SOME PROBLEMS

OF

GEODYNAMICS

Cambridge University Press
978-1-107-53647-0 - Some Problems of Geodynamics: Being an Essay to Which the Adams Prize in the University of Cambridge was Adjudged in 1911
A. E. H. Love
Frontmatter
More information

SOME PROBLEMS OF GEODYNAMICS

BEING AN ESSAY TO WHICH THE ADAMS PRIZE IN THE UNIVERSITY OF CAMBRIDGE WAS ADJUDGED IN 1911

BY

A. E. H. LOVE, M.A., D.Sc., F.R.S.

FORMERLY FELLOW OF ST JOHN'S COLLEGE, CAMBRIDGE
HONORARY FELLOW OF QUEEN'S COLLEGE, OXFORD
SEDLEIAN PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF OXFORD

Cambridge:
at the University Press
1926

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107536470

First edition 1911
Reprinted 1926
First paperback edition 2015

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-53647-0 Paperback

PREFACE

THE subject selected for the Adams' Prize of 1910 was "Some investigation connected with the physical constitution or motion of the earth." A number of questions on which it is desirable to obtain further knowledge were mentioned; among them were "The stresses in continents and mountains, when the supposition of the existence of the isostatic layer is accepted; the propagation of seismic waves." At the time when this announcement was made, March 1909, I had found that modification of previous theories concerning the effects produced by compressibility in a body of planetary dimensions which forms the basis of the investigations in Chapters VII—X of this Essay, and had sketched a programme of work dealing with the special subject cited above from the announcement. The investigations concerning the effects of the earth's rotation on earth tides did not arise as part of the original programme, but were undertaken after a discussion of the subject at the Winnipeg Meeting of the British Association for the Advancement of Science.

As the analytical investigations in the Essay are rather intricate, it has been thought advisable to prefix an Abstract, stating the special hypotheses and limitations in accordance with which the various problems are discussed, and describing the conclusions which have been reached.

My best thanks are due to the authorities of the Cambridge University Press for the readiness with which they have met all my wishes in regard to the printing.

A. E. H. L.

April, 1911.

TABLE OF CONTENTS

CHAPTER IV

PAGE

GENERAL THEORY OF EARTH TIDES

CHAPTER V

EFFECT OF INERTIA ON EARTH TIDES

CHAPTER VI

EFFECT OF THE SPHEROIDAL FIGURE OF THE EARTH ON EARTH TIDES

CHAPTER VII

GENERAL THEORY OF A GRAVITATING COMPRESSIBLE PLANET

CHAPTER VIII

EFFECT OF COMPRESSIBILITY ON EARTH TIDES

CHAPTER IX

PAGE

THE PROBLEM OF GRAVITATIONAL INSTABILITY

CHAPTER X

VIBRATIONS OF A GRAVITATING COMPRESSIBLE PLANET

CHAPTER XI

THEORY OF THE PROPAGATION OF SEISMIC WAVES

ABSTRACT

THE first three Chapters deal with the problem of determining the Stress produced in the Interior of the Earth by the Weight of Continents and Mountains. Chapter I contains a brief discussion of the distribution of land and water on the surface of the globe. This discussion is designed to evaluate roughly the amplitudes of those spherical harmonic inequalities which are most prominent in the shape of the lithosphere. By the lithosphere is here meant the surface of the land in places where there is land, and the surface at the bottom of the sea in places where there is sea. The most important deviation of this surface from a spherical form is the inequality specified by the ellipticity of the meridians; but this inequality is without influence upon the distribution of land and water, for the lithosphere and the surface of the sea both have elliptic meridians, and the difference of their ellipticities is trifling. After the ellipticity of the meridians the most prominent inequalities are those which are manifested in the existence of a single continental block, embracing all the continents, and surrounding two great areas of depression, the basin of the Pacific Ocean and the basin of the Atlantic and Indian Oceans, the portions of the Southern Ocean which lie to the south of these oceans respectively being counted as parts of them. The inequalities manifested in mountain ranges and deeps have not nearly so much importance in regard to the figure of the Earth as a whole.

In Chapter I it is explained how the shape of the lithosphere could, if the elevation or depression of every point above or below a mean level were known accurately, be expressed by equating the radius vector, which joins the centre of gravity and a point of assigned latitude and longitude, to a sum of spherical surface harmonics, which are definite functions of the latitude and longitude, each provided with a suitable coefficient. Further it is shown, on the basis of previous work by the writer, that the inequalities manifested in the continental block and ocean basins may be represented roughly by restricting the sum in question to harmonics of the first, second, and third degrees. It appears that the elevations and depressions answering to the first and third harmonics are nearly equal, the third slightly the greater, and greater than those answering to the second harmonic; and that an amplitude of 2 km., implying, in the case of the uneven harmonics, a range of 4 km. from greatest elevation to greatest depression, and, in the case of the second harmonic, a range of 3 km., would be amply sufficient to express the elevation of the actual mean surface of the land above the bottom of the sea.

Such deviations from the spherical figure as are manifested in the continental block and the ocean basins, and in mountains and deeps, imply the existence within the earth of tangential stresses. They could not be maintained if the stress at every point across every plane passing through the point were normal to the plane. The maintenance of the ellipticity due to the rotation does not require any tangential stress. In Chapter II it is explained that the problem of finding the stress required for the support of continents and mountains is strictly indeterminate, as it would admit of an infinite number of solutions founded on different hypotheses; and a solution is sought on the basis of isostasy. Even when this hypothesis is adopted the problem is still indeterminate, and that in two ways. In the first place the hypothesis, as developed by previous writers, lacks precision. In the second place the equations of equilibrium of a solid body, subjected to its own gravitation, do not suffice for the determination of the stress within it until some relation between stress and strain is introduced; and, in the present problem, the notion of strain is inappropriate, because the earth is not strained from a state without continents and mountains to a state possessing these features. It becomes necessary to make two assumptions. The first assumption amounts to assigning a special form to the hypothesis of isostasy. According to the hypothesis, the inequalities of the earth's figure (apart from the ellipticity due to the rotation) are associated with inequalities of density in a superficial layer, the thickness of which is about one-fiftieth of the radius, in such a way that the stress in the interior parts is hydrostatic pressure which is not affected by the inequalities of density. This condition could be satisfied by an infinite number of laws of density in the layer, and the special law which is chosen is dictated by analytical convenience. It proves to be convenient to assign a form to the inequalities of potential that are due to the inequalities of density, and to deduce a law of density which, it must be admitted, appears to be rather artificial. This is effected by taking the inner surface of the superficial layer, or "layer of compensation," to be a spherical surface of radius b, and the mean outer surface to be a concentric spherical surface of radius a, and supposing that the terms which are contributed to the potential at any point within the layer by the inequalities of density contain as factors the expressions $(a-r)$ and $(r-b)^2$, where r denotes the distance of the point from the centre of the spherical surfaces. The rotation is neglected. The factor $(a-r)$ secures that, in spite of the inequalities, the mean outer surface is an equipotential surface, a condition which must be fulfilled, at least approximately, if the theory is to be brought into accord with geodetic observations. The factor $(r-b)$ must occur, and be repeated, if, as is laid down in the hypothesis, the stress at every point within the inner surface of the layer across every plane passing through the point is normal to the plane. Clearly the function by which the potential due to the inequalities is expressed is not determined by the condition of possessing the two factors $(a-r)$ and $(r-b)^2$. To any spherical harmonic inequality of the surface there must answer a term of this potential which contains also, as another factor, the spherical surface harmonic expressing the inequality. This term may, without affecting the hypothesis, be multiplied by any function of r. By choosing this function in various ways we could arrive at an infinite number of laws of density in the layer, all of them equally compatible with the hypothesis of isostasy. The law actually chosen is obtained by taking this function to be the lowest power of r for which the equations of the problem can be integrated without introducing any logarithmic terms. The corresponding inequalities of density are deduced from the inequalities of potential by Poisson's rule. Apart from these inequalities, the density of the layer of compensation is taken to be uniform. No assumption is made in regard to the distribution of density of the matter within the inner surface of the layer, except that it is symmetrical about the centre. Even when the law of density is settled, in the sense described above, the problem of determining the stress remains indeterminate, and it is necessary to make another assumption in order to render it determinate. The second assumption relates to the subsidiary equations which take the place occupied by stress-strain relations in the ordinary theory of elasticity. It is assumed that, apart from a hydrostatic pressure, the stress at any point in the layer is related to a vector quantity, called the "fictitious displacement," in the same way as the stress in an isotropic elastic solid body, which is slightly strained, is related to the displacement of the body, and further that the divergence of this vector vanishes, as it would do if the vector denoted an actual displacement in an incompressible solid. This assumption must be distinguished from the assumption that the earth behaves as an incompressible solid body which undergoes a slight strain.

On the basis of the two assumptions described above the equations of equilibrium of the earth are formed, and the solution corresponding to any spherical harmonic inequality is obtained. The strength which the material of the layer must have in order to support an inequality of assigned amplitude, and specified by an assigned spherical surface harmonic, is to be determined by calculating the "stress-difference," which is the difference between the algebraically greatest and least principal stresses at a point. Formulae for calculating the stress-difference answering to any zonal harmonic inequality are obtained. The solution of the equations of equilibrium is expressed in terms of a number of definite integrals, and these are not at first evaluated analytically, but the solution is completed in an approximate fashion for inequalities which are expressed by zonal harmonics of low degrees. The work is simplified by taking the mean density of the matter within the inner surface of the layer of compensation to be twice the mean density of the matter of the layer, in accordance with the known fact that the mean density of the earth is about twice the mean density of surface rocks. The stress-difference which is calculated, in accordance with the assumptions described above, as that necessary to support an inequality, specified by a spherical harmonic inequality of the first, second or third degree, and having an amplitude of 2 km., is much smaller than the tenacity, or the crushing strength, of any ordinary solid material. In particular, in the case of the harmonic of the third degree, it is less than $\frac{1}{38}$ of a metric tonne per square cm. For the support of similar inequalities of the first and second degrees smaller tenacities would be required. Further it appears that, for harmonic inequalities of the first degree, the maximum stress-difference occurs at a depth equal to one-third of the thickness of the layer of compensation, and beneath places where the gradient of the superficial inequality is steepest. For harmonics of the second and third degrees, it occurs at a slightly smaller depth, and beneath places intermediate between those where the height of the superficial inequality is greatest and those where the gradient is steepest. As the crushing strengths of various kinds of granite have been measured as $\frac{2}{5}$ of a metric tonne per square cm. and upwards, it may be concluded that no exceptional strength is needed in the materials of the layer in order to support a continental block and ocean basins, of such dimensions as those which actually exist on the earth, but these could be maintained easily by any ordinary solid material. The theory gives no support to the doctrine that the earth is a "failing structure."

In Chapter III the analysis developed in the previous Chapter is adapted to the problem of determining a stress-system by which inequalities that may be taken to represent mountains could be supported. Such inequalities may be taken to be expressed by zonal spherical harmonics of rather high degrees. The solution of the equations of equilibrium obtained in Chapter II is completed by an analytical evaluation of the definite integrals that occur in it, and by a numerical calculation of their values in the special case where the degree of the spherical harmonic concerned is 50, which is the assumed ratio of the mean radius of the earth to the mean thickness of the layer of compensation. The corresponding stress-difference is calculated approximately. It appears that it is greatest at the mean surface, and beneath places where the height of the superficial inequality is greatest. The solution is adapted to the special case of a series of parallel mountain-ranges at distances apart equal to about 400 km., and with crests at a height of about 4 km. above the valley bottoms; and, as in the previous Chapter, the work is simplified by taking the mean density of the matter within the inner surface of the layer of compensation to be twice the mean density of the matter composing the layer. The stress-difference which is calculated, in accordance with the assumptions described above, as that necessary to support such mountains is about the tenacity of sheet-lead, or a little greater than half the crushing strength of moderately strong granite. From this theory it would appear that much stronger materials are required to support existing mountains than to support existing continents. The theory is however imperfect, because existing mountains are much less well represented by means of a few zonal harmonic inequalities than existing continents.

The next three Chapters of the Essay, Chapters IV—VI, deal with the problem of Earth Tides. In Chapter IV there is given a *résumé* of the work of previous writers. As this Chapter is explanatory, and does not contain any intricate analysis, it may suffice here to state that it is designed to bring out the points which have not been elucidated in previous discussions of the problem. One of these points is the fact, disclosed by Dr Hecker's observations, that the force which disturbs a horizontal pendulum at Potsdam is a larger fraction of the tide-generating force when it acts east or west than when it acts north or south. The suggestion, made by Sir George Darwin, that this phenomenon may be due to the rotation of the earth, seemed to demand that the effect of rotation should be investigated. An investigation of this effect is undertaken and carried out in the two following Chapters. This investigation is based upon certain simplifying assumptions which may be stated here. The first assumption is that the earth may be treated as a homogeneous solid body, the material of which is absolutely incompressible, but possesses a finite degree of rigidity. The second assumption is that, apart from the action of tide-generating forces, this body is in a state of initial stress, by which its own gravitation is balanced throughout its mass, while the body rotates uniformly about an axis passing through its centre of gravity. The third assumption is that the figure of the body, when undisturbed, is an ellipsoid of revolution about this axis, the ellipticity, supposed small, being connected with the angular velocity in the same way as it would be if the material were homogeneous incompressible fluid. This involves the further assumption that the initial stress is hydrostatic pressure. These assumptions involve the hypothesis that, when the body is disturbed, the stress at any point is compounded of the initial hydrostatic pressure and an additional stress, which depends upon the displacement produced by the disturbing forces in the same way as the stress in an incompressible solid body, which is slightly strained from a state of zero stress, depends upon the relative displacements of the parts of the body. In the problem in hand the disturbing force may be taken to be the tide-generating force of the moon; and, according to a well-known analysis, this force is derived from a potential, which can be expressed as a sum of terms, every term being the product of a spherical solid harmonic of the second degree, a simple harmonic

b 2

function of the time, and a constant coefficient. The most important term of this sum is the term which answers to the principal lunar semi-diurnal tide, and the period of that simple harmonic function of the time which is a factor of this term is half a lunar day. The investigation is restricted to determining those effects which are simple harmonic functions of the time and have this period, that is to say it is conducted as if the corresponding term of the tide-generating potential were the only one.

In Chapter V it is explained how the problem may be treated approximately, in accordance with the assumptions stated above, as that of finding a correction to the known solution of the problem of determining the displacement that would be produced in a homogeneous incompressible solid sphere by constant forces, these forces being derived from a potential which is proportional to a spherical solid harmonic of the second degree. It is shown that the desired correction must consist of two parts, one depending upon the inertia of the body, and the other upon the ellipticity of its figure. These are described as the "correction for inertia" and the "correction for ellipticity." The rest of Chapter V is occupied with the working out of the correction for inertia. It is shown that the problem can be reduced to the statical problem of determining the displacement that would be produced in a homogeneous, incompressible solid sphere by a certain system of body forces. The ordinary solution of this problem cannot, however, be adapted to the question in hand, and the solution necessarily introduces new features. For the various questions which arise, and the methods adopted for dealing with them, the reader must be referred to the Chapter itself. The result which is obtained may be stated as follows:—In addition to the inequality determined by the ordinary known theory, the periodic tide-generating force raises two inequalities in the surface, one expressed by a constant multiple of the tide-generating potential, and the other expressed by a certain spherical harmonic of the fourth degree. The potential of the forces which can disturb a horizontal pendulum contains additional terms proportional to the same two spherical harmonics. The first does not alter the ratio of the forces which act in the east-west and north-south directions, but, in consequence of the presence of the second term, there is a force acting on the pendulum against the moon's force in the north-south direction, but not in the east-west direction. The sense of the correction is therefore precisely that required by Hecker's result, as, indeed, it is obvious beforehand that it should be. The magnitude of the correction is calculated on the assumption that the rigidity is about that of steel. It turns out to be so small as to be quite outside the limits of error of observation. Without any analysis it could be expected that the correction would be proportional to the quantity $a\omega^2/g$, where a denotes the mean radius of the earth, ω the angular velocity of rotation, g the mean value of gravity at the surface; but this quantity might have been multiplied by a rather large coefficient. The result found is that it is multiplied by a rather small coefficient. No probable value of the rigidity will make the coefficient large.

Chapter VI contains an investigation on similar lines of the correction for ellipticity. It is explained how the problem may be reduced to that of expressing the boundary-conditions which hold at the surface of the disturbed ellipsoid of revolution to boundary-conditions which hold at the surface of a sphere of equal volume. This reduction requires a considerable amount of rather intricate analysis, for which the reader must be referred to the Chapter itself. The result which comes out is that, in addition to the inequality produced by tide-generating forces in a homogeneous incompressible solid *sphere*, these forces raise two inequalities in the surface of the ellipsoid of revolution, one proportional to the tide-generating potential, and the other expressed by a spherical harmonic of the fourth degree. The potential of the forces which can disturb a horizontal pendulum contains two additional terms which are proportional to the same two spherical harmonics. The correction for ellipticity affects the forces which can disturb a horizontal pendulum in two ways. In the first place the additional terms in the potential must be taken into account. In the second place the forces must be derived from the potential by forming derivatives in the directions of meridians and parallels drawn on the ellipsoid of revolution, not on the sphere of equal volume. It appears that the forces in both directions, east-west as well as north-south, are subject to correction, and both are altered by amounts which are multiples of the ellipticity. The corrections are calculated for the latitude of Potsdam on the supposition that the rigidity is about that of steel. It is found that both forces are increased, the east-west component less than the north-south component. Thus the sense of the correction is opposite to that required by Hecker's result. It is obvious beforehand that the magnitude of the correction should be proportional to the ellipticity, but, without investigation, the sense of the correction could not be guessed, and its magnitude might have been such that the ellipticity would have to be multiplied by a rather large coefficient. The coefficient by which it is actually multiplied differs but little from unity, and therefore the correction falls almost outside the limits of error of the observations.

From these investigations it appears to be unlikely that the effect observed by Hecker is due to the rotation of the earth. Although simplifying assumptions are introduced, it is improbable that they can affect the sense or that they can affect very much the order of magnitude of the corrections which should be made on account of the rotation. By making similar assumptions, and taking account of deviations from the spherical figure other than the ellipticity, and expressed by spherical harmonics of low degrees, it would be possible to work out similar corrections in order to express the effects which might be due to the distribution of land and water; but, after what has been done, it would seem to be very improbable that such effects would be large enough to be observed. But, if the cause of the observed effect is not to be sought either in the rotation of the earth or in the distribution of land and water, it may be suggested that it is possibly due to the attraction of the tide-wave in the North Atlantic and its pressure on the bed of the ocean. A rough calculation sketched at the end of Chapter VI indicates that these causes may be of about the right order of magnitude to produce the observed result if they are timed properly.

After the Essay was in type my attention was called to an investigation of Earth Tides which had been conducted by A. Orloff * at Yurief (Dorpat). He used two horizontal pendulums, hung in the meridian plane and the plane of the prime vertical, and supported in a different way from those used by Hecker, and he adopted a different method of reducing his observations. The results which he found are similar to those found by Hecker. The lunar semi-diurnal parts of the observed effects show a close agreement of phase with the corresponding part of the tide-generating force, the observed deflexions of the pendulums are on the average a little less than two-thirds of what they would be if the earth were absolutely rigid, and the force that deflects a horizontal pendulum at Dorpat is a larger fraction of the tide-generating force when it acts east or west than when it acts north or south. If the results were expressed by a diagram, as on p. 55, the inner curve would be flatter than the outer in the same direction as in that diagram, but not nearly so much, the ratio of the major axes of the inner and outer curves at Dorpat being about 0·65, and that of the minor axes about 0·55. These results appear to be in accordance with the above explanation of the phenomenon observed by Hecker; for a horizontal pendulum at Dorpat (Lat. about 60° N., Long. about 27° E.) would be much less affected by the tide-wave in the North Atlantic Ocean than a similar instrument at Potsdam (Lat. about 53° N., Long. about 13° E.).

The next four Chapters of the Essay, Chapters VII—X, are devoted to the Dynamics of a Gravitating Compressible Body of Planetary Dimensions. In the classical solutions of the problems of corporeal tides and the vibrations of the earth, considered as a spherical solid body, the assumption was made that the substance could be treated as incompressible. The remarkable effects that could be caused by compressibility were first brought to light by Jeans, but he found it necessary to regard the self-gravitation of the body as balanced in the undisturbed state by external body forces, instead of being balanced, as it must be, by internal stress. A theory of the balancing of the self-gravitation by "initial" stress, which was worked out in detail by the author, is now found to stand in need of modification. In that theory it was assumed that the stress at any point of the body when disturbed consists

* A. Orloff, "Beobachtungen über die Deformation des Erdkörpers unter dem Attraktionseinfluss des Mondes an Zöllnerschen Horizontalpendeln." *Astr. Nachrichten*, Nr. 4446, Bd. 186 (October, 1910).

of two stress-systems:—the *initial* stress and the *additional* stress. The initial stress was taken to be hydrostatic pressure; and the additional stress was taken to be related to the strain, by which the body passes from the undisturbed state to the disturbed state, by the same formulae as hold in an isotropic elastic solid body which is slightly strained from a state of zero stress. The modification which it is now proposed to make in the theory consists in a different way of assigning the value of the hydrostatic pressure (constituting the initial stress) at a point of the disturbed body. When a small portion of the undisturbed body around a geometrical point P is displaced so as to become a small portion of the disturbed body around a neighbouring geometrical point Q, it suffers dilatation and distortion. In the Essay it is regarded as carrying its initial pressure with it and acquiring an additional stress which depends upon the dilatation and distortion. Thus the stress at Q in the disturbed body is here regarded as compounded of the hydrostatic pressure at P in the undisturbed body and a stress correlated in the usual way with the displacement. In my previous theory the hydrostatic pressure at the geometrical point Q in the disturbed body was taken to be the same as the hydrostatic pressure at the same geometrical point Q (not P) in the undisturbed body. In Chapter VII this modification of the theory is explained in detail, and the equations of vibratory motion are formed in accordance with it. For simplicity it is assumed that the body in the undisturbed state is homogeneous. A typical solution of the equations is found. The typical solution contains a single spherical solid harmonic, and more general solutions can be obtained by a synthesis of typical solutions. It appears that the functions of the radius that are involved in the typical solutions are already well-known, but the parameters, by which, in these functions, the radius is multiplied, are the roots of a quadratic equation, the coefficients in which involve the frequency of vibration, the elastic constants, and the density of the body. The special equations which express the condition that the bounding surface is free from traction are obtained. A short discussion is given of the special formulae which hold when the displacement is purely radial.

The first application of the general theory of Chapter VII is to determine the Effect of Compressibility on Earth Tides. This is discussed in Chapter VIII. Solutions of this problem have been obtained by various writers, but in none of them is proper account taken of the initial stress. It was, as a matter of fact, by an attempt to apply my previous theory of initial stress to the problem that I found that that theory needed modification; for it led to the surprising result that the (compressible) earth should yield less to tide-generating forces than it would do if it were incompressible. A new solution of the problem is here obtained on the basis of the modified theory developed in Chapter VII, and the results that are obtained are entirely in accordance with what might be expected. The problem is treated as a statical one. The earth is treated as an elastic solid body, which, in the undisturbed state, is homogeneous and bounded by a spherical surface, and it is assumed that, in the undisturbed state, the gravitation of the body is balanced throughout its mass by hydrostatic pressure. The rigidity of the material composing the body is taken to be about that of steel, and it is found that, if the Poisson's ratio of the material is $\frac{1}{3}$, the height of the corporeal tide is increased, on account of the compressibility, by about 10 per cent. of itself, while, if the Poisson's ratio of the material is $\frac{1}{4}$, the increase is about 20 per cent. It is known that the earth actually yields less to tide-generating forces than it would do if it were homogeneous and incompressible, the rigidity being supposed to be adjusted in accordance with the assumptions of homogeneity and incompressibility and the results of horizontal pendulum observations. It is also known that the effect of heterogeneity (the density increasing from surface to centre) is to diminish the yielding, while the effect of compressibility is now found to be an increase of the yielding, as could be expected beforehand. It appears therefore that heterogeneity of density produces more important effects in modifying the resistance which the earth offers to disturbing forces than does the compressibility of the substance.

The next application of the general theory of Chapter VII is to the problem of Gravitational Instability. The problem arises from the fact that gravitating matter tends to condense towards any part where the density is in excess of the average. In a body of ordinary size this tendency is checked by the elastic resistance of the body, but in a body of planetary dimensions the resistance may be insufficient to hold the tendency in check. For example, it might be impossible for a body of the size and mass of the earth to exist in a homogeneous state. If such a body existed for an instant it might be unstable, and then the slightest change of density in any part would be followed by a large progressive change which would only come to an end when equilibrium in a new configuration, differing appreciably from the original one, was reached. To investigate the question for a body of given constitution in a given configuration, we have to begin by forming the equations of vibration of the body, supposed to be slightly disturbed from that configuration, and then to seek the conditions that must be satisfied if there can be a vibration of zero frequency. For the sake of simplicity we may begin by considering a body of the same size and mass as the earth to be formed of homogeneous material, and seek the conditions that that body may be gravitationally stable. A first solution of this problem was given some years ago by J. H. Jeans. He avoided all questions of initial stress by assuming that, in the undisturbed state, the self-gravitation of the body was balanced by an external system of body forces. He found that when the body is disturbed, so that the radial displacement at a point is proportional to a spherical harmonic of an assigned degree, the condition for the existence of a vibration of zero frequency became a transcendental equation to determine a certain modulus of elasticity of the material as a multiple of $g\rho_0 a$, where g denotes the mean value of gravity at the surface, ρ_0 the density in the undisturbed state, and a the mean radius. The equation in question contained also, as a parameter, the ratio of the rigidity to the incompressibility. It was found that the elastic resistance necessary to render the body stable in regard to disturbances specified by spherical harmonics of the first degree would be sufficient to render it stable in regard to disturbances specified by spherical harmonics of any higher degree. The conditions necessary to secure stability in regard to radial disturbances were not discussed. It was taken as probable that, if a homogeneous body with certain elastic constants is unstable in respect of disturbances specified by spherical harmonics of any degree, a heterogeneous body, with not very different values of the *average* elastic resistances to compression and distortion, would be unstable in respect of the same type of disturbances. It was found that in accordance with the assumptions here described the earth would be gravitationally stable if it were homogeneous, the average rigidity and incompressibility of its substance being those deduced from the theory of seismic waves; but it was suggested that, if the earth was once in such a condition as regards elastic resistance to compression and distortion that, if homogeneous, it would have been gravitationally unstable, it should now exhibit some traces of this past state, and that such traces might be found in the existing distribution of land and water on the surface of the globe. In particular, it was pointed out that the geographical fact of the land and water hemispheres was in accordance with the result that the instability would manifest itself in respect of disturbances specified by spherical harmonics of the first degree.

A second solution of the problem was afterwards given by the present writer on the basis of that theory of initial stress which has already been mentioned. In its main results this solution did not differ very much from that given by Jeans, but one result that was found was that if the body, supposed homogeneous, was unstable at all it would be unstable as regards radial displacements. On this theory therefore, if the land and water hemispheres could be traces of a past state, in which the earth would have been unstable unless the mass of one hemisphere had been greater than that of the other, it would be necessary that the argument stated above as to the average values of the elastic resistances in a heterogeneous body should be sound. At the same time it was shown how the rotation could be taken into account. The modes of vibration of a rotating body in the form of a planetary ellipsoid can be correlated with those of a sphere at rest; and it was proved that to such modes of the sphere as are specified by spherical harmonics of the first degree there would answer, in the ellipsoid, modes specified by harmonics of the first, second, and third degrees properly superposed. It was pointed out that the earth does exhibit prominent inequalities of these degrees. The idea that the main features in the shape of the earth might be due to its having once been in such a state that, if its mass had been arranged symmetrically around its centre, it would have been unstable, seemed to be of sufficient interest to make it desirable to obtain a fresh solution on the basis of the new theory of initial stress. This is given in Chapter IX of this Essay. The new results throw some further light upon the question. It remains true, in so far as the problem has been examined, that elastic resistances which are sufficient to secure stability in respect of disturbances specified by spherical harmonics of the first degree are also sufficient to secure stability in respect of disturbances specified by spherical harmonics of any higher degree; and it also remains true that, if the ratio of the elastic resistances to compression and distortion is neither large nor small compared with the values which it has for ordinary solid materials, subjected to experiment at the earth's surface, then elastic resistances which are sufficient to secure stability in respect of radial displacements are amply sufficient to secure stability in respect of displacements specified by spherical harmonics of the first or any higher degree. But the interesting result is found that, if the rigidity is rather small compared with the resistance to compression, the body may be unstable in respect of displacements specified by spherical harmonics of the first degree although stable as regards radial displacements. This result is distinctly favourable to the hypothesis that the division of the earth's surface into a land hemisphere and a water hemisphere may be a survival from a past state in which a symmetrical arrangement of the matter about the centre would have been unstable. The superficial displacements which occur in any mode of vibration of the body, whether of zero frequency or not, are associated with inequalities in the density, and these are of the same spherical harmonic type as the radial inequality of figure, so that, if the actual inequalities of the figure of the earth, or any large part of them, can be traced to the cause under consideration, the elevations and depressions of the surface should be compensated by defects or excesses of density in the underlying material, as is assumed in the theory of isostasy. It seemed therefore to be worth while to examine the distribution of density which a sphere would take up if it were unstable when homogeneous, with no tendency to condense towards the centre (stability as regards radial displacements), but with a tendency to sway to one side, so that one hemisphere would have a preponderant mass (instability as regards displacements specified by spherical harmonics of the first degree). The result is not very favourable to the hypothesis. It is found that the inequalities of density would be deep-seated, instead of being practically confined to a superficial layer, as the doctrine of isostasy lays it down that they should be. It must, however, be understood that the results have been obtained by making several simplifying assumptions. The body of which the gravitational stability is examined is assumed to have the same size and mass as the earth, it is assumed to be homogeneous as regards the distribution of its density and as regards its elastic resistances to compression and distortion, it is assumed that, in the undisturbed state, the initial stress by which the self-gravitation of the body is balanced throughout its mass is simply hydrostatic pressure. It is possible that the result might be different if the problem could be solved for a body of which the density increases from surface to centre and the elastic resistances to compression and distortion are different at different depths. It is also possible that the part of the earth's volume within which there is compensation of the superficial inequalities of figure by inequalities of density may now be more restricted than it was once, or, in other words, that in the course of long ages an inequality which was once appreciable at great depths may have been progressively diminishing, and diminishing faster near the centre than near the surface. This suggestion is, however, rather speculative. In Chapter IX the problem of gravitational instability is solved for an initially homogeneous sphere, the material of which is supposed to possess resistances to compression and distortion which are in one or other of certain definite ratios, and the displacements of which, with their accompanying inequalities of density, are taken to be either symmetrical about the centre or specified by spherical harmonics of the first, second, or third degree.

Chapter X is devoted to a determination of the Normal Modes of Vibration of a Gravitating Compressible Sphere. The sphere is taken to be homogeneous in the undisturbed state, and to be of sufficient rigidity to secure gravitational stability. The modes fall into two classes in exactly the same way as those of a sphere which is free from gravitation; and the modes of the first class, characterized by the absence of radial displacement, are unaffected by gravitation. If the rigidity is small enough, the modes of the second class are of two kinds, which may be described roughly as vibrations governed mainly by elasticity and vibrations governed mainly by gravity. The latter have the smaller frequencies, and the two kinds of modes are described in the Essay as being of "quick types" and "slow types" respectively. If the rigidity has one or other of a certain determinate set of values, there may be vibrations of intermediate types. The particular case of vibrations specified by spherical harmonics of the second degree is worked out in detail, and it is found that, for a body of the size and mass of the earth, there are no vibrations in modes of this degree, which are of slow or intermediate types, if the Poisson's ratio of the material is $\frac{1}{4}$, and the rigidity is great enough to secure stability in respect of radial displacements. It is known that, for a homogeneous sphere which is free from gravitation, the gravest of all the normal modes of vibration is of a type in which the sphere becomes an harmonic spheroid of the second degree, and that, if the sphere has the same size and mass as the earth, and the material is incompressible and as rigid as steel, the period of these vibrations is about 66 minutes. It is known also that, if gravitation is taken into account, but the other conditions, including that of incompressibility, are maintained, the period is reduced to about 55 minutes. It is now found that, when account is taken of compressibility as well as gravitation, the period is almost exactly one hour, the Poisson's ratio of the material being taken to be $\frac{1}{4}$.

The theory of the vibrations of a body of planetary dimensions leads naturally to a discussion of the theory of Seismic Waves. This theory is considered in Chapter XI. The Chapter begins with a description of the most important steps that have been taken in the interpretation of seismic records, accompanied by a statement of the chief points in respect of which the existing theory seems to require extension. These relate to the oscillatory character of the recorded movements and the nature of the Large Waves. To elucidate these matters a series of problems are solved. The first of these problems is to determine the laws of transmission of waves through a gravitating compressible planet. On the basis of the dynamical theory developed in Chapter VII it is shown that, if the planet in the undisturbed state is spherical and homogeneous, waves of pure distortion, characterized by rotation of the elementary portions without change of volume, can be transmitted in precisely the same way as if the body were free from gravitation, but that the law of propagation of dilatational waves is affected by gravitation. In the first place the waves cannot be purely dilatational, but there must be a small rotation accompanying the change of volume. In the second place the velocity of propagation is not constant, but it depends partly on the locality and partly on the wave-length, the shorter waves travelling faster than the longer ones. This result indicates dispersion, and suggests that the displacement observed at any place during the preliminary tremors should be oscillatory, with gradually increasing intervals between successive maxima. Both these characters are in accordance with observation.

The second problem discussed in Chapter XI is that of the limiting form to which the frequency equation, obtained in Chapter X for a vibrating sphere, tends when the degree of the spherical harmonic involved is high. The result gives the wave-velocity with which a train of simple harmonic straight-crested waves can be transmitted over the surface without penetrating far into the interior. The sphere being assumed to be homogeneous when undisturbed, the waves must, except for a small correction depending on gravity, belong to the type discovered by Lord Rayleigh, and since known as "Rayleigh-waves." The correction has the effect of introducing a slight amount of dispersion, and if, as is probable, the average Poisson's ratio of surface rocks is not less than $\frac{1}{4}$, the wave-velocity increases slightly as the wave-length increases.

The observed fact that, in the earlier phases of the large seismic waves, the motion of the ground is mainly in a direction at right angles to the direction of propagation of the waves, cannot be brought under any theory by which the Large Waves are identified with Rayleigh-waves; and, so long as the earth is treated as homogeneous, it is theoretically impossible for waves of any other type to be transmitted over the surface without penetrating far into the interior. To introduce the possibility of waves which shall have the two characters: (1) transverse horizontal movement, (2) superficial transmission, it has been proposed by more than one writer to assume that the body of the earth is covered by a rather thin layer of matter having different mechanical properties from the matter beneath it; but the conditions necessary to secure these characters in the transmitted waves appear not to have been investigated hitherto. The third problem discussed in Chapter XI is that of the transmission of transverse waves through a superficial layer, such waves to be practically confined to the layer, the motion in the subjacent material diminishing rapidly as the depth increases. The problem is discussed under the simplifying assumptions that the surface may be treated as plane and the waves as straight-crested; and it is proved that the essential condition for the existence of waves having the desired characters is that the velocity of simple distortional waves in the layer should be decidedly less than that in the subjacent material. It is proved further that the wave-velocity of a simple harmonic wave-train cannot be less than the velocity of simple distortional waves in the layer, and that it increases with the wave-length, approaching the velocity of simple distortional waves in the subjacent material as a limit. The analogy of waves on deep water, the only example of waves subject to dispersion which has been worked out fully, suggests that, on account of the relation between wave-velocity and wave-length, the disturbance received at a place should be oscillatory, and the intervals between successive maxima should diminish as time goes on. This result is in accordance with observation of the earlier phases of the Large Waves.

If we invoke a superficial layer, or crust of the earth, to help us to explain the phenomena presented by the earlier phases of the Large Waves, we must not neglect to consider the effect of such a layer in modifying the laws of transmission of those superficial waves in which the horizontal displacement is parallel to the direction of propagation. The problem of the transmission of such waves through a superficial layer is the fourth problem considered in Chapter XI. The surface is treated as plane, the material as incompressible, and the waves as straight-crested. It is proved that there necessarily must be a class of waves similar in type to Rayleigh-waves, and that simple harmonic waves of this class have a wave-velocity which, for very short waves, is the velocity of Rayleigh-waves, but increases as the wave-length increases, approaching the velocity of simple distortional waves in the layer as a limit. A result of some theoretical interest is that these waves, analogous to Rayleigh-waves, may not be the only type of waves which, while they do not penetrate far beneath the layer, have their horizontal displacements parallel to the direction of propagation. Under suitable conditions there may be a second type. The condition that waves of the second type may exist is found to be that the difference between the velocities of simple distortional waves in the two media should be small. As this condition is opposed to the condition which was found to be essential if waves with transverse horizontal displacement are to be transmitted through the layer without penetrating far into the subjacent material, the possible existence of the new type of waves under suitable conditions would seem to have no bearing on the interpretation of seismic records. The waves in the layer which are analogous to Rayleigh-waves are subject to slight dispersion, both on account of gravity, as was seen in the solution of the second problem, and on account of the change of mechanical properties at the under surface of the layer, and, on both accounts, the wave-velocity of a simple harmonic wave-train increases as the wave-length increases. The analogy of waves on deep water leads us to expect that the movement which can be observed at any place should be oscillatory, and that the intervals between successive maxima should diminish as time goes on. These results are in accordance with observations of the central phases of the Large Waves.

The view as to the nature of the Large Waves which is put forward in the Essay is that these waves are of two distinct types. The motion of either type is regarded as an aggregate of motions corresponding to standing simple harmonic waves, which combine to form progressive waves, or, what comes to the same thing, as an aggregate of motions transmitted by simple harmonic wave-trains, the period of any simple harmonic wave depending upon the wave-length according to rather complex laws. The waves of the first type are waves of transverse displacement, transmitted through a superficial layer, and not penetrating far into the matter beneath it. The second type of waves are analogues of Rayleigh-waves, and differ from these only by the modifications that are necessary on account of gravity and the change of mechanical properties at the under surface of the layer. The wave-velocities of simple harmonic waves of both types increase as the wave-lengths increase, and there is for each type a maximum and a minimum wave-velocity, but the minimum of the first type is identical with the maximum of the second. Since the propagation is practically two-dimensional, minima of wave-velocity are less important than maxima, for it is known that waves propagated in two dimensions are prolonged in a sort of "tail," even in the simplest case, that in which all simple harmonic wave-trains travel with the same velocity. If the above-stated view as to the nature of the Large Waves is correct, we should expect that there would be a marked change of type in the observed movement, and that this change would occur at different places at such times as would correspond to the passage over the surface of a phase travelling with the velocity of simple distortional waves in the layer, the horizontal displacement before the change being mainly transverse to the direction of propagation, and after the change mainly parallel to this direction. The existence of such a marked change of type is well established by observation. The proposed view would also account for the facts in regard to the gradual changes in the observed "periods." It suggests, in fact, that these so-called periods are not genuine periods of simple harmonic wave-trains, but intervals of time separating successive instants at which the displacement attains a maximum, the displacement being an aggregate of simple harmonic displacements. This suggestion also furnishes a possible explanation of the apparent discrepancy between theory and observation which arises from the fact that, whereas in Rayleigh-waves the vertical displacement is larger than the horizontal, the vertical displacements observed by seismologists are always smaller than the horizontal. In an aggregate of standing simple harmonic waves, in each of which the theoretical relation between the two components of displacement holds, but the periods are not proportional to the wave-lengths, the relative magnitudes of the maxima of the two components may depend upon the initial circumstances.

The results obtained in Chapter XI, like those obtained in Chapters II and III, suggest that there is a veritable "crust of the earth," or superficial layer, the mechanical properties of which differ from those of the matter composing the interior portions. The results are, in fact, obtained by assuming that such a layer exists. But a little consideration shows that the results could not be very different if the constitution were such that all the quantities, density, rigidity, and so on, by which it is specified, were expressed by continuous functions of the depth, or, more generally, by continuous functions of the position of a point within the earth. Heterogeneity there certainly is, and the simplest heterogeneous constitution that can be imagined is that specified by a nucleus and a superficial layer; but a constitution specified by continuously varying quantities might very well be quite as consistent with the results of observations made at the surface as this discontinuous structure, especially if the quantities should vary rather rapidly near the surface and more gradually at greater depths.