CHAPTER I

RATIONAL QUARTIC CURVE C IN [4]

Linear transformations

1·1. If x_i, y_i $(i=0, 1, 2, 3, 4)$ are the projective coordinates of points (x), (y) in four-dimensional spaces X, Y, the linear transformation

$$x_i = \sum_k a_{ik} y_k \quad (i, k = 0, 1, 2, 3, 4),$$

where the determinant of the coefficients $|a_{ik}|$ does not vanish, is interpreted geometrically as a collineation, i.e. a one-one correspondence between the points such that the points of a line, plane, prime of X correspond to points of a line, plane, prime of Y. There are twenty-four essential constants in the transformation; it is possible to determine a collineation (uniquely) so that six given points of X transform into six given points of Y, each five of the six points in both cases being linearly independent but otherwise arbitrary. If the six points of the reference system in X correspond to the points of the reference system in Y, the collineation is simply $x_i = y_i$. Hence the general linear transformation above may equally well be regarded as the change of reference system in X, where x_i and y_i are now the old and new coordinates of the same point of X.

If the spaces X and Y are superposed there are in general five linearly independent points which are transformed into themselves. These are called the double points of the collineation. Referred to the pentad of double points and any point as unit point, the equations of the collineation become

$$x_0 : x_1 : x_2 : x_3 : x_4 = \rho_0 y_0 : \rho_1 y_1 : \rho_2 y_2 : \rho_3 y_3 : \rho_4 y_4; \qquad 1·11$$

and the ratios $\rho_0 : \rho_1 : \rho_2 : \rho_3 : \rho_4$, which are the coordinates of the point corresponding to the unit point, are called the invariants of the collineation.

Particular cases of special importance arise when in 1·11:

$$\text{(i)} \quad \rho_0 = \rho_1 = \rho_2 = 1, \quad \rho_3 = \rho_4 = \rho,$$

in which case every point of the line $x_0 = x_1 = x_2 = 0$, and every point of the plane $x_3=x_4=0$ are double points. The line joining any two corresponding points (x) and (y) meets this line and plane; and ρ, the one invariant, is the cross-ratio of the four points thus obtained upon the joining line. In the case when $\rho=-1$, this collineation is called *harmonic inversion*. If the line and plane of double points are given, the point (y) corresponding in the harmonic inversion to any point (x) may be obtained by drawing through (x) the unique line which meets the line and plane of double points, and taking (y) on this line such that (x) and (y) are harmonically separated by the line and plane.

$$\text{(ii)} \quad \rho_0=\rho_1=\rho_2=\rho_3=1, \quad \rho_4=\rho,$$

in which case every point of the prime $x_4=0$ and the point $(0, 0, 0, 0, 1)$ are double points. This is the case of perspective: the line joining (x) and (y) passes through a fixed point, the isolated double point, O; the four points, (x), (y), O and the intersection of the line with $x_4=0$, have cross-ratio ρ. When $\rho=-1$, the collineation is called *harmonic perspective*.

It should be noted that if in 1·11 the collineation is involutory, so that to (x) corresponds (y) and in the same sense to (y) corresponds (x), we must have

$$\rho_0^2=\rho_1^2=\rho_2^2=\rho_3^2=\rho_4^2=1,$$

and the only cases which arise apart from identical transformation are the harmonic cases above.

Definition of C

2·1. A curve is said to be *rational* when the coordinates of its points can be expressed as rational functions of one parameter. The parameter can always be chosen so that to each value of the parameter corresponds one and only one point of the curve,* i.e. there is a one-one algebraic correspondence between the points of the curve and the points of a line.

Any algebraic curve of order four lies in a space of n dimensions, $n \leqslant 4$, for if the curve were in [5] a prime could be drawn through five general points of the curve and would then contain it entirely.

* Lüroth, *Math. Ann.* **9** (1876) 163.

Every algebraic curve of order four in [4] is rational and without singularities, its points being in one-one correspondence with the points of a line: for a pencil of primes may be drawn through three general points of the curve, each prime having one further intersection with the curve and a corresponding point of intersection with the line; moreover, if the curve has a node, the prime determined by any three points of the curve together with the node has five intersections with the curve and therefore contains it.

The rational quartic curve in [4], C, is given by

$$x_0 : x_1 : x_2 : x_3 : x_4 = f_0(t) : f_1(t) : f_2(t) : f_3(t) : f_4(t),$$

where $f_0, \ldots, f_4$ are polynomials in t, and since the curve is of order four it is met by an arbitrary prime $\Sigma\xi_i x_i = 0$, in four points whose parameters are the roots of $\Sigma\xi_i f_i = 0$, so that the polynomials must be such that at least one is of the fourth order in the parameter t, and none is of higher order. Thus the coordinates of a point (x) of C are given by

$$x_i = \sum_{k=0}^{4} a_{ik} t^k \quad (i = 0, \ldots, 4),$$

wherein at least one of the coefficients a_{i4} is not zero; further, the condition $|a_{ik}| \neq 0$ is necessary to ensure that the curve C does not lie in a space of less than four dimensions.

If $|a_{ik}| = 0$ but the matrix of coefficients is of rank four, then the polynomials $f_i(t)$ are linearly dependent and there is a single identical relation $\Sigma\lambda_i x_i = 0$, which is the equation to a prime in which the curve lies. This prime may then be taken as the fifth prime of the pentad of reference, and the curve lying therein is referred to the tetrad $x_0 = 0$, $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, of planes in which the other primes of the pentad meet the fifth. The curve then has equations

$$x_i = \sum_{k=0}^{4} a_{ik} t^k \quad (i = 0, \ldots, 3),$$

in the prime in which it lies. This curve, which is the general rational quartic curve in [3], may be considered as the projection from the point (0, 0, 0, 0, 1) in [4] upon the prime $x_4' = 0$ of any rational quartic curve in [4] which is given by the above equations together with $x'_4 = \Sigma a'_{4k} t^k$, where the coefficients a'_{4k} are arbitrary subject to the non-vanishing of the new determinant of the coefficients.

Further, if the matrix of the coefficients in the equations of C is of rank three, there are two linear identities in the polynomials and the curve lies in a plane. If the first three polynomials are those which are linearly independent, the curve when referred to a system of coordinates in the plane in which it lies has equations

$$x_i = \sum_{k=0}^{4} a_{ik} t^k \quad (i = 0, 1, 2).$$

This curve, the general rational quartic in the plane, may be considered as the projection of a rational quartic in [4] given by the above equations together with $x'_i = \Sigma a'_{ik} t^k$ $(i = 3, 4)$, the projection being from the line $x_0 = x_1 = x_2 = 0$ upon the plane $x'_3 = x'_4 = 0$.

The rational quartic curve in [4] is *normal* in that it is not the projection of any quartic curve in space of higher dimensions. *Every irreducible rational quartic curve which is not normal is a projection of the normal curve.*

Osculating system of reference

2·2. The equations of the curve C,

$$y_i = \Sigma a_{ik} t^k, \quad |a_{ik}| \neq 0,$$

are reduced by the linear transformation

$$y_i = \Sigma (-1)^k a_{ik} x_k$$

to the form

$$x_i = (-1)^i t^i.$$

The linear transformation is equivalent to the choice of an *osculating system of reference*, and we proceed to find the relation of such a system to the curve. The pentad of reference is $A_0, \ldots, A_4$, where $A_0 = (1, 0, 0, 0, 0)$, etc.; A_0, A_4 and the unit point are on the curve at the points whose parameters are $0, \infty, -1$ respectively. At A_0 $(t = 0)$ we have

the osculating prime $A_0A_1A_2A_3 : x_4 = 0$,
the osculating plane $A_0A_1A_2 : x_3 = x_4 = 0$,
the tangent $A_0A_1 : x_2 = x_3 = x_4 = 0$;

and similarly at A_4 $(t = \infty)$. The point A_2 is that common to the osculating planes at A_0 and A_4, and $A_0A_1A_3A_4$ ($x_2=0$) is the prime bitangent to C at A_0 and A_4; while A_0A_4 ($x_1=x_2=x_3=0$) is the chord joining A_0 and A_4, and $A_1A_2A_3$ ($x_0=x_4=0$) is the axis plane common to the two osculating primes.

A linear transformation of the parameter, $\bar{t}=(at+b)/(ct+d)$, admits of the choice of $a:b:c:d$, and thus any three points of the curve can be given the parameters 0, ∞, -1. By taking the first two of these as A_0, A_4, an osculating pentad of reference can be constructed; the third point is taken as unit point. Hence there are ∞^3 osculating systems of reference; for each of these the coordinates of the curve are given as $x_i=(-t)^i$, and to pass from one to the other is equivalent to a linear transformation of t and a consequent linear transformation of the coordinates (x).

Projective generation of C

2·3. From the equations $x_i=(-t)^i$, it follows that C lies on six linearly independent quadrics whose equations are the determinants of the array

$$\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \right\| = 0. \qquad 2{\cdot}31$$

In [4] there are in all ∞^{14} quadrics; to pass through C imposes on a quadric nine linear conditions, so that the complete system of quadrics through C is a linear combination of those given by 2·31.

The array suggests two complementary generations of the curve C. On the one hand the curve is the locus of the point of intersection of the related pencils of primes

$$x_0+tx_1=0, \quad x_1+tx_2=0, \quad x_2+tx_3=0, \quad x_3+tx_4=0.$$

On the other hand corresponding primes of the two projective stars of primes, through the points (0, 0, 0, 0, 1) and (1, 0, 0, 0, 0), given by

$$\left.\begin{aligned} \lambda_0x_0+\lambda_1x_1+\lambda_2x_2+\lambda_3x_3=0, \\ \lambda_0x_1+\lambda_1x_2+\lambda_2x_3+\lambda_3x_4=0, \end{aligned}\right\} \qquad 2{\cdot}32$$

intersect in planes; then (x) being given subject to 2·31 we have ∞^2 (and thus three linearly independent) solutions for $\lambda_0:\lambda_1:\lambda_2:\lambda_3$; thus of the one star the primes so obtained meet in a line, and the corresponding primes of the other star meet in the corresponding line; and these two lines meet in the point (x) of C. Thus the quartic curve is the locus of points of intersection of corresponding lines of two related stars of primes.

These complementary methods of generation arise in a slightly more general form from the array

$$\left\| \begin{matrix} y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{matrix} \right\| = 0, \qquad 2{\cdot}33$$

where y_i, z_i are any linear functions of the coordinates. In this case the base planes, $y_i = z_i$, of the pencils are four trisecant planes of C, while $y_1 = y_2 = y_3 = y_4 = 0$ and $z_1 = z_2 = z_3 = z_4 = 0$ are two points on C which are the base points of the stars.

A more descriptive method of exhibiting the generation by projective stars is obtained by taking a section of the stars by a general prime. In the section the corresponding lines, planes and primes of the stars appear as corresponding points, lines and planes of a collineation. There are in general four double points of this collineation and they are given by corresponding lines of the stars which meet in the section. Thus the locus of the intersection of corresponding lines is a quartic curve through the base points of the stars. The corresponding planes of the stars are determined by pairs of corresponding lines and thus meet in chords of the quartic curve; the corresponding primes are determined by triads of corresponding lines and hence meet in planes each containing three points of the curve.

A quartic curve can be described through seven general points A_i, by either of the foregoing complementary methods of generating the curve. With A_1 and A_2 as centres, the five lines A_1A_3, A_1A_4, ..., A_1A_7, corresponding to the lines A_2A_3, A_2A_4, ..., A_2A_7, determine related stars whose corresponding lines intersect in points of the curve. On the other hand four projective pencils of primes with bases $A_2A_3A_4$, $A_3A_1A_4$, $A_1A_2A_4$, $A_1A_2A_3$, are determined by three corresponding sets given by the primes to A_5, A_6, A_7, respectively, and the intersection of these primes generates a quartic curve through these three points and A_1, A_2,

A_3, A_4. *Thus seven general points determine uniquely a quartic curve through them.*

2·3, 1. A unique quartic curve can be constructed having given r trisecant planes and $7-r$ points of the curve, in the cases when $r=0, 1, 2, 3, 4, 5$.*

2. The trisecant planes of C meet a general fixed prime in the lines of a tetrahedral complex.

3. The cross-ratio of the four primes joining a variable point on C to four fixed trisecant planes is constant.

4. A variable tangent of C meets four fixed osculating primes in a constant cross-ratio.

5. Corresponding lines of a collineation in [3] which meet do so on a four-nodal cubic surface.

6. If ten points on C be 1, 2, 3, 4, 5, 6, 7, 8, 9, 0, the primes 1234 and 6789, 2345 and 7890, 3456 and 8901, 4567 and 9012, 5678 and 0123, meet in five planes which are associated.†

7. If ten points are the intersections of a quadric with an elliptic quintic curve in [4], the quartic curve determined by seven of them meets the plane through the remaining three in three points.

8. If $P_i=0$, $(i=1, \ldots, 10)$, are the equations of ten points of C, then there is a relation $\sum_1^{10} P_i^2 \equiv 0$.

If ten points in [4] are related by $\sum_1^{10} P_i^2 \equiv 0$, the quartic curve through seven of the points has the plane containing the other three as trisecant plane. The proposition 2·3, 6 holds for any ten points for which $\sum_1^{10} P_i^2 \equiv 0$.‡

9. Quartic curves in [4] which pass through six fixed points meet a general prime in tetrads which are polar with respect to a fixed quadric Q in that prime.

The prime is met by the lines, planes and primes joining the six points in the configuration arising from two tetrahedra in perspective; this figure is self-reciprocal with respect to Q.

* White, *Jour. L.M.S.* 4 (1929) 11. The result is also true when $r=6$; Todd, *Proc. Camb. Phil. Soc.* 26 (1930) 323; for $r=7$ there are ten curves; Babbage, *Jour. L.M.S.* 8 (1933) 9.

† Cf. Weddle's theorem in Reye, *Geometrie der Lage*, 3 (1923) 229.

‡ Cf. P. Serret, *Géométrie de direction* (1869) 311.

10. If (x) and (y) are two points on the same quartic curve through the six reference points in [4], then t and θ can be found such that

$$t/x_i + \theta/y_i = 1 \quad (i = 0, 1, \ldots, 4).$$

If (x) and (y) are in the prime $\Sigma\xi_i x_i = 0$, they are conjugate with respect to the quadric $\Sigma\xi_i x_i^2 = 0 = \Sigma\xi_i x_i$.

Fundamental polarity

2·4. In what follows the equations of the curve C are taken as $x_i = (-t)^i$ and a point whose parameter on C is t is spoken of as the point t.

A prime (ξ) meets C at points given by the roots of the equation

$$\xi_0 - \xi_1 t + \xi_2 t^2 - \xi_3 t^3 + \xi_4 t^4 = 0.$$

The coordinates of the prime are thus proportional to the elementary symmetric functions of these roots, so that if t_1, t_2, t_3, t_4 are four points of the curve, the equation of the prime containing them is

$$x_0 t_1 t_2 t_3 t_4 + x_1 \Sigma t_1 t_2 t_3 + x_2 \Sigma t_1 t_2 + x_3 \Sigma t_1 + x_4 = 0. \qquad 2{\cdot}41$$

If $t_1 = t_2 = t_3 = t_4 = t$, this prime is the osculating prime at t; its equation is

$$(xt)^4 \equiv x_0 t^4 + 4x_1 t^3 + 6x_2 t^2 + 4x_3 t + x_4 = 0, \qquad 2{\cdot}42$$

and its coordinates are

$$\xi_0 : \xi_1 : \xi_2 : \xi_3 : \xi_4 = t^4 : 4t^3 : 6t^2 : 4t : 1.$$

There are therefore *four osculating primes through a given point* (x), their points of contact being the roots of the equation 2·42. The prime (ξ) determined by the points of contact is called the *polar* prime of (x) and (x) the *pole* of (ξ), the formulae connecting (x) and (ξ) being

$$\xi_0 = x_4, \quad \xi_1 = -4x_3, \quad \xi_2 = 6x_2, \quad \xi_3 = -4x_1, \quad \xi_4 = x_0.$$

Two points (x) and (y) are conjugate in the polarity when

$$x_0 y_4 - 4x_1 y_3 + 6x_2 y_2 - 4x_3 y_1 + x_4 y_0 = 0;$$

and pole and polar prime are incident for points and primes of the quadric

$$i \equiv x_0 x_4 - 4x_1 x_3 + 3x_2^2 = 0. \qquad 2{\cdot}43$$

This polarity is fundamental in the geometry of C; we shall refer to the quadric which is the nucleus of the polarity as I.

In consequence of the polarity the complete figure of C consisting of its points, tangents, osculating planes and osculating primes is self-dual; to a chord corresponds an axis plane common to two osculating primes, to a trisecant plane an axis line, and to a bitangent prime a point common to two osculating planes.

The polar prime of any point of C is the osculating prime at the point and the tangent prime of I. The polar plane of a tangent of C is that common to the two consecutive osculating primes at the two consecutive points which determine the tangent, and is thus the osculating plane at the point of contact of the tangent. As the osculating plane contains the tangent, it follows that any tangent of C lies on I, and this may be verified by direct substitution, the points of the tangent at t being given by

$$(1,\ -t-\lambda,\ t^2+2\lambda t,\ -t^3-3\lambda t^2,\ t^4+4\lambda t^3).$$

Thus as a locus the quadric I contains the curve C and its tangents; and as an envelope contains the osculating primes and osculating planes of C.

Trisecant planes

2·5. A plane containing three points t_1, t_2, t_3 lies in a prime with any fourth point t_4. If (x) is a point on this plane, the formula 2·41 holds for any value of t_4. Hence the equations of the trisecant plane are

$$\left.\begin{aligned} x_0 t_1 t_2 t_3 + x_1 \Sigma t_1 t_2 + x_2 \Sigma t_1 + x_3 = 0, \\ x_1 t_1 t_2 t_3 + x_2 \Sigma t_1 t_2 + x_3 \Sigma t_1 + x_4 = 0, \end{aligned}\right\} \qquad 2{\cdot}51$$

where the summations are taken over t_1, t_2, t_3, or

$$\lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0,$$
$$\lambda_0 x_1 + \lambda_1 x_2 + \lambda_2 x_3 + \lambda_3 x_4 = 0$$

(as in 2·32), and the points of intersection t_1, t_2, t_3 with the curve are the roots of

$$\lambda_0 - \lambda_1 t + \lambda_2 t^2 - \lambda_3 t^3 = 0.$$

The two equations are linearly independent except when

$$\left\| \begin{matrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \end{matrix} \right\| = 0;$$

consequently there are ∞^1 trisecant planes through any point (x)

which is not on C. If a trisecant plane passes also through a second point (y), then (x) and (y) are connected by the relation

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ y_0 & y_1 & y_2 & y_3 \\ y_1 & y_2 & y_3 & y_4 \end{vmatrix} = 0, \qquad 2\cdot52$$

which is therefore the equation to the locus generated by the trisecant planes through a fixed point (y). Thus *the* ∞^1 *trisecant planes through a point generate a quadric cone.*

The equations of the osculating plane at t are obtained by putting $t_1 = t_2 = t_3 = t$ in 2·51 or by regarding the plane as the intersection of consecutive osculating primes; they are thus

$$\left.\begin{aligned} x_0 t^3 + 3x_1 t^2 + 3x_2 t + x_3 = 0, \\ x_1 t^3 + 3x_2 t^2 + 3x_3 t + x_4 = 0. \end{aligned}\right\} \qquad 2\cdot53$$

The elimination of t between these two equations shews the locus of osculating planes to be a sextic primal; thus six osculating planes meet a general line.

Chords

2·6. A chord joining the points t_1, t_2 lies in the same prime with any other two points t_3, t_4; thus from 2·41, any point (x) on the chord satisfies the equations

$$\left.\begin{aligned} x_0 t_1 t_2 + x_1 (t_1 + t_2) + x_2 = 0, \\ x_1 t_1 t_2 + x_2 (t_1 + t_2) + x_3 = 0, \\ x_2 t_1 t_2 + x_3 (t_1 + t_2) + x_4 = 0, \end{aligned}\right\} \qquad 2\cdot61$$

or

$$\begin{aligned} \lambda_0 x_0 + \lambda_1 x_1 + \lambda_2 x_2 = 0, \\ \lambda_0 x_1 + \lambda_1 x_2 + \lambda_2 x_3 = 0, \\ \lambda_0 x_2 + \lambda_1 x_3 + \lambda_2 x_4 = 0, \end{aligned}$$

its intersections t_1, t_2 with the curve being given by

$$\lambda_0 - \lambda_1 t + \lambda_2 t^2 = 0.$$

Thus the chords of C generate the cubic primal, J, whose equation is

$$j \equiv \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0; \qquad 2\cdot62$$

and the primal appears as the locus of the lines of intersection of