2 . 3 . 7 . 11 . 19 . 31=272118
B_{110}	... 1, 2, 10, 22, 110	2 . 3 . 11 . 23=1518

8. Again taking the fractional part of any B and splitting it into partial fractions, we see that:

the fractional part of $B_n = (-1)^{\frac{1}{2}n}$ {the sum of the reciprocals of the prime factors of the denominator of B_n} $-(-1)^{\frac{1}{2}n}$. ....(20)

Thus the fractional part of $B_{16} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} - 1 = \frac{47}{510}$;

that of $B_{22} = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{23} = \frac{17}{138}$;

that of $B_{28} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{29} - 1 = \frac{59}{870}$;

and so on.

9. It can be inferred from (20) that:

if G be the G.C.M. and L the L.C.M. of the denominators of B_m and B_n, then L/G is the denominator of $B_m - (-1)^{\frac{1}{2}(m-n)} B_n$, and hence, if the denominators of B_m and B_n are equal, then $B_m - (-1)^{\frac{1}{2}(m-n)} B_n$ is an integer. ....(21)

Example: $B_{24}-B_{12}$ and $B_{32}-B_{16}$ are integers, while the denominator of $B_{10}+B_{20}$ is 5.

It will be observed that:

(1) if n is a multiple of 4, then the numerator of $B_n-\frac{1}{30}$ in its lowest terms is divisible by 20; but if n is not a multiple of 4 then that of $\frac{B_n}{n}-\frac{1}{12}$ in its lowest terms is divisible by 5; ....(22)

(2) if n is any integer, then

$$2(2^{4n+2}-1)\frac{B_{4n+2}}{2n+1},\quad 2(2^{8n+4}-1)\frac{B_{8n+4}}{2n+1},\quad 2(2^{8n+4}-1)\frac{B_{16n+8}}{2n+1}$$

are integers of the form $30p+1$. ....(23)

10. If a B is known to lie between certain limits, then it is possible to find its exact value from the above properties.

Suppose we know that B_{22} lies between 6084 and 6244; its exact value can be found as follows.

The fractional part of $B_{22}=\frac{17}{138}$ by (20), also B_{22} is divisible by 11 by (18). And by (22) $B_{22}-\frac{11}{6}$ must be divisible by 5. To satisfy these conditions B_{23} must be either $6137\frac{17}{138}$ or $6192\frac{17}{138}$.

But according to (18) the numerator of B_{22} should be a *prime* number after it is divided by 11; and consequently B_{22} must be equal to $6192\frac{17}{138}$ or $\frac{854513}{138}$, since the numerator of $6137\frac{17}{138}$ is divisible not only by 11 but also by 7 and 17.

11. It is known (Edwards' *Differential Calculus*, Ch. V, Ex. 29) that

$$B_n=\frac{2.n!}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots\right),$$

or

$$\frac{2.n!}{(2\pi)^n}=B_n\left(1-\frac{1}{2^n}\right)\left(1-\frac{1}{3^n}\right)\left(1-\frac{1}{5^n}\right)\dots, \quad \text{...........(24)}$$

where 2, 3, 5, ... are prime numbers.

Also

$$\frac{B_n}{2n}=\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}\,dx. \quad \text{..........................(25)}$$

For

$$\int_0^\infty \frac{x^{n-1}}{e^{2\pi x}-1}\,dx=\int_0^\infty x^{n-1}(e^{-2\pi x}+e^{-4\pi x}+\dots)\,dx$$
$$=\frac{(n-1)!}{(2\pi)^n}\left(\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\dots\right)=\frac{B_n}{2n}$$

by (24). In a similar manner

$$\int_0^\infty \frac{x^n}{(e^{\pi x}-e^{-\pi x})^2}\,dx=\frac{B_n}{4\pi},$$

and

$$\int^\infty x^{n-2}\log(1-e^{-2\pi x})\,dx=-\frac{\pi B_n}{n(n-1)}. \quad \text{...........(26)}$$

Take logarithms of both sides in (24) and write for $\log_e n!$ the well-known expansion of $\log_e \Gamma(n+1)$, as in Carr's *Synopsis*, viz.

$$(n+\tfrac{1}{2})\log n - n + \tfrac{1}{2}\log 2\pi + \frac{B_2}{1.2n} - \frac{B_4}{3.4n^3} + \frac{B_6}{5.6n^5} - \dots$$

$$- (-1)^p \frac{B_{2p}\theta}{(2p-1)\, 2pn^{2p-1}}, \quad \dots\dots\dots\dots\dots(27)$$

where $0 < \theta < 1$, and where

$$\frac{B_{2p}\theta}{(2p-1)\,2pn^{2p-1}} = \frac{B_{2p}}{(2p-1)\,2pn^{2p-1}} - \frac{B_{2p+2}}{(2p+1)(2p+2)\,n^{2p+1}} + \dots$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-1}} \log(1-e^{-2\pi x})\,dx + \frac{1}{\pi}\int_0^\infty \frac{x^{2p}}{n^{2p+1}}\log(1-e^{-2\pi x})\,dx - \dots$$

$$= -\frac{1}{\pi}\int_0^\infty \left(\frac{x^{2p-2}}{n^{2p-1}} - \frac{x^{2p}}{n^{2p+1}} + \dots\right)\log(1-e^{-2\pi x})\,dx$$

$$= -\frac{1}{\pi}\int_0^\infty \frac{x^{2p-2}}{n^{2p-3}(n^2+x^2)}\log(1-e^{-2\pi x})\,dx$$

$$= -\int_0^\infty \frac{x^{2p-2}\log(1-e^{-2\pi n x})}{\pi(1+x^2)}\,dx.$$

We can find the integral part of B_n, and since the fractional part can be found, as shewn in § 8, the exact value of B_n is known. Unless the calculation is made to depend upon the values of $\log_{10} e$, $\log_e 10$, π, ..., which are known to a great number of decimal places, we should have to find the logarithms of certain numbers whose values are not found in the tables to as many places of decimals as we require. Such difficulties are removed by the method given in § 13.

12. Results (14) to (17), (20) and (21) can be obtained as follows. We have

$$\frac{1}{2x^2} + \frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \dots$$

$$-\frac{1}{x} - \frac{1}{6(x^3-x)} + \frac{1}{5(x^5-x)} + \frac{1}{7(x^7-x)} + \frac{1}{11(x^{11}-x)} + \dots$$

$$= \frac{1}{x^{15}} - \frac{7}{x^{17}} + \frac{55}{x^{19}} - \frac{529}{x^{21}} + \dots, \quad \dots\dots\dots\dots\dots\dots(28)$$

where 5, 7, 11, 13, are prime numbers above 3. If we can prove that the left-hand side of (28) can be expanded in ascending powers of $1/x$ with integral coefficients, then (20) and (21) are at once deduced as follows.

From (27) we have

$$\frac{d^2 \log \Gamma(n+1)}{dn^2} = \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \dots$$

$$= \frac{1}{n} - \frac{1}{2n^2} + \frac{B_2}{n^3} - \frac{B_4}{n^5} + \frac{B_6}{n^7} - \frac{B_8}{n^9} + \dots - (-)^p \frac{B_{2p}\theta}{n^{2p+1}}, \quad \dots\dots\dots\dots\dots(29)$$

where

$$\frac{B_{2p}\theta}{n^{2p+1}} = \frac{B_{2p}}{n^{2p+1}} - \frac{B_{2p+2}}{n^{2p+3}} + \dots$$

$$= 4\pi \int_0^\infty \frac{x^{2p}}{n^{2p+1}(e^{\pi x} - e^{-\pi x})^2} dx - 4\pi \int_0^\infty \frac{x^{2p+2}}{n^{2p+3}(e^{\pi x} - e^{-\pi x})^2} dx + \dots$$

$$= \pi \int_0^\infty \left(\frac{x^{2p}}{n^{2p+1}} - \frac{x^{2p+2}}{n^{2p+3}} + \dots \right) \frac{dx}{\sinh^2 \pi x}$$

$$= \pi \int_0^\infty \frac{x^{2p}}{n^{2p-1}(n^2 + x^2)} \frac{dx}{\sinh^2 \pi x} = \int_0^\infty \frac{\pi x^{2p}}{(1+x^2)\sinh^2 \pi n x} dx.$$

Substituting the result of (29) in (28) we see that

$$\frac{B_2}{x^3} - \frac{B_4}{x^5} + \frac{B_6}{x^7} - \dots - \frac{1}{6(x^3 - x)} + \frac{1}{5(x^5 - x)} + \frac{1}{7(x^7 - x)} + \frac{1}{11(x^{11} - x)} + \dots,$$

where 5, 7, 11, ... are prime numbers, can be expanded in ascending powers of $1/x$ with integral coefficients.

Therefore $B_2 - \frac{1}{6}$, $-B_4 - \frac{1}{6} + \frac{1}{5}$, $B_6 - \frac{1}{6} + \frac{1}{7}$, $-B_8 - \frac{1}{6} + \frac{1}{5}$, $B_{10} - \frac{1}{6} + \frac{1}{11}$, ..., which are the coefficients of $1/x^3$, $1/x^5$, $1/x^7$, ..., are integers.

Writing $\frac{1}{2} + \frac{1}{3} - 1$ for $-\frac{1}{6}$ we get the results of (20) and (21).

Again changing n to $\frac{1}{2}n$ in (29), and subtracting half of the result from (29), we have

$$\frac{1}{(n+1)^2} - \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} - \dots = \frac{1}{2n^2} - \frac{(2^2-1)B_2}{n^3} + \frac{(2^4-1)B_4}{n^5}$$

$$- \frac{(2^6-1)B_6}{n^7} + \dots + (-1)^p (2^{2p} - 1) \frac{B_{2p}\theta}{n^{2p+1}}, \quad \dots\dots(30)$$

where $0 < \theta < 1$, and also, by (29),

$$(2^{2p} - 1)\frac{B_{2p}\theta}{n^{2p+1}} = \int_0^\infty \frac{\pi x^{2p} \cosh \pi n x}{(1+x^2)\sinh^2 \pi n x} dx.$$

Thus we see that, if we can prove that twice the left-hand side of (30) can be expanded in ascending powers of $1/n$ with integral coefficients, then the second part of (16) is at once proved.

Again from (27) we have

$$\frac{d \log \Gamma(n+1)}{dn} = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \ldots + \frac{1}{n} - \gamma$$

$$= \log n + \frac{1}{2n} - \frac{B_2}{2n^2} + \frac{B_4}{4n^4} - \frac{B_6}{6n^6} + \frac{B_8}{8n^8} - \ldots + (-1)^p \frac{B_{2p}\,\theta}{2pn^{2p}}, \quad \ldots(31)$$

where $0 < \theta < 1$; and also, by (25),

$$\frac{B_{2p}\,\theta}{2pn^{2p}} = \int_0^\infty \frac{2x^{2p-1}}{(1+x^2)(e^{2\pi n x}-1)}\,dx,$$

from which it can easily be shewn that

$$\frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+6} - \frac{1}{n+8} + \frac{1}{n+10} - \ldots$$

$$= \frac{1}{2n} - 2(2^2-1)\frac{B_2}{2n^2} + 2^3(2^4-1)\frac{B_4}{4n^4} - 2^5(2^6-1)\frac{B_6}{6n^6} + \ldots$$

$$+ (-1)^p\, 2^{2p-1}(2^{2p}-1)\frac{B_{2p}\,\theta}{2pn^{2p}} - \ldots, \quad \ldots\ldots(32)$$

where $0 < \theta < 1$; and also, by (31),

$$2^{2p-1}(2^{2p}-1)\frac{B_{2p}\,\theta}{2pn^{2p}} = \int_0^\infty \frac{x^{2p-1}}{2(1+x^2)\sinh \frac{1}{2}(\pi n x)}\,dx.$$

From the above theorem we see that, if we can prove that

$$2\left(\frac{1}{n+2} - \frac{1}{n+4} + \frac{1}{n+6} - \ldots\right),$$

can be expanded in ascending powers of $1/n$ with integral coefficients, then the first part of (16) at once follows.

13. The first few digits, and the number of digits in the integral part as well as in the numerator of B_n, can be found from the approximate formula:

$$\log_{10} B_n = (n + \tfrac{1}{2}) \log_{10} n - 1{\cdot}2324743503n + 0{\cdot}700120,$$

the true value being greater by about $0{\cdot}0362/n$ when n is great. ......(33)

This formula is proved as follows: taking logarithms of both sides in (24),

$$\log_e B_n = (n + \tfrac{1}{2}) \log_e n - n(1 + \log_e 2\pi) + \tfrac{1}{2} \log_e 8\pi$$

nearly. Multiplying both sides by $\log_{10} e$ or $\cdot 4342944819$, and reducing, we can get the result.

14. Changing n to $n-2$ in (24) and taking the ratio of the two results, we have

$$B_n = \frac{n(n-1)}{4\pi^2} B_{n-2}\left(1 - \frac{2^2-1}{2^n-1}\right)\left(1 - \frac{3^2-1}{3^n-1}\right)\left(1 - \frac{5^2-1}{5^n-1}\right)\ldots, \quad (34)$$

where 2, 3, 5, ... are prime numbers.