MATHEMATICAL AND PHYSICAL PAPERS

IN TWO VOLUMES

VOLUME II

Cambridge University Press
978-1-107-53640-1 - Mathematical and Physical Papers: Volume II
Sir Joseph Larmor
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MATHEMATICAL AND PHYSICAL PAPERS

BY

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HON. F.R.S. EDIN.: HON. MEM. R. IRISH ACAD.: HON. MEM. ASIATIC SOC. OF BENGAL, MANCHESTER LIT. AND PHIL. SOC.: HON. FOR. MEM. U.S. NATIONAL ACAD. OF SCIENCES, AMERICAN ACAD. OF SCIENCE AND ARTS, BOSTON, AMERICAN PHILOSOPHICAL SOCIETY, PHILADELPHIA, WASHINGTON ACAD., R. ACCADEMIA DEI LINCEI, ROME, ISTITUTO DI BOLOGNA: CORRESPONDANT OF THE INSTITUTE OF FRANCE: LUCASIAN PROFESSOR IN THE UNIVERSITY OF CAMBRIDGE, AND FELLOW OF ST JOHN'S COLLEGE

VOLUME II

CAMBRIDGE
AT THE UNIVERSITY PRESS
1929

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107536401

First published 1929
First paperback edition 2015

A catalogue record for this publication is available from the British Library

ISBN 978-1-107-53640-1 Paperback

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PREFACE TO VOLUME II

EXTENSIVE groups of omissions have been found to be necessary in the present volume. This has arisen mainly from unforeseen insertion of new material in both volumes during the last two years. The fascination of the origins of thermodynamics, the science which essays to make the connections between practical mechanical principles and the underlying atomism, has again asserted itself: and further attempt has been made in Appendices and Notes to disentangle the romantic history of the evolution of foundations in that domain, with the mainly statistical outlook which its generality imposes. Here the names of Carnot, Kelvin, Clausius, Maxwell, Boltzmann, Rayleigh, Willard Gibbs are predominant. Also the temptation to probe closer into the nature of natural radiation, its specification and its mode of dispersal, was not to be resisted. But there it has been confined mainly to verifying just how far, in various respects, the original well-worn ideas of electrical radiation, based on the limited analogy of transmission of sound, yet perhaps in many ways as will appear holding the kernels of recent abstractions, can carry. Apology for such restriction is hardly necessary: the modern constructs in the problems of quantified spectroscopy, with their atomic storage of energy, many of them highly successful in their special fields, have hardly been entered upon, because the vast and tentative literature could not be justly appreciated except by a critic closely cognizant of the diverse evolutions of the last fifteen years in this field of knowledge. That is the modern penalty against expansion of interests: yet complete knowledge up to date is not indispensable to profitable consideration. Even in the astrophysical domain history can repeat itself (cf. p. 586) on wider foundation of facts: while by aid of analysis, of transcendental types as introduced originally through Hamilton's Quaternions, constructive thought had long ago essayed to pioneer beyond the range of formal proof, bringing back to physical reality its harvest of results. Meantime the secured territory behind the ragged frontiers of knowledge repays intensive cultivation.

In these subjects that are for the future to consolidate, as in all other departments of physical theory, the absence of Professor Lorentz, recently removed from science, leaves a blank that cannot be filled: which adjoins to other recent severe losses, including Arrhenius, Dewar, K. Onnes, Wien.

As before, in addition to various Postscripts and eight Appendices, the extensive new footnotes are again indicated by marks such as asterisks instead of numbers: other substantial additions or modifications are put in square brackets. The index, in addition to being a record, may help to consolidate the correlations of the variety of subjects that come under discussion.

Corrections are not infrequent. It has been judged to be informing, as well as historically incumbent, to amend without obliterating the original mistakes, except in trivial cases. This course may tend to interpose delays on a rapid survey of the conclusions. But clarity and finality are reached largely through the clash of alternatives, all kept well in view: there is authority for the opinion that systematic effort towards due articulation of the historical course of progress in an intricate domain, not always improved by division into chapters, may instruct the writer himself and the critic, at least as much as it is calculated to inform the general reader in quest of final results.

The additions are thus largely of the nature of general survey, not encumbered by details of partial developments. Especially in topics of widespread physical concatenation, a condensed phrasing, even though elliptical, in ordinary language, may be a time-saving compromise: exploring students, requiring rapid recovery at will of a field of view as a single whole, may prefer it to ramifying general description or to algebraic formulation with its apparatus of special schedules, however ill it may be adapted for more cursory reading. Algebraic analysis, outside the realm of computation, has indeed to run in blinkers, though they may be ultramundane as in the multiplex geometry: only in combination with a general physical intuition does it become the source and the expression of expanded outlook.

Among omissions that need not be specified, including a set of Articles around 1902 for a supplement of the *Encyclopaedia Britannica*, the writer cannot help regretting the absence of a series of scientific biographical Notices prepared at various times: especially an intensive account of the activities of Lord Kelvin, drawn up with considerable care and research for the Royal Society (*Proceedings*, 1908, pp. i–lxxvi), and intended to assist in exhibiting a synopsis of his career in relation to the scientific progress of his age, from an evolutionary point of view hardly perhaps now much in sight.

The other main omission is a series of papers aiming towards an understanding of the scope and significance, as distinct from the mathematical development, of the modern doctrines of relativity, identified here with invariance of physical aspect as regards all frames of reference belonging to the proper group,—those that are permissible in conformity with the slowness of the messenger rays of light,—for the external world in its electric and its astronomical features. On the astronomical side this problem may claim to have been set once for all by Bradley, the great founder of modern practical astronomy of precision, as far back as Newton's day: he provided the practical solution which has remained adequate ever since. The exploration towards an ideal solution which would be mathematically exact, without any need for approximation, has been stimulated afresh in recent times by the null results of very close physical experimentation starting with Michelson, ultimately evolving towards a formal mathematical calculus through novel avenues of approach to the subject opened out mainly by Einstein. Perhaps in its present stage the theory may be held to lay alongside the actual world which we know, as in recent times most closely explored and formulated in the marvellously precise records of the astronomers, a self-contained artificial map of reference, made up solely of vectors and tensors after the manner of line-geometry, thus subject to the rules of the intricate special discipline known as the Mathematical Theory of Relativity. In this auxiliary cosmos space and time and motion do not occur: yet it can be of great value in a geographical sense, after the manner of a spherical map of the Earth, for unravelling the intricacies of relations connecting regions in our actual world, which astronomers are permitted to formulate only in terms of the complications of delay in the messages of the informing rays of light. The parallelism of relations between the world of actual perception and memory and this particular conglomerated fourfold assembly of permanent ray-models, so to say, extends over a prominent yet necessarily limited range whose boundaries have hardly yet been very closely examined. There may be, however, people who aim at transferring their whole life into this new cosmos. The papers now omitted are concerned largely with various general aspects of this correlation: naturally much of their contents has now become transcended, or modified into improved presentations. A list of these papers has been inserted after the Table of Contents.

In such ideal constructions one may readily continue to go astray: yet the writer has ventured now, partly for his own clarification, on a sketch of what appears to him at present to be the effective mode of formulation in this field, describable perhaps as complete coherence between recipient minds and a unique external world. It is postulated as a basis for systematic exploration that there may be potential astronomers everywhere in the universe, each group of them active only in their own local environment, except so far as information from outside arrives by the delayed rays of light and in no other way. It is postulated also that such astronomers can be all in instantaneous understanding with one another as regards results thus acquired—a state of affairs at first sight prohibited*, which yet is approached indirectly through the mental process of memory in relation to the adjusted records of the observations of actual astronomers made in all positions along their extensive terrestrial orbit round the Sun, this being in fact the method of formulation tacitly adopted for the construction of the actual science. The question is whether the worlds that these groups of astronomers separately analyse out for themselves could be recognized by them, in joint ideal conference across space and time, as the same extraneous world for all, whether in fact there is an unambiguous external world. Absolutely, such recognition is presumably always possible, if there is a world, for signals can be conceived as instantaneous; but the complexity of an actual identification, mathematically exact, based on the delayed light signals, might be overwhelming. Thus practically the procedure of optical relativity—much more involved than the original instantaneous inertial relativity of Newton—would be to explore this field, with a view to verify that no overt discrepancies appear that are definitely outside feasible modes of adjustment. To this limited extent may astronomy and general physics perhaps make contact with psychology. A summary on these lines has been included in the final Appendix: the subject is our mode of apprehension of the Universe, as valid for all observers in space and time, and some degree of repetition may be pardoned.

The main feature that is involved practically is, as would be expected, a closer formulation as regards the specification of times. Here also a different and more direct problem has been ripening for the last half-century, to which attention is paid in various papers in this collection; namely, how to provide a scale of Newtonian time for dynamical astronomy that shall include correction for the ascertained irregularity in the Earth's axial rotation (cf. p. 479). The scale of time thus sought for would on these ideas (p. 769) be the one belonging to astronomers attached to the Sun, on the presumption that the Solar motion in space has negligible acceleration.

The survey, undertaken systematically after a long interval, of procedure in the early fundamental constructs, has involved some reconsideration of the resulting classical scheme of electrical and optical theory, a delicate undertaking as now conducted in the high light of its modern incapacities. Important and surprising as these are, and fruitful as are the mobile tentative syntheses in spectroscopy and thermionics

* This however actually expresses the situation in the cognate problem of ranging for positions in modern artillery practice: the observers acquire direct knowledge only through the delayed sound-waves, but they can confer with one another instantaneously (though not with the sources) by telephone.

and other domains internal to the atoms that they have promoted, the impression may be allowed to persist that, neither in extent as compared with the coherent territory of our inherited science—with its lacunae concerned with defect of convergence to a limit in atomic radiative phenomena, which were never out of sight—nor in inevitableness as regards a comprehensive understanding of the new phenomena that are involved, do they justify a general negative attitude such as it is easy to acquire. The question indeed rather forces itself, whether the load of duly verified discrepancy, especially in regions involving statistical thermodynamic theory, has not been somewhat exaggerated.

In another aspect, the view that modern physical science had erred by basing itself on a self-sufficient narrow materialism is historically one-sided, as has been recognized for example long ago (1899) by the psychologist James Ward in the Preface to his treatise *Naturalism and Agnosticism*. In recent years the magnificent and in many ways startling output in constructive astrophysical results, mainly from the great observatories of the Pacific Coast, Mount Hamilton and Mount Wilson and now Victoria, following the early leads of Herschel, Rosse and I. Roberts, has in its turn promoted the new exploring trend in dynamical molecular studies, thus stimulated by the spectacle of vastness interacting with primordial simplicity of law. Brief discussions in this direction have here been adventured, for example as regards transmission through nebulae and also the Solar magnetic fields.

Even the working hypothesis of the earlier papers of this collection, involving a rotationally elastic aether with its essential innate fluidity, which towards the end of last century had been eliminated largely as too materialist and naïve as a representation of the electro-optical field, can now find more to be advanced on its behalf than would have been at first anticipated. It implied naturally an intrinsic isotropic pressure in the fluent aether. The problems of physical optics, then the most prominent, did not however require its intervention: the dynamical laws of refraction after Fresnel involved no excitation of pressure in the optical medium, a fact that became familiar in the physical school of Stokes and Kelvin. That was on the postulate that the optical elastic aether was to be taken as virtually incompressible, so that any initiation of pressure however arising became adjusted throughout with speed relatively infinite: finite pressural waves propagated in time had to be excluded in optics. The standard Maxwellian equations of the electric field, supporting themselves in this theory on its electronic singularities, were naturally restricted to this implied condition. But the wider possibilities that became obvious in the rotational model could not of course entirely be put aside. If the elastic aether were almost but not quite incompressible, pressure originating anywhere, atomically if not optically, would be promptly equalized universally, not indeed to a static distribution after the manner of an instantaneous Laplacian potential, but by elastic wave-adjustment with velocity very great yet definite, and therefore proportionately difficult to excite or sustain except in intimate processes within the very intense field of the atom itself. The electric vector potential is thus to have divergence, which is elastically propagated. And questions then presented themselves obviously, such as whether this hydrodynamic pressure, an essential feature of the aether model, might not even involve a clue to the universal gravitation, so potent in phenomena on a large scale yet locally so slight. Cf. Vol. I, p. 500.

The Maxwellian electric scheme permitted a principle of relativity to be imposed upon it, as interpreted here into the postulate that there is an external world, of electrons and atoms composed of them, which is identifiable as the same world by all observers who are informed by the messages of radiation, however rapidly these observers may be travelling. The origin of this theory was the ascertained null effects, in various directions, of the convection of the observer's field, up to the second order. In contrast to this permissive relativity, is the fact that in the rotationally elastic model such optical relativity is compelled, being essential to the coherence of the scheme: the necessary aether pressure there demands the convective shrinkage of the system. Moreover this intrinsic pressure imparts work to the shrinking electron, putting into its system energy of amount proportional to v^2 in addition to that required for its electrodynamic field, making up in all $\frac{1}{2}e^2/C$, where C is the electrostatic capacity of the shrunken system, with inertia in proportion; thus interpreting dynamically Vol. I, p. 672.

When the detection of free travelling electrons thirty years ago, and their enormous speeds as β-rays, led to a closer scrutiny of the original working model of an electron (cf. *supra*, Vol. I, p. 520, in relation to Vol. II, p. 226) as a small spherical region on which lines of the elastic twist representing an electric field abutted, at right angles on account of complete loss of elasticity inside the sphere, the rotational aether scheme still remained effective for that extension of the picture. Under the influence of rapid convection through the fluid aether this spherical core still managed to subsist, deforming itself into a steady ellipsoid, thus becoming in fact, and of its own inner necessity without possibility of any alternative, the Lorentz model of a relativity electron (cf. p. 276). But it has been emphasized at various times, among others by Poincaré and by Lorentz, that even the mere existence of a mechanical model of this kind should be regarded as debarred, because forces would have to be introduced from outside in order to sustain it which are foreign to the electric field. The forces that are needed prove, as they ought, to be none other than an excess in the model,—necessarily constant all over each surface of an electron,—of this hydrostatic aethereal pressure on the outside of the ellipsoidal boundary over that on the inside. So far then from a dynamical scheme failing to include electrons on account of such necessity for extraneous force, this rotational model of the aether, from which the concept of an electron was originally derived, proved to be precisely adapted to the new emergency, the sole type of adjusting internal stress which it could supply in addition to the Maxwellian electric forces being just what was required. Any initial inclination to restrict the core of the electron by constraint to a constant volume had, on the initiation of Lorentz, to be definitely forgone, optical relativity forbidding: yet here again, this feature conforms, for the excess of fluid pressure can then remain constant as regards the same electron however convected. Cf. p. 810: also the more general statement in *Aether and Matter*, § 96. For different electrons, once the charge is somehow determined, this difference of pressures fixes the radii and the masses. If then the cores are regarded as vacuous spaces devoid of pressure the conditions are thus satisfied for identical electrons simply by a constitutive pressure unchanging throughout the aether. The drawback, however, which has always been in mind, is the circumstance that this hollow electron model would have to be made secure against obvious instability as regards change of form: though this can be effected by modification relatively very slight provided the disturbances are not too great,—*e.g.* conceivably by putting a limit to the intensities for which the linear equations of the field hold good, which would be exceeded in the enormously intense fields close to an atom, a procedure which amounts to falling back upon local dispersive quality such as the more recent developments in atomic dynamics have strikingly demanded. But the problem of the positive electron remains.

Coming down now to recent atomic schemes, this old idea of hydrodynamic pressure in the aether, as a natural supplement to the Maxwellian field of electrodynamic force, is again forcibly brought to mind. The most arresting of the recent constructions, in interpretation of the *corpus* of empirical numerical *quantum* relations built up on the spectroscopic side, has been the wave equation set up by E. Schrödinger, involving only one scalar independent variable, emerging from earlier more geometrical constructs by L. de Broglie, as on p. 552; he discovers that one suitably adapted differential equation could prescribe the complete *corpus* of precise *quantum* numbers for a vibrating atomic system in very various tractable cases, much as the single wave equation of aereal pressure determines frequency numbers for the various types of sound-vibration in an organ-pipe or other bounded aereal region. This equation regulates the distribution of one scalar quantity, at first sight very foreign to the vectors which describe the electric field: and being successful it does not contradict the formal presentations by matrices. Yet let us revert again to the problem of modes of vibration of a set of atomic singularities imbedded in an isotropic elastic medium, resisting deformation, or rotation its field equivalent in the electric aether, now in combination with strong but not infinite resistance to compression. The procedure for the treatment of such problems for an isotropic medium, familiar since the early days of Stokes and Lamé, manages to separate out by itself the pressural vibration: the scalar field pressure satisfies its own differential equation, recalling therefore this esoteric equation of Schrödinger. For, pursuing the analogy, if the electric singular points of the aethereal vibrating system of the atom are not to be also singularities, after the manner of the actual vesicles filled with compressed gas that occur in crystals, from or to which intense hydrostatic pressure radiates other than the intrinsic electrically sustained constant universal pressure aforesaid—and any such added source of pressure would probably spoil the type of model of a convected electron—we are invited to seek out forms of solution for disturbance superposed on this intrinsic aethereal pressure, adapted to the local field of the atom, of types which indeed converge radially on the positions of its electrons or other nuclei in a way prescribed by their electric field, but without tendency towards values there increasing without limit; for stable modes of local vibration of the ambient aether could only occur around the definite configurations of the material atom permitting of such solutions of the pressure-equation, which alone could subsist. The selective radiation which escapes from an atom would be presumed to arise, in the manner now frequently envisaged on the basis of the combinatory law of spectral series, from temporary interference excited between pairs of states of pure vibration, after the manner of the Helmholtz theory of difference tones in acoustics except that these states are themselves now self-contained and non-radiating. This train of ideas hardly amounts to more than illustration, possibly far removed in its simplicity from the necessities of the picture: though the mental analysis of Nature has never yet turned out to be other than simple in the long run. Half the Schrödinger procedure is left still in obscurity. Yet it may reasonably give some support to the position that the original working electronic model, as directly compelled by a rotational aether, and now recognizing explicitly and adapting pressural propagation in conjunction with electric, may still—crude perhaps but within its range wholly concatenated—have claims to be retained in mind alongside other clues to consistent theoretical formulation.

There is another impressive, because less special yet wholly coherent, analytic procedure for linking up the sources with the field through the universal Action formula, generalized if need be so as to include aethereal pressure—a method which originated indirectly in Maxwell's stress-formulations for the transfer into precise analytic form of the general idea of Faraday (dating from the earliest beginnings of electrodynamics in 1821, cf. p. 763) of a connecting continuous medium in internal tension. If this elastic ultimate medium is imagined mathematically to be severed along an interface, such stress would be determinate as the type required to be applied over the severed face to replace the previous support of the medium beyond, itself already kinetically equilibrated internally by conforming to the equations of the field: the stress, which is quadratic, is relative to a frame of reference though it presents a fourfold invariance, because the variational virtual displacement from which it arises (p. 800) is a movement in that frame. As thus inherent directly in the ultimate formulation by the Action, this quadratic stress, which became the essential foundation of the relativist fourfold dynamical construct by tensors, does now acquire significance other than merely algebraic, by appearing as a necessary element in the interlaced systematization, based on the single local Action function, of the physical manifestations of the field which are otherwise mainly of undulatory type.

With avenues such as these as yet not fully explored, it must be premature to conclude that the classical foundations of electric and optical science are to be regarded as undermined. It has never of course been suggested that any mechanical model is more than an aid to insight, by providing a vivid picture of the particular group of interacting relations of the phenomena that is in hand: the oft-repeated demur, that if there could be one model there would be an infinite number, can only be a different mode of expression of the familiar common ground that, after the groups of relations relevant to the model have been reduced by its aid to coherent and condensed algebraic statement, the scheme of equations that survives may be interpreted as the manifestation of a unique abstract dynamical Action of local structure, independent of models altogether.

Recently there has been a revival of emphasis on the limitations which must affect the representation of a molecular world by the machinery of a continuous mathematical analysis. The foundations here have been an affair of the Fourier theorem: for natural radiation the ultimate analysis is into groups of waves, each presenting itself as a differential element in the Fourier integral, and the restrictions that arise have come into view in various connections (cf. p. 549). As physical knowledge has developed in recent years, the analysis has trended empirically towards arithmetization, under statistical treatment concerned with elements or *quanta* of correlated magnitudes grouped into cells, dynamically consistent and of definite extents (cf. p. 401). It would thus seem that any prospect of assistance from further improvement of the abstract Fourier analysis has vanished, just because the transition from atomism to continuous fields is possible only in the rough. But why the continuous presentation of the physical world has to stop at a definite scale of coarseness remains an unfathomed problem.

The principles of Hydrodynamics, in the hands of Stokes, Helmholtz and Kelvin, and later of an active school headed by Osborne Reynolds and Rayleigh, developed at first largely in contact with the practical problems of propulsion of ships. Besides exerting a conspicuous reaction with general physical theory, these ascertained laws with their empirical limitations have now come to an apotheosis of their own, as the rational foundation, notwithstanding some disparagement, of all improvement in the principles of aviation. In later papers attempt is made to bring this historical mode of discussion, resting on general principles, under concise review in that domain, so far as the requisite technical information has been found to be readily accessible in systematic form.

The writer is again greatly indebted to the skill and attention of his collaborators at the Cambridge University Press.

J. L.

CAMBRIDGE
September 1928

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PAPERS RELATING TO PHYSICAL RELATIVITY NOT HERE REPRINTED· (*See* p. vii.)

On the Essence of Physical Relativity.
[*Proc. U.S. National Academy of Sciences*, Vol. IV, Nov. 1918, pp. 334–337.]

On generalized Relativity in connexion with Mr W. J. Johnston's Calculus.
[*Proc. Roy. Soc.* Vol. XCVI A, Aug. 1919, pp. 334–363.]

The Relativity of the Forces of Nature.
[*Monthly Notices of the R. Astron. Soc.* Vol. LXXX, Dec. 1919, pp. 109–138.]

Gravitation and Light.
[*Proc. Camb. Phil. Soc.* Vol. XIX, Jan. 1920, pp. 324–344.]

Questions in Physical Inter-Determination.
[Lecture to International Congress of Mathematicians, Strasbourg, Sept. 1920, *Comptes rendus du Congrès*, pp. 3–40.]

On the nature and amount of the Gravitational Deflection of Light.
[*Philosophical Magazine*, Vol. XLV, Jan. 1923, pp. 243–256.]

Can Gravitation really be absorbed into the Frame of Space and Time?
[*Nature*, Feb. 10, 1923, p. 200.]

Newtonian Time essential to Astronomy.
[Supplement to *Nature*, April 9, 1927, pp. 1–12.]

The Grasp of Mind on Nature.
[James Scott Lecture, *Proc. Roy. Soc. Edin.* Vol. XLVII, Aug. 1927, pp. 307–325.]

Addenda *and* Corrigenda (*Jan.* 1929).

Vol. I, p. 81. The adaptation of this screw-theory to aerodynamic practice would involve inclusion of the action of the propeller in steady transfer of angular momentum from the machine to the surrounding air, thus originating the instabilities of spin. **Spin of aircraft.**

p. 106, footnote. The criterion for validity is that the free paths of the ions should be a multiple of the molecular interval large enough, say 10^3, to enable the Poisson averaging for potentials to be independent of the statistical averaging for densities. **Electrolytes.**

p. 130, line 4 from foot, *for* r^2 *read* r *twice*.

p. 293. If the two sheets of a wave-surface are the outer and inner boundaries of one expanding pulse, it would perhaps be anomalous for them to intersect. The general elastic wave-surface has however three sheets. **The wave-surface in actuality.**

p. 430. Kelvin's elastic paradox, correcting Green, that a medium of negative modulus of compression may be stable as regards internal disturbance of density, is still intriguing when thus expressed. **Elastic paradox.**

p. 641. A list of dates is significant as regards progress in scientific formulations. The General Dynamics of Lagrange, as also the Calculus of Variations, arose from the idea of Least Action in 1758: the path-breaking activities of Hamilton in regard to Systems of Rays and General Dynamics ended with the two great dynamical memoirs, *Phil. Trans.* 1834–5: Jacobi's Lectures on Dynamics were delivered at Königsberg in 1842–3, and published by Clebsch from Borchardt's script as a hearer in 1866. From another side, the Dioptrics of Gauss, following the less formal initiatives of Huygens, Cotes, Möbius, appeared in 1853. Thomson and Tait's *Nat. Phil.*, which placed fresh emphasis on the fundamental character of the Action principle, appeared in 1867, and was soon improved in formal rigour as regards the hydrodynamic analysis by Kirchhoff and by Boltzmann: use had been made, in more tentative and exploratory manner, of the cognate generalized dynamical ideas relating to momentum in Maxwell's final constructive electric memoir of *Phil. Trans.* 1863. **History of Generalized Dynamics.**

p. 641, footnote. The key to the argument (p. 189) is that if $(p - s)^2 + (q + r)^2$ were invariant for the rotations of axes round the central ray at the two ends, then if these rotations were chosen so as to make $p = s$, $q = 0$, the further condition $r = 0$ would be satisfied so that U would reduce to the form (21). But this invariance does not hold, for all that has been there done is to express the central terms in bipolar coordinates, the two angles being any whatever.

Let us then make another attempt. The two relations $p = s$, $q = -r$ can be attained by two different rotations at the two ends: and then the middle terms in p. 188 take on the form $\rho_1 \rho_2$ multiplied by $p \cos(\theta_1 - \theta_2) + q \sin(\theta_1 - \theta_2)$, that is by $A \cos(\theta_1 - \theta_2 + \alpha)$: thus if there is a rotation imposed through α at the second end so that $\theta_2 - \alpha$ replaces θ_2, they assume the form $A(x_1 x_2 + y_1 y_2)$ appropriate to an axial optical system. Indeed they were already of that type, in the extended sense that such a system is one that (when straightened out) is not altered by rotation round its axis. And this wider sense appears to be the proper optical one, for if two successive reflexions occurring at the second terminal are part of the system they are equivalent to a rotation round the axis parallel to both their planes. With this extension of the definition of an axial system, and simple verbal modifications, the discussions referred to in the footnote appear to stand. **Analysis of an asymmetric optical system.**

p. 643. Cf. Cotes' "noble and beautiful theorem," his "last invention," that there is an apparent distance of any pair of points across an optical system as a reciprocal property of the pair, with wealth of consequent general relations **Cotes' theorem.**

for image points, in his cousin R. Smith's *Opticks*, Vol. II, notes, p. 78, where reference is made to Huygens' much earlier but more special propositions.

p. 652, line 11 from foot, *read* $r\,(1 - v'/\mathrm{C})$.

p. 664, line 17, *for* $\frac{1}{45}$ *read* $\frac{1}{27}$

p. 667, line 13 from foot, corrected in Vol. II, p. 752.

p. 679, line 4, *read* $\frac{1}{2}\rho c \varpi v$

The Fresnel convection contrasted with relativity.

Vol. II, p. 39, line 7. The procedure for the Fresnel convection term in phase velocity of wave-trains, here indicated, was followed out in *Aether and Matter* (1900), § 113, under the heading "The Correlation between a stationary and a moving Medium as regards trains of Radiation." The result was that the Fresnel expression was found to require correction by a term of the second order $-(\mu^{-1} - \mu^{-3})\,v^2/\mathrm{C}$. The relativity formula gives the same correction to the second order: but as was to be expected on present ideas (p. 40) where matter is concerned, the two disagree widely when v/C is considerable. As however the Fresnel result without any change has more recently been widely appealed to, as one of the most arresting of the exact illustrations of an unrestricted relativity, it seems worth while to probe that conclusion further, more directly. Unless the discrepancy can be adjusted, it will illustrate the precariousness of the fourfold as a substitute for direct comparison of frames of reference (p. 803). The Fresnel term is an addition to the group velocity, as Lorentz pointed out, thus removing experimental discrepancies.

The problem before us is simply to transfer a wave-train

$$A \cos \frac{2\pi}{\lambda}\left(\frac{\mathrm{C}}{\mu} t - lx - my - nz\right), \quad l^2 + m^2 + n^2 = 1,$$

referred to the frame (t, x, y, z) of the material medium which carries it, into the form

$$A' \cos \frac{2\pi}{\lambda'}\left(\frac{\mathrm{C}}{\mu'} t' - l'x' - m'y' - n'z'\right), \quad l'^2 + m'^2 + n'^2 = 1$$

appropriate to the observer's frame (t', x', y', z') with regard to which the carrying medium is in motion with velocity $(v, 0, 0)$. The formulae of transformation (p. 39) are

$$t = \epsilon^{-\frac{1}{2}}\left(t' - \frac{v}{\mathrm{C}^2} x'\right), \quad x = \epsilon^{-\frac{1}{2}}(x' - vt'), \quad y = y', \quad z = z',$$

with a transformation from A to A' which is not here required. The wave-train referred to the observer's frame is thus expressed by

$$A' \cos \frac{2\pi}{\lambda}\left\{\epsilon^{-\frac{1}{2}} \frac{\mathrm{C}}{\mu}\left(t' - \frac{v}{\mathrm{C}^2} x'\right) - \epsilon^{-\frac{1}{2}} l\,(x' - vt') - my' - nz'\right\},$$

which is $$A' \cos \frac{2\pi}{\lambda}\left\{\epsilon^{-\frac{1}{2}}\left(\frac{\mathrm{C}}{\mu} + lv\right) t' - \epsilon^{-\frac{1}{2}}\left(l + \frac{v}{\mu\mathrm{C}}\right) x' - my' - nz'\right\}.$$

We are at present concerned only with the phase velocity in this frame, with regard to which the material system is convected with velocity v: for a wave-train travelling in the direction of v for which $(l, m, n) = (1, 0, 0)$ it is

$$\left(\frac{\mathrm{C}}{\mu} + v\right) \Big/ \left(\mu + \frac{v}{\mathrm{C}}\right), \text{ which is } \frac{\mathrm{C}}{\mu} + (1 - \mu^{-2})\, v \left(1 + \frac{v}{\mu\mathrm{C}}\right)^{-1},$$

the addition to C/μ thus differing from the Fresnel expression by the last factor. The value deduced as an example of composition of velocities in relativity is

$$\left(\frac{\mathrm{C}}{v} + v\right) \Big/ \left(1 + \frac{v}{\mu\mathrm{C}}\right).$$

When the wave-train is oblique to the velocity of convection, there is change of direction as well as change of velocity when the frame is thus changed. The former, for the change from the observer's to the Solar frame, is the Bradley aberration of the rays; for a ray at inclination θ with v, changed to θ' by aberration, simplifying by taking n null, we have

$$\cot\theta = l/m, \quad \cot\theta' = \epsilon^{-\frac{1}{2}}\left(l + \frac{v}{\mu \mathrm{C}}\right)\Big/ m;$$

so that the aberration $\theta - \theta'$ is given by the law

$$\epsilon^{\frac{1}{2}}\cot\theta' - \cot\theta = v/\mu\mathrm{C}\,.\sin\theta,$$

or approximately $\sin(\theta - \theta') = \mu^{-1}\dfrac{v}{\mathrm{C}}\sin\theta' - \frac{1}{2}\dfrac{v^2}{\mathrm{C}^2}\cot\theta'.$

This is the aberration for an observing system in an atmosphere of index μ: its value to the first order appears to involve the factor μ^{-1}, not μ as considerations of velocity would suggest, which can be neglected for air. But the index of the aqueous humour of the eye, or of the water in Airy's water telescope, is not relevant, because it belongs to the observing system. **Bradley aberration: atmospheric influence.**

The Fresnel problem refers to a system in motion without change of configuration: the formula has been finally verified by Zeeman, with the necessary intricate adjustments, for water running through a tube. The difference for transmission across a slab of glass at rest and set in motion has also been verified by Zeeman. Here the theory is simpler, though an incident ray gives rise to a reflected ray and a transmitted ray each made up of contributions from successive to-and-fro reflexions in the slab: for each of these resultants is a simple train of waves, and we have only to determine, by change to a moving frame, how its phase is modified from the value that holds for a slab at rest. Cf. Lorentz, *Pasadena Lectures*, § 35. **Effect of convection of a slab of glass.**

p. 45. It is to be observed that the inverse square law which is essential to molecular refractions, as here analysed in terms of equivalent cavitations, derives ultimately from the general ideas of elastic propagation: after the manner of the cognate, but more deep-seated, relation of regular propagation to internal molecular scattering considered on p. 753. **Continuous optical analysis implies inverse square laws.**

p. 50. For a medium whose atoms admit of storage of part of the incident radiant energy, as in modern theories inspired by spectroscopic relations, the direct way of arriving at the formula of dispersion would be, following Rayleigh's remark (p. 759), to calculate the vibration dispersed or emitted by the atoms in the direction of the beam, coherently but in quadrantal phase, and deduce the velocity of the compound transmitted beam by combining it with the original incident beam. **The slowing of light waves by matter.**

p. 119. Dust or gas released from the head of a comet, even by explosion for the velocity is relatively slight, will accompany the comet in its gravitational orbit, while it is also exposed to the full blast of repulsion by the Solar radiation: thus whatever be the size of the particles, or the velocity of the head, the comet and dust would be expected to form a system with one tail pointing at first directly away from the Sun, however it be broken up later. A massive planet would hold back the dust, so could have no tail. **Comet's tail at first away from Sun: depends on its small mass.**

pp. 251, 261, footnote. For propagation in n dimensions the solution of the final equation in r can be derived, when n is odd (after Gaskin), by differentiation from the result for $n = 3$: the pulses can therefore travel without trail in any medium of odd dimensions, in agreement with general theory, after Volterra, *Acta Math.* XVI, as quoted, p. 261. **Pulses can travel only in spaces of odd dimensions.**

p. 321. Cf. "Note on...Theory of the Action of Magnetism on Light," *American J. of Math.* Vol. XIX, pp. 371–376 (1897), not here reprinted.

Constants of Nature as of variational origin.

p. 401. In one of the alternative modes of connecting up the Planck formula for natural radiation with light *quanta*, Einstein (1917) has asserted a threefold balance between incident and scattered radiation and spontaneous atomic emission. This introduces by the variational method undetermined constants in excess, each also presumably absolute for all types of atoms: the number appears to be reduced to the one that is representative of temperature by equating two of them.

Momentum of waves.

p. 434. The relation of travelling momentum to energy is instructively different if the cord illustrates a dispersive medium, with its group transmission. Cf. Rayleigh, *Papers*, Vol. VI, p. 236 (1914).

p. 448, last line, *for* double *read* is another aspect of *and delete* twice

Variation of latitude.

p. 485. Annual features are found to emerge on comparing the international Kimura term in the latitude with that derived from Greenwich observations alone: see the Greenwich reduction (1928), p. 29, for the records of the zenith telescope.

Critical range for water: usual scheme too simple.

p. 511. According to H. L. Callendar's preliminary account of experimental exploration for water substance, claimed to be very coherent over a wide field, in the correlative scheme of curves connecting total heat, instead of volume, with pressure at the various temperatures (*Engineering*, Nov. 30, 1928, from *Roy. Soc. Proc.* Aug. p. 466) the unstable region separating liquid from gas ends in a sharp cusp, instead of bending round gradually as the usual illustrative equations of state connecting volume with pressure have required. The system was found to be very sensitive to dissolved air. The properties of the critical transition are thereby upset fundamentally: *e.g.* just above the critical point the elastic reaction to compression would no longer tend to vanish, with resulting instability of density. No single equation, such as that of van der Waals, seemed possible to even illustrate both the phases, which were treated separately with satisfactory results on a hypothesis of gradual coaggregation of molecules of the vapour.

An anticipation confirmed.

But more than this is involved. The collapse of sharp transition appears to conform to what was predicted, with reluctance, *supra*, p. 511: even extremely rapid transition in density would involve abolition of surface energy, and so the disappearance of a curved meniscus which with Callendar is the criterion.

Stellar distances.

p. 587. It appears that this principle had been used also by Newton himself, after Huygens, with Jupiter as the intermediary, to estimate the distance of the stars, in a posthumous tract on the "System of the World."

Sun's general magnetic field:

p. 615, line 17 *seq.* By confusion of thought these estimates have been attempted for free electrons at rest, and l is free path of the electron not of the radiation. For actual circumstances, when the electrons are in motion like a gas, the electronization required to produce the Solar magnetic fields proves to be much smaller, as follows.

as due to rotation and radiation:

From a stream of radiation of intensity ϵ it appears (Vol. I, p. 663) that a free electron disperses $\frac{8}{3}\pi\,(e/m)^2e^2\epsilon$, being the same for all wave-lengths. The momentum in a ray is energy divided by C, and momentum dispersed by the electron per unit time measures its repulsion by the incident radiation. If l is mean free path and v_0 velocity of the electron, this repulsion F may be regarded as operating for the free time of order l/v_0 (cf. p. 338), thus generating velocity v along its direction, of rough mean value for purpose of reckoning steady electric flux, equal to half its final amount Fl/mv_0, which is not held up by reaction but ends by constantly degrading into the fortuitous motions of temperature by the collisions with obstructing atoms. The aberration factor of the emergent Solar radiation, at its maximum at the Sun's equator, is $\frac{7}{10}10^{-5}$: this gives for a numerical density N of electrons an electric flux of intensity $\frac{7}{10}10^{-5}Nev$ in the direction of the motion of the electrons with the rotating Sun: this is at the Sun's equator

$$\tfrac{8}{3}\pi\left(\frac{e}{m}\right)^3 e\,\frac{\epsilon}{c}\,\tfrac{1}{2}\,\frac{l}{v_0}\,\tfrac{7}{10}10^{-5}Ne.$$

If v_0 has the temperature value it is 6.10^6 at six thousand degrees, and only 30 times greater at six million: for the former surface value this electric flux appears to figure out to intensity $\frac{1}{2}10^{-31}Nl$, for the latter to $\frac{1}{6}10^{-32}Nl$. It changes with position in the Sun as indicated *loc. cit.* To produce a magnetic field H at the equator of the Sun, of radius a, would require a total magnetic moment of order $\frac{1}{2}Ha^3$, involving axial electric flux within the Sun of averaged order of intensity Ha^{-1}: attending only to very rough averages, to produce a general field of order 30, as observed, Nl in the Sun would have to range around 10^{12} for this lower temperature and around $\frac{1}{3}10^{14}$ for the higher, values which are both abundantly within the facts for electronization of gases. The value of l would be determined mainly by collisions with the atoms; but if the electronization were nearly complete Nl, and therefore the electric flux, would tend to be the same at all densities, a rarefied atmosphere being thus very potent magnetically.

the electronization required.

It appears (p. 616) that for a large sunspot of radius 10^9 with velocity of whirl as much as 10^6 a magnetic field as high as 10^4 would require N to be increased from this lower estimate appropriate to surface temperature about 10 times: thus the vortical velocity in sunspots seems to be adequate without any call for abnormally intense electronization.

Also the intense fields of sunspots.

This formulation of the effects of aberration of the emergent radiation implies the view of the relations of propagation and scattering in atomic media, advanced on p. 759. The outward radiation in the Sun is to be regarded as on the whole coherent, within bounds as regards wave-length, however great the opacity of the medium may be. It is not as if the reinforcement from each atom it passes over were sporadic, relative to that atom and its motion alone: then there could be no regularity in the propagation and no velocity such as local thermal equilibrium demands. On the other hand, viscosity is an affair of the atoms alone, affected by their relative motion in pairs, and outside any influence of a stream of radiation passing across the medium. The contributions from the elements and ions conspire in phase in a certain direction, that of the ray along which energy travels, while their quadrantal difference is the cause of the diminution of velocity from c to c/μ. A convection of the medium affects this reduced velocity in free space by adding the Fresnel term $(1-\mu^{-2})\,v$, modifying the paths of the rays by Fermat's principle to an extent which for the Solar gas is negligible.

Coherence of stream of radiation is involved.

p. 752. Subject to unimportant corrections, Poynting's conclusions and suggestions in two lectures of dates 1904–6 (*Scientific Papers*, pp. 629–720) on the astronomical effects of pressure of radiation, deserve closer attention. For example, he points out how, in a mass of cometary dust, the incident Solar radiation would sift away the smaller particles into a trail which, he suggests, might be the replenisher of the haze within the Solar system which reveals itself by its illumination as the zodiacal light, as it is constantly being drawn into the Sun by his radiation. On modern ideas the radiation from an adjacent bright star might even be imagined to sift out to some degree the different kinds of gas in a nebula (cf. p. 755) if time were long enough.

Sifting effect of radiation.

pp. 615, 811. One is tempted to speculate further from the appearances presented by sunspots. In a vortex underneath the surface the pressure and density of the Solar gas would diminish inward on account of the centrifugal force: at depths where the density is considerable the stream of radiation near the vortex will thus be bent by refraction, after the manner of *mirage* in the Earth's atmosphere. Although this outward stream is everywhere being intensely absorbed or scattered, and renewed by fresh contributions from the atoms over which it passes, the Fermat principle of least time of transit may still be expected to be applicable, at any rate in a general way, for its rays; on the foundation of the Rayleigh synthesis (p. 757) of reduced velocity of propagation in material media, and consequent refraction, as itself a result of molecular scattering in direction along the ray where it is completely coherent in phase and thus far more potent (p. 758) than the sideway scattering which weakens the beam. In usual circumstances, bending can thus be important while scattering is negligible. For a vortex in the radial direction, thus roughly parallel to the emergent stream of radiation, the rays will therefore (pp. 486, 645) curve outwards from the vortex core, because the time of transit between two points on the path will be shortened by approach of the middle part to the core where the velocity of radiation is greater. The upward stream would thus be diverted sideways by the vortex, so that there would be a deficiency of radiation emerging centrally that would show as a dark spot. For a perfect radiator the temperature at the surface is determined solely by the intensity of emission into free space; likewise the strata at the dark spot, though not perfect radiators, must be of lower temperature, which will annul to some degree, near the surface, the centrifugal diminution of density. The effort to adjustment of temperatures by internal radiation is thus an essential part of the complex scheme of interactions. When the vortex is radial, so that all the features are of cylindrical type, the reduced density towards its core will be in adjustment vertically with Solar gravity; the mathematical analysis of this, if it were feasible, might require to some degree a local depression or dimple at the surface as well as reduced density along a column. The penumbra is a conspicuous feature in the appearance of sunspots, with its radial structure. If we may regard the outward radiation as made up of component parallel streams in definite directions each nearly radial, they will be deviated differently in travelling along the vortex: the dark regions will overlap for them in the central region of the spot, but only partially on the outside where striation may conceivably be established, on the lines of an analogy that follows.

Fermat's principle applicable in the Sun. — *A radial vortex diverts the stream of radiation sideways,* — *producing a dark spot: of lower temperature.* — *The penumbra.*

For a strikingly vivid illustration of this train of ideas, from which in fact they in part arose, can readily be arranged. A bath filled with water is illuminated by a lamp some distance above it, and the usual turbulent fluid motion is set up: as the motion subsides the more permanent residual vortical features become localized in time into regular whirls: these are projected on to the matt white surface of the bottom of the bath as round dark spots which may be surrounded by spiral dark converging streaks, the whole simulating rather closely the patterns revealed by Hale in the rarefied hydrogen gas high over sunspots. Here the dark spots arise from refraction of the downward illumination at the sides of the vortical dimples on the surface of the water, the analogue of the axial change of density in the Solar gas: while the patterns converging spirally around them have been impressed on the beam of light by refraction at the spirally ribbed pattern on the surface, being able to persist in it until the bottom is reached. Granular patterns recalling small Solar faculae may also be noted, due possibly to consolidation of residual undulatory disturbance between the vortices. In the Solar pattern the spirality is due to emission from the hydrogen itself: thus its dark ribs might arise either from diminished aggregate column of the gas in uniform illumination or from a uniform distribution of gas scattering a ribbed illumination from below, after the example of the bath: and further speculation becomes too precarious. Especially the phenomena of sunspots near the edge of the disc may present

Spot formation simulated by refraction at water whirls, — *especially the hydrogen spirals.*

difficulties. But the general combination of vortical effects with shadows due to refraction may invite further scrutiny by astronomers in this connection.

As another instance, the dark bands along a spiral nebula need not perhaps always be due to opaque matter in front of it: they might be indications of clefts of reduced density along the line of sight, not extremely deep, though smallness of refraction* has to be compensated by length of path so that the deviation, equal to transverse gradient of index of refraction multiplied by length over which it is present along the ray, ought to be of the order of the angular breadth of the observed dark band. If we imagine reversal of the light, drawing a sheaf of rays *from* the observer, which are refracted away from the cleft, a boundary would be obtained of the region from which alone radiation whether of stars or nebular matter could reach him along this direction, all the depths beyond this sheaf of rays being shut off by refraction†. If this order of ideas were tolerable some nebulae that are apparently flat spirals might come under suspicion of being elongated spindles. Here study is essential of the wealth of material and discussions for judgment on such questions, which constitutes Vol. XIII (1918) of *Lick Observatory Publications*.

Dark bands on nebulae need not be opaque.

p. 667. *For* line 13 from foot *read* "yet capable of dispersal by a grating into independent beams," and cf. p. 758.

p. 692, footnote. This remark may profitably be expanded. The exact dynamical equations of vorticity (ξ, η, ζ) in fluid are

$$(D/dt - \nu\nabla^2)(\xi, \eta, \zeta) = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)(u, v, w),$$

where ν is viscosity divided by density: they are cognate, where gradient of the motion is small, to diffusion of temperature θ from sources, $(d/dt - k\nabla^2)\,\theta = 0$, where k is conductivity divided by specific heat and density. If ν were null, vorticity would be simply convected, after the manner explored by Helmholtz: but actually while it is being convected it diffuses into the surrounding fluid according to a kinematic modulus ν, thus comparable with thermal diffusion.

Diffusion of vorticity,

Some orders of magnitude will vivify this mode of representation. It appears that for metals such as copper k is of the order $15 \cdot 10^{-1}$, for rock such as granite it is of the order $2 \cdot 10^{-2}$, for wood of the order 10^{-3}: while ν for water is of the order 10^{-2} at 20° C. and for air is of the order $15 \cdot 10^{-2}$. Thus, for example, heat diffuses in granite on about the same scale as vorticity diffuses in water.

as compared with heat.

Consider a sphere anchored in a stream of water. If the surface conditions could adapt themselves to perfect slip, viscosity would not come into play at all and the stream lines would be of the familiar frictionless type. This type

* On this Rayleigh order of ideas the particles to which the refraction is due need not be gaseous, provided they are densely distributed in the cubic wave-length (p. 758): which is corroborated by the molecular refraction persisting nearly constant into the liquid and solid states.

Optical refraction possible by fine dust.

† The generalization to cones of vision, of the Cotes theorem of apparent distance (p. xxi) which is of the very essence of practical relativity, involves that in a refracting cosmic field where apparent extent is increased the brightness of each of the objects in it is increased in the same proportion, save for a factor the square of the index. This appears to afford a sort of probe, with whatever results, for the rays bending slowly over vast distances within nebulae. From a different aspect, if the Cotes principle did not hold good natural radiation in an isolated region could never attain to an equilibrium. This aspect involves that the illumination per unit area at each position is not altered by the visual distortion though its direction is altered, which concurs only partially with the surmise in the text. Yet the total brightness of each luminous object appears to be increased in proportion to transverse areal shrinkage; which involves partial obliteration of the individual stars in lanes or other regions where the nebula is seen expanded. (Cf. p. 519.)

Distances in nebulae as apparent.

of flow can be established throughout instantaneously by the interaction of fluid pressure and inertia, for waves of compression travel with the speed of sound which in this connection may be regarded as infinite. But actually such establishment is inhibited locally by the surface of the sphere gripping the fluid, with the result that the adjacent layer of fluid tumbles over upon itself after the manner of a tidal bore advancing up a river, giving rise ultimately to a cushion of vortices which ease off, like friction wheels, the slip between solid and fluid. These vortices are, by Helmholtz's principle, carried away with the stream, and form the wake behind the obstacle: at the same time they are modified by diffusing of the vorticity into the adjacent parts of the fluid with which they travel, for the equations of vorticity are unaltered by uniform convection. Cf. various relevant investigations of surface flow by Lord Rayleigh. But if the stream is moderately fast the water, except close to the obstacle, has flowed past before vorticity can have diffused to any sensible degree out to it from the surface layer in which it originates. The fluid motion around the obstacle is thus very nearly the irrotational type as originally established instantly by propagated pressure, except in this easing vortical layer close to the surface of the obstacle: but behind it the vorticity may gradually spread. This description of the character of the motion near the solid accords with what is seen on looking down over the side of a ship travelling through smooth water: the resistance, other than that due to surface waves, arises from energy expended in the continual creation of the wake. Cf. p. 675.

Vortical bore over surface of obstacle, carried away as a wake: while flow in front is irrotational.

Two different problems.

It seems reasonable that the small deviation from irrotational flow will be established differently, according as it is a sphere being set in motion in resting fluid or a spherical portion of a stream suddenly solidified and reduced to rest.

Cf. a systematic discussion, with references, by Rayleigh, in *Papers*, Vol. VI (1911), pp. 28–40: also various later papers on stability of flow.

Influence of spin in gas theory.

p. 741. These ideas may receive apt illustration in the theory of the magnetism of gases (p. 729). If the magnetic molecules are endowed with intrinsic momentum of spin round their axes, it must stiffen the reaction against rotation in the magnetic field. The modified Action density with this rotational velocity eliminated contains terms of the first order in the remaining component velocities (cf. Vol. I, pp. 47, 67). It may be reduced, as in this gravity problem, to a quadratic in terms of local momentoids, together with a modified form for the positional part. This will alter the Langevin expression for induced magnetization, in a way that may be readily worked out.

Note also that if one, or several, of the internal coordinates of a system is locked by pure constraint, thus giving rise to a new system, that system also has its Liouville differential invariant. The original invariance thus involves invariance as regards variation of every partial set of its coordinates and their momenta.

Scattering of radiation.

pp. 753, 760. Cf., and contrast, as regards theory of scattering in general, the ideas of a discussion by C. V. Raman, "A Classical Derivation of the Compton Effect," *Indian Journal of Physics*, Dec. 15, 1928.

Nebulae show no colour.

p. 759. The stoppage of the light from streaks in a nebula, whether gaseous or composed of dust, by bending of the rays after the manner of terrestrial *mirage*, would be non-selective, being deviation rather than absorption. The former would in any case be potent before the latter becomes effective.

p. 761. *For* vd/dt − *read* − vd/dt.

Thresholds.

p. 766. Cf. the memoir of B. van der Pol and J. van der Maek on "The Heartbeat considered as a Relaxation Oscillation...," *Phil. Mag.* Nov. 1928, where the mode of establishment of a regular beat is illustrated from special types of equations.

p. 795. The value of $|\eta_s|$ has for the purpose of the argument to be independent of the distance, a condition which appears on examination to be satisfied. **Retarded potentials.**

p. 807. To elucidate in another direction the essential distinction at the beginning of the next note, between the field of algebraic formulae for continuous electric density superposed on aethereal structure in space, and the physical theory of one complex medium of the type of aether associated with and controlled by atomic matter, it seems profitable to revert to the difficulties that were encountered in the discussions relating to the stress-energy tensor, as made fundamental with the aim of its absorbing into itself all mechanical science. It was noted by Hilbert and also Lorentz, followed by Einstein and by F. Klein, that this mechanical tensor theory could be adapted so as to arise out of an Action procedure, after the electrodynamic model, for the continuous single extension, with its appropriate continuous densities as above, provided various restrictions and adjustments, perhaps essentially empirical, held good as regards the character of the permitted variations: the perplexity that showed itself in this regard, amidst the variety of tentative processes of abstract analysis, with converging agreement, may be illustrated from the important exchange of Notes between Klein and Hilbert within the Göttingen Academy (*loc. cit.*, F. Klein, *Abhandl.* I, pp. 553–612). The annulling of the various types of variation in the Action, after the manner perhaps suggested by the type of result thus aimed at, leads to a large scheme of equations of control for this continuous field, of which four are proved algebraically to be superfluous as being involved in the others, doubtless ultimately on account in fact of the invariance of the form of the Action: a result which can be expressed in the form that the divergence of a certain quadratic tensor is identically null. The science of mechanics thus appears to present itself in the vacuous guise of a set of algebraic identities involving vectors and tensors belonging to a geometric continuum: and this point of view has even been widely stressed. **A stress tensor for a continuous medium: identically balanced:**

The discussions above referred to leave the impression that this scheme, like various *quantum* constructs, works by virtue of special algebraic devices introduced solely with a view to that end, in fact taking the place of physical imagery with its natural limitations. For example, the restriction required on the variation as regards densities and their field is to make it (with Klein, p. 570) a tangential transformation subject to the equations of that field: a postulate there tentative, but foreshadowing some more coherent procedure which the physically duplex structure of the medium can reveal. **the Action procedure then empirical.**

The dynamic of free aether, as it has evolved historically, is in line with the primitive dynamic, after Lagrange and Green, of a continuous elastic vibrating medium: the variation of the distribution of Action is first to be annulled with regard to the most general variation of the strain, with no regard to whether this variation is a connected geometric displacement in the frame of reference or has a value at each place entirely arbitrary. In fact a purely geometric displacement for ordinary space ought not to alter the essential physical strain, when no electric field is involved: so if the Action function has been constructed subject to the proper invariances, the result ought for this particular type of variation to be a tensorial identity, like the one here in question: as is not unfamiliar in elastic theory: it is not however connected with the Poynting formulation for flux of energy. But the variation of the Action has to vanish as above for unrestricted variation, not however now as an identity but as providing the aggregate of conditions determining local equilibration in the free aether. The free aether is then taken to have settled to equilibrium in the wider kinetic sense, as regards its field, which abuts on and is sustained by its electrons and other atomic nuclei; and their own equilibration, again in the full sense of kinetic balance, remains to be secured by annulling the further **Evolution of elastic theory by Action: identities: equilibration of medium: interaction of sources.**

variation of the Action, thus internally determined when their positions are given, arising from mutual displacement of these positions. The variation thus produced in the field is not now uncontrolled and so arbitrary at each point in free aether, but is continuous, subject to the equilibrium conditions in the field as already secured (cf. Klein, p. 570) as illustrated here on p. 801: the procedure there shown is that each variation of the field is split into one arising by way of gradient from an unconditioned displacement together with terms involving the gradient of the equilibrated displacement. But for both types of displacement, of which they are the difference, the variation of the Action is already annulled: therefore for them alone it is null identically, as on p. 801. For any finite region in free aether it can be expressed by integration by parts of this gradient of displacement, as the equilibration of a stress tensor over its boundary. But each electron or other nucleus is a singular locality of stress locked into the aether: if any such are within the region each of them must be cut out, after the manner introduced by Green, by inserting a boundary closely surrounding it, and a stress acting over that boundary then takes part in the equilibration. Expressed conversely, the nucleus makes its local contribution to the balancing of the stress tensor over the outer boundary as regards that place, its reversed resultant being the reaction on the nucleus in the form of a set of local forces acting there and arising from its structure as a centre of intrinsic stress. In the usual circumstances, contemplating an interlocked dense atomic system forming a material body of ordinary experience, these atomic forces themselves consolidate into the expression of a material stress, as in the standard theory of an elastic atomic medium; and this stress can be added to the stress tensor making the customary aggregate which has now null divergence everywhere. It is in this way that the internal equilibrium of the free aether involves its own stress which is linear and is for example operative in the transmission of radiation, while interaction of the electrons or other sources is also involved in the Action and can be appropriately expressed by a formal quadratic field stress between them, equilibrating over every boundary which does not contain such sources, in identical manner, on account of the equilibrium of the aether previously secured by its own linear equations. But the essential here is that the variation is to be made as that of a straining frame along with which the sources are displaced, so that they have themselves the same virtual displacements as the frame in general, the equilibration of the free aether between the sources, as already secured, involving that the displacements throughout it are not arbitrary, but are determined by the virtual displacements of the sources, thus having at each point definite gradients which, as above, particularize the result.

Emergence of the stress tensor:

it gives the interaction with each source:

which conversely can be absorbed into the tensor.

The Action analysis involves more than propagation by contact.

The stress tensor belongs to the sources.

This would explain how the Faraday-Maxwell stress theory, promoted by Einstein to be the concentrated expression of all dynamics, arises on an atomic theory: if the universe is not made up of aether and matter, it can appear only as a plausible algebraic process, with accidentally resulting identities, avowedly tentative, and liable to internal misfits of the type that can arise from undue narrowness of the foundation scheme.

Historical evolution of the method of variations:

now generalized.

The method of virtual displacements or variations, for a mechanical system, originated in systematic manner with Newton in the *Scholium* to his Third Law of Motion, which laid down the general principle of interaction. It required a century for its full development and consolidation, largely in the hands of d'Alembert and Lagrange. If, to gain a coherent view of the dynamics of electric and radiational science, it has to be envisaged as an affair of a duplex medium, aether and its atomic matter, it is only to be expected that the application of the variational calculus will require a closer formulation, whether on the foundation of a physical scheme in touch with the other branches of natural science, or of empirical mathematical adaptations which may alterna-