# CHAPTER I

## THE DISTRIBUTION OF LAND AND WATER

**1.** The purpose of this Chapter is to explain how the surface of the earth may be represented by means of spherical harmonics, and to estimate the amplitudes of those harmonics which are concerned in the representation of the most important features.

If we wish to discuss the distribution of land and water with greater precision than is customary in books on Geography we may adopt either of two points of view:—those of Mathematical Geography and Geophysics. In Mathematical Geography the object aimed at is a precise geometrical or analytical description of the actual shape of the earth's surface. In Geophysics we seek the causes which have led to the shape being what it is. Before we can make any progress with the geophysical enquiry we must know what the shape of the earth's surface really is. To know this is to know the equation of the surface referred to some assigned axes and origin. But here it is necessary to distinguish one from another various surfaces which are all equally regarded as being "the surface of the earth" when different matters are discussed. The *visible* surface is that surface on which the atmosphere rests, the matter of which it is the bounding surface being land in some parts and water in others. The surface of the ocean is disturbed by tidal and other waves, but the mean surface of the ocean, which is a level surface of the earth's attraction (the rotation being taken into account), enables us to define the surface which is called the *geoid*. This is a closed surface which is everywhere a level surface of the earth's gravity modified by the rotation, and coincides with the mean undisturbed surface of the ocean wherever there is ocean. In treatises on the "Figure of the Earth" the problem to which most attention is paid is the problem of determining the geoid. The height of any place above the geoid is its "height above sea-level," and the depth of the bottom of the sea at a spot "below sea-level" is the depth below the geoid. The geoid is the surface that is always used as a zero in determining levels. But when we speak of the surface of the earth we may, and often do, mean a surface which is neither the visible surface nor the geoid, but the surface of the land, in places where there is land, and the surface at the bottom of the sea, in places where there is sea. A name sometimes used for this surface is the *lithosphere*, and this name will be adopted here*. We regard the question that is posed when precise information as to the shape of the earth is sought as the question of determining the shape of the lithosphere. To know the shape of the lithosphere we must first find the shape of the geoid, next find the height of every spot of land above sea-level, and then find the depth of the sea at every locality in the sea (determined by latitude and longitude). This is the course that is necessary in practice, but an abstract geometrical description need not introduce the sea at all, it would be concerned only with determining the shape of the somewhat irregular round body on which the ocean and the atmosphere rest. The result of the enquiry, if it could be obtained, would be expressed by writing down the equation of the lithosphere, which is the surface of this somewhat irregular round body.

**2.** The shape of the geoid is known with considerable exactness. It is very nearly an oblate ellipsoid of revolution of ellipticity 1/297, the axis of revolution being the polar axis of the earth. For determining the lithosphere the best origin is the centre of this ellipsoid, and the most appropriate coordinates are polar coordinates, the co-latitude and longitude of a point. Let these be denoted by $r$, $\theta$, $\phi$. The equation of the lithosphere would express $r$ as a function of $\theta$ and $\phi$.

Let the equation of the nearly spherical harmonic spheroid of the second degree which most nearly coincides with the geoid be

$$r = a_0 - \tfrac{2}{3} a_0 \epsilon_0 P_2(\cos\theta),$$

where $a_0$ denotes the mean radius ($6{\cdot}37 \times 10^8$ cm.), $\epsilon_0$ the ellipticity of the meridians ($\frac{1}{297}$), and $P_2$ the zonal surface harmonic of the second degree given by the formula

$$P_2(\cos\theta) = \tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}.$$

The actual surface of the geoid must be expressible by an equation of the form

$$r = a_0 - \tfrac{2}{3} a_0 \epsilon_0 P_2(\cos\theta) + F_0(\theta, \phi),$$

in which $F_0(\theta, \phi)$ could be expanded, if it were known, in a series of surface harmonics, and the series would contain no term in $P_2(\cos\theta)$. In other words $F_0(\theta, \phi)$ must be such that the equation

$$\int_0^{2\pi} d\phi \int_0^{\pi} P_2(\cos\theta) F_0(\theta, \phi) \sin\theta \, d\theta = 0$$

is satisfied. The function $F_0(\theta, \phi)$ is small compared with $a_0$ for all values of $\theta$ and $\phi$. What is more important is that it is small compared with $a_0\epsilon_0$. The deviations of the geoid from an harmonic spheroid of the second degree

* The word "lithosphere" is sometimes used as a synonym for the "crust of the earth," whatever that may be.

are everywhere quite trivial compared with the deviations of the harmonic spheroid (of the second degree) of closest fit from a sphere of equal volume.

**3.** In like manner the equation of the lithosphere must be of the form

$$r = a - \tfrac{2}{3} a_0 \epsilon_0 P_2 (\cos \theta) + F(\theta, \phi),$$

in which $a$ denotes the mean radius of the lithosphere. It is known that $a < a_0$, for the volume within the lithosphere is slightly smaller than that within the geoid. The distance denoted by $a_0 - a$ is between 3 and 4 km., and the level at depth $a_0 - a$ below the level of the sea is known as "mean sphere level." The elevation, or depression, of the lithosphere above, or below, the geoid is expressed by the formula

$$F(\theta, \phi) - F_0(\theta, \phi) - (a_0 - a),$$

and we may say that where this expression is positive there is land, and where it is negative there is sea. But this statement needs qualification if there is land below sea-level; the land around the shores of the Caspian Sea may serve as an example. Since all the values of $F_0(\theta, \phi)$ are small compared with $a_0 - a$, the main features of the distribution of land and water are seen to be expressed by the function $F(\theta, \phi)$, which represents the elevation of a point on the lithosphere above mean sphere level (depression when the function is negative). Since, however, $F(\theta, \phi)$ may be positive without being so great as $a_0 - a$, large tracts of the lithosphere which ought to be regarded as places of elevation, because they are above mean sphere level, are actually submerged. This description applies not only to the whole of the continental shelf but also to the beds of nearly land-locked seas, such as the Mediterranean, and even to some parts of the open ocean.

**4.** The elevated portions of the lithosphere, the portions that are above mean sphere level, form the "continental block*," and the remaining parts the "ocean basins." The lithosphere protrudes beyond the mean sphere $r = a$ in some parts and lies inside it in others, but the characteristic property of the mean sphere is that the volume contained between the sphere and the protruding part of the lithosphere is equal to the volume contained between the sphere and the parts of the lithosphere which lie inside it. The continental block is believed to form a single continuous region† of elevation, the great continents being all connected together beneath the sea at depths which do not much exceed 3 km. at any place. A map of the world at mean

* The name is often given to the portions of the lithosphere which are actually land or continental shelf (within the hundred-fathom line). The usage adopted in the text seems more appropriate to the present discussion.

† This was not the case in the map of the world at mean sphere level drawn by H. R. Mill in *The Scottish Geographical Magazine* (Edinburgh), vol. VI. 1890, p. 184, where the Antarctic land is shown separated from the rest of the block; but the writer has been informed by Dr Mill that the depth of mean sphere level below sea-level was underestimated in 1890.

sphere level, with merely local irregularities smoothed out, shows an extremely simple plan. The contour line at this depth gives a very good indication of those features of the shape of the lithosphere which must be regarded as the most important. It seems to the writer that a geometrical plan of the earth, to be acceptable, must show a contour line actually or nearly coinciding with this line. In other words the function denoted above by $F(\theta, \phi)$ must vanish at all points of a curve which lies everywhere close to this curve, and it must be positive on the side towards the continental block, and negative on the side towards the two great ocean basins.

**5.** Now whatever the function $F(\theta, \phi)$ may be, it can be expanded in a series of surface harmonics, and the most important features of the shape ought to be represented by the first few terms of the series. We should expect therefore that an expression, consisting of surface harmonics of the first three or four degrees, could be constructed to vanish along a curve which nearly coincides with the outline of the continental block, and to be positive within the block. It has been shown that the first three degrees suffice for the purpose*. Thus a first approximation to the shape of the lithosphere is given by a formula of the type

$$r = a + S_1 + S_2 + S_3,$$

where $S_1$, $S_2$, $S_3$ denote surface harmonics of degrees indicated by the suffixes.

**6.** According to the paper cited above we may take

$$S_1 = \{(16{\cdot}5)\cos\phi + (9{\cdot}5)\sin\phi\}\sin\theta + 8\cos\theta,$$

$$S_2 = \{(1{\cdot}5)\cos\phi + (2{\cdot}5)\sin\phi\}\sin 2\theta + \{(-7)\cos 2\phi + (-4)\sin 2\phi\}\sin^2\theta + \{3\cos 2\theta + 1\},$$

$$S_3 = (-5)\{\cos 3\theta + (0{\cdot}6)\cos\theta\} + \{(-1{\cdot}25)\cos\phi + (-0{\cdot}5)\sin\phi\}(\sin\theta + 5\sin 3\theta) + (6{\cdot}5)\sin 2\phi(\cos\theta - \cos 3\theta) + \{(-0{\cdot}25)\cos 3\phi + (3{\cdot}5)\sin 3\phi\}(3\sin\theta - \sin 3\theta).$$

$S_1$ is a zonal harmonic with a maximum value (about 20·5) near to the point $\theta = 67°$, $\phi = 30°$. $S_2$ is not exactly a zonal harmonic, but is very nearly a zonal harmonic having a maximum numerical value (about 10) near to $\theta = 105°$, $\phi = 15°$. At this point, and at its antipodes, $S_2$ is negative. $S_3$ is not exactly a zonal harmonic, but does not differ much from a zonal harmonic having its pole near to $\theta = 75°$, $\phi = 35°$; the maximum value (at the pole) is about 25. The actual values of the coefficients in the expressions for $S_1$, $S_2$, $S_3$ do not express any fact about the shape, as the scale of the inequalities was arbitrary in the paper cited. The ratios of the coefficients are alone significant. It appears that the harmonics of the first and third degrees are much more important than those of the second degree, the

* A. E. H. Love, *Proc. R. Soc. Lond.* (Ser. A), vol. 80, 1908, p. 555.

harmonics of the third degree slightly more important than those of the first degree. The greatest elevations and depressions corresponding to harmonics of the first and third degrees occur near the same places (Northern Africa and its antipodes in the Pacific Ocean). Now the average depth of the ocean may be taken roughly to be about 4 km.*, and the average height of the continents above sea-level is less than $\frac{1}{2}$ km., so that, if we allow that a large part of the continental elevations and the oceanic depressions can be represented by harmonics of the first three degrees, it would seem that an amplitude of 2 km. would be more than sufficient for any one harmonic, since this gives an elevation of 4 km. for the highest point above the lowest point in the case of any harmonic of uneven degree.

It is proper to observe that, although the formulae given in the paper cited above furnish a fair representation of the *outline* of the continental block, they do not adequately represent the amount of elevation or depression at a place. In particular, they make the Pacific Ocean much deeper than any other ocean, and they make the northern part of the continent of Africa much higher than any other land. This defect does not seem to render them ineffective as approximations to the first three terms of the series by which the radius of the lithosphere would be expressed if it were known accurately.

* A more exact estimate is not needed for the purpose in hand.

# CHAPTER II

## THE PROBLEM OF THE ISOSTATIC SUPPORT OF THE CONTINENTS

**7.** The existence of the continental elevations and oceanic depressions proves decisively that the earth as a whole cannot be in a state of fluid equilibrium, that is to say a state such that the stress at any point, across any plane passing through the point, is normal to the plane. For, if this were so, the stress at any point would be the same in all directions round the point, or it would have the character of hydrostatic pressure; and then the surfaces of equal pressure would coincide with the equipotential surfaces, and, in particular, the surface of the earth would be an equipotential surface, everywhere at right angles to the direction of gravity. To an observer anywhere on the earth's surface the ground would appear to be a level plain. Since this is not the case, it is certain that the stress at a point within the body of the earth cannot have the character of hydrostatic pressure; there must be tangential tractions as well as normal tractions.

**8.** The question to be discussed is: How are the great inequalities, the continental elevations and the oceanic depressions, supported? The idea which one naturally forms is something like this: one imagines a perfectly spherical or spheroidal solid model of the earth to be deformed by paring away material from the parts that are to form the oceanic depressions and heaping it up to form the continental elevations. In fact, one naturally thinks of the continental block as if it were stuck on to the earth like a postage-stamp on an envelope. But this notion is quite erroneous. In the first place the attraction of the block would probably be so great that the sea would be drawn up over it and it would be almost submerged*. In the second place it is doubtful if the material of which the earth is composed could be strong enough to stand the strain†. But the decisive reason for rejecting this notion is that the values of gravity, as observed at places in the interior of the continents and in the open ocean, or on oceanic islands, cannot be reconciled with the values that would be deduced by assuming the notion to be correct‡.

* This is the result obtained by F. R. Helmert, *Math. u. phys. Theorien d. höheren Geodäsie*, Teil 2, Kap. 4 (Leipzig, 1884).

† This is the general result of the calculation made by G. H. Darwin, "On the stresses caused in the interior of the earth by the weight of continents and mountains," *Phil. Trans. R. S.* vol. 173 (1882), revised in *Scientific Papers*, vol. II. p. 457 (Cambridge, 1908).

‡ See § 38 of Teil 2 of the treatise by Helmert cited above, and also his article "Die Schwerkraft u. d. Massenverteilung d. Erde" in *Ency. d. math. Wissenschaften*, Bd. VI. Teil I. Nr. 7 (Leipzig, 1910).

Most writers who have rejected the idea above described have supposed that the superficial inequalities of shape are correlated with internal inequalities of density, so that the elevated portions are, as it were, floated up and kept in position by hydrostatic pressure. This hypothesis under various forms is known as the "hypothesis of compensation" or the "hypothesis of isostasy," and is ascribed to J. H. Pratt. Certain anomalies observed in the measurements of gravity in Northern India were interpreted by him as pointing to a compensation of the mass of the Himalaya by a comparatively light layer of matter beneath them; and he also pointed out that the geographical fact of the land and water hemispheres indicated a displacement of the centre of gravity of the earth from the centre of the geoid towards the middle of the Pacific Ocean*. The hypothesis was adopted by Helmert in 1884 for the reason already stated as decisive against older notions. In recent times it has been revived and developed very much in America by C. E. Dutton† and by O. H. Tittmann and J. F. Hayford‡. It has also been tested by Helmert§ in discussions of various series of geodetic observations. The forms of the hypothesis which have proved to be adequate for Geodesy seem to be not quite sufficiently precise for the purpose of determining a system of stresses by which the inequalities can be supported, and a rather special form will presently be proposed. It must, however, be understood that the special form is introduced for the sake of analytical simplicity rather than physical appropriateness.

## Special form of the hypothesis of isostasy.

**9.** According to the hypothesis of isostasy, as developed by Hayford, the earth consists of a central core coated over with a rocky crust. Within the core it is assumed that there are no tangential stresses, but the matter is in a state of fluid equilibrium; the tangential stresses necessary to maintain the continents and mountains are supposed to be confined to the crust. Within the thickness of the crust the mass is assumed to be so distributed as not to affect the hydrostatic equilibrium of the core. This condition would imply the same amount of mass in every vertical column of the crust,

* J. H. Pratt, "On the deflection of the plumb-line...," *Phil. Trans. R. S.*, vol. 149 (1859), p. 745; and "A treatise on...the Figure of the Earth," 3rd edition, 1865, pp. 135, 159.

† C. E Dutton, "Some of the greater problems of physical geology," *Bull. Phil. Soc. Washington*, vol. XI. 1892.

‡ Tittmann and Hayford, "United States geodetic operations in the years 1903—1906," *Comptes Rendus de la 15me. conférence générale de l'association géodésique internationale*, 1908. See also J. F. Hayford, "The figure of the earth and isostasy from measurements in the United States," Washington, 1909.

§ F. R. Helmert, "Die Schwerkraft in Hochgebirge," *Veröff. k. preuss. geodät. Inst.* Berlin, 1890; see also L. Haasemann, "Bestimmung d. Intensität d. Schwerkraft im Harze," *Veröff. k. preuss. geodät. Inst.* Berlin, 1905, and F. R. Helmert, "Die Tiefe d. Ausgleichsfläche..." *Berlin Sitzungsberichte*, 1909.

if the thickness of the crust could be neglected, and the core were truly spherical; for a layer of uniform surface-density on a sphere gives rise to no attraction at an internal point. The height of the elevated parts of the crust is thus assumed to be compensated by defect of density. For this reason the crust is described as the "layer of compensation." When account is taken of the thickness of the layer it appears that the law of density stated above is only a first approximation. The thickness of the layer is estimated by Hayford to be about 120 km., and this estimate is supported by Helmert. As this is not far from $\frac{1}{50}$ of the radius of the earth, the numerical work in this Chapter and the following will be performed on the supposition that the mean thickness of the layer is $\frac{1}{50}$ of the earth's radius.

**10.** Among the considerations which led to the hypothesis of isostasy one of the most important was the fact, established by geodetic observation, that the actual forms of the equipotential surfaces near the surface of the earth are very approximately oblate spheroids, as they would be if the whole earth were in a state of fluid equilibrium under gravitation and rotation. This result implies that the inequalities of potential which are due to the inequalities of density in the crust, and to the deviations of the outer surface of the crust from an equipotential surface, are very small in the neighbourhood of this outer surface. With a view to a precise formulation of the hypothesis of isostasy, it is convenient to assume that these inequalities of potential actually vanish at the mean outer surface of the crust. It is part of the hypothesis that they vanish within the core. We shall therefore take them to vanish at both the mean outer and the inner surfaces of the layer of compensation. Further the gravitational attraction within the earth varies continuously from point to point. Within the core it must be independent of the inequalities of density which occur in the layer of compensation. At any point within the layer of compensation the gravitational attraction depends partly on the inequalities of density. To secure continuity at the internal surface of the layer it is necessary that those terms in the expression for the attraction which arise from these inequalities should vanish at this surface.

**11.** In order to formulate this theory analytically it will be sufficient to neglect the rotation of the earth. If a body of the size and mass of the earth, at rest, could support assigned continental elevations and oceanic depressions without requiring an improbable degree of tenacity in its materials, a body of similar constitution rotating once in a day could almost certainly support similar elevations and depressions. We shall therefore take the core to be spherical, and the outer surface of the layer of compensation to be a nearly spherical surface concentric with the surface of the core, and shall suppose the radial elevation of the outer surface to be expanded in a series of spherical surface harmonics, and we shall write the equations of these two surfaces in the forms:

$$\text{for the core} \quad r = b,$$
$$\text{for the crust} \quad r = a + \Sigma \epsilon_n S_n,$$

where the suffix $n$ denotes the degree of the surface harmonic $S_n$, and $\epsilon_n$ is a small constant indicating the magnitude of the inequality. In general it will be sufficient to discuss the case where $\Sigma \epsilon_n S_n$ reduces to a single term. The inequalities of density within the layer of compensation are then to be correlated with this term.

**12.** The density at any point within the core will be a quantity which can be expressed as a function of $r$ only. The expression for the density at any point within the layer of compensation will consist of two terms, the first term being a function of $r$, and the second term the product of a function of $r$ and the spherical surface harmonic $S_n$. The potential at any point in the core will be a function of $r$ only. The potential at any point within the layer of compensation will be the sum of two terms, one of them a function of $r$ only, and the other the product of a function of $r$ and $S_n$. The second term is the inequality of potential above mentioned; we shall denote it by $V'$. It is convenient to assume a form for $V'$ and deduce a form for the density. To give effect to the considerations already adduced in regard to the form of $V'$ we must suppose that the $r$-factor of $V'$ contains $(a - r)$ and $(r - b)^2$ as factors. The factor $(a - r)$ secures that the mean surface is an equipotential, the factor $(r - b)^2$ secures that the inequality of potential and the corresponding inequality of attraction shall both vanish at the surface of the core. Accordingly we assume for $V'$ an expression of the form

$$(r - a)(r - b)^2 f(r)\, r^n \epsilon_n S_n.$$

The factor $r^n$ has been introduced in order that we may have to deal with a spherical solid harmonic $r^n S_n$. The factor $f(r)$ is in our power; all forms for it except such as become infinite at $a$, or $b$, or at an intermediate value of $r$, are equally compatible with the hypothesis of isostasy, according to the statement of this hypothesis made above. It might be possible to choose it so as to diminish the amounts of the calculated tangential stresses, but, for the present, it is better to choose it with a view to analytical simplicity. It turns out to be convenient to assume that $f(r)$ is simply proportional to $r^4$. See p. 18 *infra*, ftn. As the outcome of this discussion we put

$$V' = A_n (r - a)(r - b)^2 r^4 W_n \quad \text{.......................(1)},$$

where $W_n$ is written for the spherical solid harmonic $r^n S_n$, and $A_n$ is a constant to be determined in terms of $\epsilon_n$.

**13.** To simplify the problem to the utmost we are going to assume that the mean density of the layer of compensation is independent of $r$. Within the layer of compensation the density $\rho$ is assumed to be given by

$$\rho = \rho_1 + \rho' \qquad \text{.................................(2)},$$

where $\rho_1$ is a constant, and $\rho'$ is the inequality of density mentioned above. Then the potential $V'$ is that due to (i) a volume distribution of density $\rho'$ in the region $a > r > b$, (ii) a surface distribution of density $\rho_1 \epsilon_n S_n$ on the surface $r = a$. The potential $V_0$ at any point within the core is a function of $r$ only. The potential $V$ at any point within the layer of compensation is expressed by the equation

$$V = V_1 + V',$$

where $$V_1 = \tfrac{4}{3}\pi\gamma(\rho_0 - \rho_1)\frac{b^3}{r} + \tfrac{2}{3}\pi\gamma\rho_1(3a^2 - r^2) \qquad \text{..................(3)},$$

$\gamma$ denotes the constant of gravitation, and $\rho_0$ is the mean density of the core.

To determine $\rho'$ we have the equation

$$\nabla^2 V' = -4\pi\gamma\rho' \qquad \text{..............................(4)},$$

and in accordance with (1) this gives

$$-4\pi\gamma\rho' = A_n [7(2n+8)r^5 - 6(2n+7)r^4(a+2b) + 5(2n+6)r^3 b(2a+b) - 4(2n+5)r^2 ab^2] W_n \qquad \text{...(5)}.$$

$V'$ is the potential of a certain volume density and a certain surface density, as explained above. $V'$ vanishes at $r = a$, and therefore the potential at any point outside the surface $r = a$, due to the same volume density and surface density, is zero. The surface characteristic equation for the potential at the surface $r = a$ therefore becomes

$$\left(\frac{\partial V'}{\partial r}\right)_{r=a} = 4\pi\gamma\rho_1\epsilon_n S_n,$$

and this gives $$A_n(a-b)^2 a^{n+4} = 4\pi\gamma\rho_1\epsilon_n \qquad \text{........................(6)}.$$

**14.** Corresponding to the superficial inequality expressed by $\epsilon_n S_n$ we have the inequalities of potential and density expressed by

$$V' = \frac{(r-a)(r-b)^2}{(a-b)^2}\frac{r^4}{a^{n+4}} 4\pi\gamma\rho_1\epsilon_n W_n \qquad \text{..................(7)}$$

and $$\rho' = -\rho_1 [7(2n+8)r^5 - 6(2n+7)(a+2b)r^4 + 5(2n+6)b(2a+b)r^3 - 4(2n+5)ab^2r^2] \frac{\epsilon_n W_n}{a^{n+4}(a-b)^2} \text{...(8)}.$$

It may be shown without much difficulty that this somewhat complicated law of density accords with the suggestion that, in the layer of compensation, the product of density and thickness should be constant. If the thickness is small compared with the radius, this relation holds to a first approximation.