CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

INTRODUCTION

1. THROUGHOUT the tract, wherever it has seemed advisable, for the sake of clearness and brevity, to use the language of geometry, I have not hesitated to do so; but the reader should convince himself that all the arguments employed in Chapters I—IV are really arithmetical arguments, and are not based on geometrical intuitions. Thus, no use is made of the geometrical conception of an angle; when it is necessary to define an angle in Chapter I, a purely analytical definition is given. The fundamental theorems of the arithmetical theory of limits are assumed.

A number of obvious theorems are implicitly left to the reader; e.g. that a circle is a 'simple' curve (the coordinates of any point on $x^2 + y^2 = 1$ may be written $x = \cos t$, $y = \sin t$, $0 \leqslant t \leqslant 2\pi$); that two 'simple' curves with a common end-point, but with no other common point, together form one 'simple' curve; and several others of a like nature.

It is to be noted that almost all the difficulties, which arise in those problems of *Analysis Situs* which are discussed in Chapter I, disappear if the curves which are employed in the following chapters are restricted to be straight lines or circles. This fact is of some practical importance, since, in applications of Cauchy's Theorem, it is usually possible to employ only straight lines and circular arcs as contours of integration.

2. NOTATION. If z be a complex number, we shall invariably write

$$z = x + iy,$$

where x and y are real; with this definition of x and y, we write[1]

$$x = R(z), \quad y = I(z).$$

If a complex number be denoted by z with some suffix, its real and imaginary parts will be denoted by x and y, respectively, with the same suffix; e.g.

$$z_n = x_n + iy_n;$$

[1] The symbols R and I are read 'real part of' and 'imaginary part of' respectively.

© in this web service Cambridge University Press www.cambridge.org

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

further, if ζ be a complex number, we write

$$R(\zeta) = \xi, \quad I(\zeta) = \eta.$$

DEFINITIONS. *Point.* A 'point' is a value of the complex variable, z; it is therefore determined by a complex number, z, or by two real numbers (x, y). It is represented geometrically by means of the Argand diagram.

Variation and Limited Variation[2]. If $f(x)$ be a function of a real variable x defined when $a \leqslant x \leqslant b$ and if numbers $x_1, x_2, \ldots x_n$ be chosen such that $a \leqslant x_1 \leqslant x_2 \ldots \leqslant x_n \leqslant b$, then the sum

$$|f(x_1)-f(a)|+|f(x_2)-f(x_1)|+|f(x_3)-f(x_2)|+\ldots+|f(b)-f(x_n)|$$

is called the *variation of* $f(x)$ *for the set of values* $a, x_1, x_2, \ldots x_n, b$. If for every choice of $x_1, x_2, \ldots x_n$, the variation is always less than some finite number λ (independent of n), $f(x)$ is said to have *limited variation* in the interval a to b; and the upper limit of the variation is called the *total variation* in the interval.

[The notion of the variation of $f(x)$ in an interval a to b is very much more fundamental than that of the length of the curve $y=f(x)$; and throughout the tract propositions will be proved by making use of the notion of *variation* and not of the notion of *length*.]

[2] Jordan, *Cours d'Analyse*, §§ 105 *et seq*.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

CHAPTER I

ANALYSIS SITUS

§ 3. Problems of *Analysis situs* to be discussed.—§ 4. Definitions.—§ 5. Properties of continua.—§ 6. Theorems concerning the order of a point.—§ 7. Main theorem; a regular closed curve has an interior and an exterior.—§ 8. Miscellaneous theorems; definitions of counterclockwise and orientation.

3. The object of the present chapter is to give formal analytical proofs of various theorems of which simple cases seem more or less obvious from geometrical considerations. It is convenient to summarise, for purposes of reference, the general course of the theorems which will be proved:

A *simple curve* is determined by the equations $x=x(t), y=y(t)$ (where t varies from t_0 to T), the functions $x(t)$, $y(t)$ being continuous; and the curve has no double points save (possibly) its end points; if these coincide, the curve is said to be *closed*. The *order* of a point Q with respect to a closed curve is defined to be n, where $2\pi n$ is the amount by which the angle between QP and Ox increases as P describes the curve once. It is then shewn that points in the plane, not on the curve, can be divided into two sets; points of the first set have order ± 1 with respect to the curve, points of the second set have order zero; the first set is called the interior of the curve, and the second the exterior. It is shewn that *every* simple curve joining an interior point to an exterior point must meet the given curve, but that simple curves can be drawn, joining any two interior points (or exterior points), which have no point in common with the given curve. It is, of course, not obvious that a closed curve (defined as a curve with coincident end points) divides the plane into two regions possessing these properties.

It is then possible to distinguish the direction in which P describes the curve (viz. counterclockwise or clockwise); the criterion which determines the direction is the sign of the order of an interior point.

The investigation just summarised is that due to Ames[1]; the analysis which will be given follows his memoir closely. Other proofs that a closed curve

[1] Ames, *American Journal of Mathematics*, Vol. XXVII. (1905), pp. 343–380.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

possesses an interior and an exterior have been given by Jordan[2], Schoenflies[3], Bliss[4], and de la Vallée Poussin[5]. It has been pointed out that Jordan's proof is incomplete, as it assumes that the theorem is true for closed polygons; the other proofs mentioned are of less fundamental character than that of Ames.

4. Definitions. A *simple curve* joining two points z_0 and Z is defined as follows:

Let[6] $$x = x(t), \quad y = y(t),$$

where $x(t)$, $y(t)$ are continuous one-valued functions of a real parameter t for all values of t such that[7] $t_0 \leqslant t \leqslant T$; the functions $x(t)$, $y(t)$ are such that they do not assume the same pair of values for any two different values of t in the range $t_0 < t < T$; and

$$z_0 = x(t_0) + iy(t_0), \quad Z = x(T) + iy(T).$$

Then we say that the *set of points* (x, y), determined by the set of values of t for which $t_0 \leqslant t \leqslant T$, is a *simple curve* joining the points z_0 and Z. If $z_0 = Z$, the simple curve is said to be *closed*[8].

To render the notation as simple as possible, if the parameter of any particular point on the curve be called t with some suffix, the complex coordinate of that point will always be called z with the same suffix; thus, if

$$t_0 \leqslant t_r^{(n)} \leqslant T,$$

we write $$z_r^{(n)} = x(t_r^{(n)}) + iy(t_r^{(n)}) = x_r^{(n)} + iy_r^{(n)}.$$

Regular curves. A simple curve is said to be *regular*[9], if it can be divided into a *finite* number of parts, say at the points whose parameters are $t_1, t_2, \ldots t_m$ where $t_0 \leqslant t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m \leqslant T$, such that when

[2] Jordan, *Cours d'Analyse* (1893), Vol. I. §§ 96–103.

[3] Schoenflies, *Göttingen Nachrichten*, Math.-Phys. Kl. (1896), p. 79.

[4] Bliss, *American Bulletin*, Vol. x. (1904), p. 398.

[5] de la Vallée Poussin, *Cours d'Analyse* (1914), Vol. I. §§ 342–344.

[6] The use of x, y in two senses, as coordinates and as functional symbols, simplifies the notation.

[7] We can always choose such a parameter, t, that $t_0 < T$; for if this inequality were not satisfied, we should put $t = -t'$ and work with the parameter t'.

[8] The word 'closed' except in the phrase 'closed curve' is used in a different sense; a *closed* set of points is a set which contains all the limiting points of the set; an open set is a set which is not a closed set.

[9] We do not follow Ames in assuming that $x(t)$, $y(t)$ possess derivatives with regard to t.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

$t_{r-1} \leqslant t \leqslant t_r$, the relation between x and y given by the equations $x = x(t)$, $y = y(t)$ is equivalent to an equation $y = f(x)$ *or else* $x = \phi(y)$, where f *or* ϕ denotes a continuous one-valued function of its argument, and r takes in turn the values 1, 2, ... $m+1$, while $t_{m+1} = T$.

It is easy to see that a chain of a finite number of curves, given by the equations

$$\left.\begin{array}{ll} y = f_1(x), & a_1 \leqslant x \leqslant a_2 \\ x = f_2(y), & b_2 \leqslant y \leqslant b_3 \\ y = f_3(x), & a_3 \leqslant x \leqslant a_4 \\ \cdots\cdots & \cdots\cdots \end{array}\right\} \quad \text{.............................(A)}$$

(where $b_2 = f_1(a_2)$, $a_3 = f_2(b_3)$, ... and f_1, f_2, ... are continuous one-valued functions of their arguments), forms a simple curve, if the chain has no double points; for we may choose a parameter t, such that

$$x = t, \quad y = f_1(t), \quad a_1 \leqslant t \leqslant a_2;$$
$$x = f_2(b_2 - a_2 + t), \quad y = b_2 - a_2 + t, \quad a_2 \leqslant t \leqslant a_2 - b_2 + b_3;$$
$$x = a + t, \quad y = f_3(a + t), \quad a_3 - a \leqslant t \leqslant a_4 - a, \quad a = a_3 - a_2 + b_2 - b_3;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

If some of the inequalities in equations (A) be reversed, it is possible to shew in the same manner that the chain forms a simple curve.

Elementary curves. Each of the two curves whose equations are (i) $y = f(x)$, $(x_0 \leqslant x \leqslant x_1)$ and (ii) $x = \phi(y)$, $(y_0 \leqslant y \leqslant y_1)$, where f and ϕ denote one-valued continuous functions of their respective arguments, is called an *elementary* curve.

Primitive period. In the case of a closed simple curve let $\omega = T - t_0$; we define the functions $x(t)$, $y(t)$ for *all* real values of t by the relations

$$x(t + n\omega) = x(t), \quad y(t + n\omega) = y(t),$$

where n is any integer; ω is called the primitive period of the pair of functions $x(t)$, $y(t)$.

Angles. If z_0, z_1 be the complex coordinates of two distinct points P_0, P_1, we say that 'P_0P_1 makes an angle θ with the axis of x' if θ satisfies *both* the equations[10]

$$\cos\theta = \kappa(x_1 - x_0), \quad \sin\theta = \kappa(y_1 - y_0),$$

where κ is the positive number $\{(x_1 - x_0)^2 + (y_1 - y_0)^2\}^{-\frac{1}{2}}$. This pair of equations has an infinite number of solutions such that if θ, θ' be any

[10] It is supposed that the sine and cosine are defined by the method indicated by Bromwich, *Theory of Infinite Series*, § 60, (2); it is easy to deduce the statements made concerning the solutions of the two equations in question.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

two different solutions, then $(\theta' - \theta)/2\pi$ is an integer, positive or negative.

Order of a point. Let a regular closed curve be defined by the equations $x = x(t)$, $y = y(t)$, $(t_0 \leqslant t \leqslant T)$ and let ω be the primitive period of $x(t)$, $y(t)$. Let Q be a point *not* on the curve and let P be the point on the curve whose parameter is t. Let $\theta(t)$ be the angle which QP makes with the axis of x; since every branch of $\arccos\{\kappa(x_1 - x_0)\}$ and of $\arcsin\{\kappa(y_1 - y_0)\}$ is a continuous function of t, it is possible to choose $\theta(t)$ so that $\theta(t)$ is a continuous function of t reducing to a definite number $\theta(t_0)$ when t equals t_0. The points represented by the parameters t and $t + \omega$ are the same, and hence $\theta(t)$, $\theta(t+\omega)$ are two of the values of the angle which QP makes with the axis of x; therefore

$$\theta(t+\omega) - \theta(t) = 2n\pi,$$

where n is an integer; n is called the *order* of Q with respect to the curve. To shew that n depends only on Q and not on the particular point, P, taken on the curve, let t vary continuously; then $\theta(t)$, $\theta(t+\omega)$ vary continuously; but since n is an integer n can only vary *per saltus*. Hence n is constant[11].

5. Continua. A two-dimensional continuum is a set of points such that (i) if z_0 be the complex coordinate of any point of the set, a positive number δ can be found such that all points whose complex coordinates satisfy the condition $|z - z_0| < \delta$ belong to the set; δ is a number depending on z_0, (ii) any two points of the set can be joined by a simple curve such that all points of it belong to the set.

Example. The points such that $|z| < 1$ form a continuum.

[11] This argument *really* assumes what is known as Goursat's lemma (see § 12) for functions of a real variable. It is proved by Bromwich, *Theory of Infinite Series*, p. 394, example 18, that if an interval has the property that round every point P of the interval we can mark off a sub-interval such that a certain inequality denoted by $\{Q, P\}$ is satisfied for every point Q of the sub-interval, then we can divide the whole interval into a *finite* number of closed parts such that each part contains at least one point P_1 such that the inequality $\{Q, P_1\}$ is satisfied for all points Q of the part in which P_1 lies.

In the case under consideration, we have a function, $\phi(t) = \theta(t+\omega) - \theta(t)$, of t, which is given continuous; the inequality is therefore $|\phi(t) - \phi(t')| < \epsilon$, where ϵ is an arbitrary positive number; by the lemma, taking $\epsilon < 2\pi$, we can divide the range of values of t into a finite number of parts in each of which $|\phi(t) - \phi(t_1)| < 2\pi$ and is therefore zero; $\phi(t)$ is therefore constant throughout each part and is therefore constant throughout the sub-interval.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE
Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

Neighbourhood, Near. If a point ζ be connected with a set of points in such a way that a sequence (z_n), consisting of points of the set, can be chosen such that ζ is a limiting point of the sequence, then the point ζ is said to have points of the set *in its neighbourhood.*

The statement 'all points *sufficiently near* a point ζ have a certain property' means that a positive number h exists such that all points z satisfying the inequality $|z-\zeta|<h$ have the property.

Boundaries, Interior and Exterior Points. Any point of a continuum is called an *interior* point. A point is said to be a *boundary point* if it is not a point of the continuum, but has points of the continuum in its neighbourhood.

A point z_0, such that $|z_0|=1$, is a boundary point of the continuum defined by $|z|<1$.

A point which is not an interior point or a boundary point is called an *exterior* point.

If (z_n) be a sequence of points belonging to a continuum, then, if this sequence has a limiting point ζ, the point ζ is either an interior point or a boundary point; for, even if ζ is not an interior point, it has points of the continuum in its neighbourhood, viz. points of the sequence, and is therefore a boundary point.

All points sufficiently near an exterior point are exterior points; for let z_0 be an exterior point; then, if *no* positive number h exists such that *all* points satisfying the inequality $|z-z_0|<h$ are exterior points, it is possible to find a sequence (ζ_n) such that ζ_n is an interior point or a boundary point and $|\zeta_n-z_0|<2^{-n}$; and, whether ζ_n is an interior point or a boundary point, it is possible to find an *interior* point ζ_n' such that $|\zeta_n'-\zeta_n|<2^{-n}$; so that $|\zeta_n'-z_0|<2^{1-n}$, and z_0 is the limiting point of the sequence ζ_n'; therefore z_0 is an interior point or a boundary point; this is contrary to hypothesis; therefore, corresponding to any particular point z_0, a number h exists. The theorem is therefore proved.

A continuum is called an[12] *open region*, a continuum with its boundary is a *closed region.*

Example. Let S be a set of points $z\ (=x+iy)$ defined by the relations

$$x_0 < x < x_1, \qquad y=f(x)+r \quad \text{..........................(1)},$$

where f is one-valued and continuous, r takes all values such that $0<r<k$, and k is constant. Then the set of points S forms a continuum.

[12] See note 8 on p. 4.

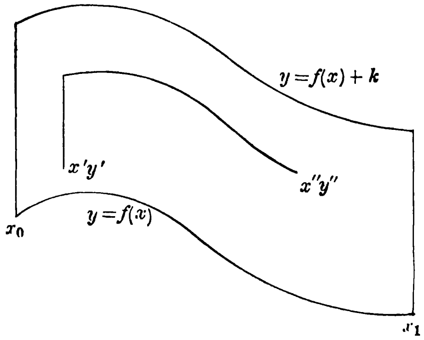
© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

Let z' be any point of S, so that

$$x_0 < x' < x_1, \quad y' = f(x') + r', \quad \text{where } 0 < r' < k.$$



Choose $\epsilon > 0$, so that

$$2\epsilon < r' < k - 2\epsilon \quad \text{.................................(2).}$$

Since f is continuous we may choose $\delta > 0$, so that

$$|f(x) - f(x')| < \epsilon \quad \text{.................................(2a),}$$

when $|x - x'| < \delta$. It is convenient to take δ so small that

$$x_0 + \delta < x' < x_1 - \delta \quad \text{.................................(3).}$$

Then $x_0 < x < x_1$ since $|x - x'| < \delta$.

Also, when $|x - x'| < \delta$,

$$f(x) - \epsilon < f(x') < f(x) + \epsilon \quad \text{..........................(3a),}$$

so that if y be any number such that

$$y' - \epsilon < y < y' + \epsilon \quad \text{...................................(4),}$$

then
$$f(x') + r' - \epsilon < y < f(x') + r' + \epsilon \quad \text{........................(4a).}$$

Adding (2), (3a) and (4a), we see that

$$f(x) < y < f(x) + k.$$

Therefore the point $z = x + iy$, chosen in this manner, is a point of the set S. Hence, if δ' be the smaller of δ and ϵ, and if

$$|z - z'| < \delta',$$

the conditions (2a) and (4) are both satisfied, and hence z is a member of the set. The first condition for a continuum is, consequently, satisfied.

Further, the points z', z'' (for which $r' \leqslant r''$), belonging to S, can be joined by the simple curve made up of the two curves defined by the relations

(i) $x = x'$, $(y' \leqslant y \leqslant y' + r'' - r')$,

(ii) $y = f(x) + r''$, $(x' \leqslant x \leqslant x''$ or $x'' \leqslant x \leqslant x')$.

Hence S is a continuum.

© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

6. Lemma. *Any limiting point Q of a set of points on a simple curve lies on the curve.*

Take any sequence of the set which has Q as its unique limiting point; let the parameters of the points of the sequence be $t_1, t_2, \ldots$. Then the sequence (t_n) has at least[13] one limit τ, and $t_0 \leqslant \tau \leqslant T$. Since $x(t)$, $y(t)$ are continuous functions, $\lim x(t_n) = x(\tau)$, $\lim y(t_n) = y(\tau)$; and $(x(\tau), y(\tau))$ is on the curve since $t_0 \leqslant \tau \leqslant T$; i.e. Q is on the curve.

Corollary. If Q_0 be a fixed point not on the curve, the distance of Q_0 from points on the curve has a positive lower limit δ. For if δ did not exist we could find a sequence (P_n) of points on the curve such that $Q_0 P_{n+1} < \frac{1}{2} Q_0 P_n$, so that Q_0 would be a limiting point of the sequence and would therefore lie on the curve.

Theorem I. *If a point is of order n with respect to a closed simple curve, all points sufficiently near it are of order n.*

Let Q_0 be a point not on the curve and Q_1 any other point. Then the distance of points on the curve from Q_0 has a positive lower limit, δ; so that, if $Q_0 Q_1 \leqslant \frac{1}{2}\delta$, the line $Q_0 Q_1$ cannot meet the curve.

Let t be the parameter of any point, P, on the given curve, and τ the parameter of a point, Q, on $Q_0 Q_1$, and $\theta(t, \tau)$ the angle QP makes with the axis of x; then $\theta(t, \tau)$ is a continuous function of τ, when t is fixed; therefore

$$\theta(t + \omega, \tau) - \theta(t, \tau)$$

is a continuous function[14] of τ; but the order of a point (being an integer) can only vary *per saltus*; therefore $\theta(t+\omega, \tau) - \theta(t, \tau)$ is a constant, so far as variations of τ are concerned; therefore the orders of Q_0, Q_1 are the same.

The above argument has obviously proved the following more general theorem:

Theorem II. *If two points Q_0, Q_1 can be joined by a simple curve having no point in common with a given closed simple curve, the orders of Q_0, Q_1 with regard to the closed curve are the same.*

The following theorem is now evident:

Theorem III. *If two points Q_0, Q_1 have* different *orders with regard to a given closed simple curve*, every *simple curve joining them has at least one point in common with the given closed curve.*

Theorem IV. *Within an arbitrarily small distance of any point, P_0, of a regular closed curve, there are two points whose orders differ by unity.*

The curve consists of a finite number of parts, each of which can be

[13] Young, *Sets of Points*, pp. 18, 19.

[14] See note 11 on p. 6.

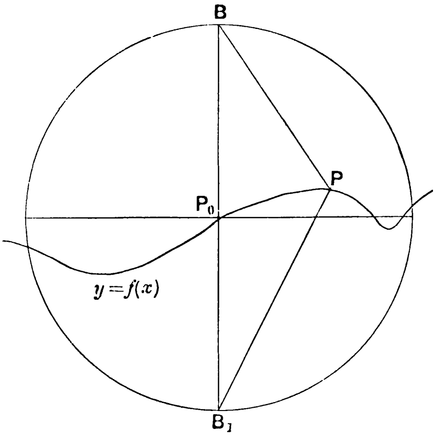
© in this web service Cambridge University Press www.cambridge.org

CAMBRIDGE

Cambridge University Press
978-1-107-49395-7 - Complex Integration and Cauchy's Theorem
G. N. Watson
Excerpt
More information

represented either by an equation of the form $y=f(x)$ or else by one of the form $x=f(y)$, where f is *single-valued* and continuous. First, let P_0 be not an end point of one of these parts.

Let the part on which P_0 lies be represented by an equation of the form $y=f(x)$; if the equation be $x=f(y)$, the proof is similar.



The lower limit of the distance of P_0 from any other part of the curve[15] is, say, r_1, where $r_1>0$.

Hence if $0<r<r_1$, a circle of radius r, centre P_0, contains no point of the complete curve except points on the curve $y=f(x)$; and the curve $y=f(x)$ meets the ordinate of P_0 in no point except P_0.

Let B be the point (x_0, y_0+r), B_1 the point (x_0, y_0-r).

If P be any point of the curve whose parameter is t and if $\theta(t)$, $\theta_1(t)$ be the angles which BP, B_1P make with the x axis, it is easily verified that if $BP=\rho$, $B_1P=\rho_1$ and $\phi=\theta(t)-\theta_1(t)$,

$$\sin\phi=-\frac{2r(x-x_0)}{\rho\rho_1}, \quad \cos\phi=\frac{(x-x_0)^2+(y-y_0)^2-r^2}{\rho\rho_1}.$$

If ω be the period of the pair of functions $x(t)$, $y(t)$ and if δ be so small that the distances from P_0 of the points whose parameters are $t_0\pm\delta$ are less than r, then[16], if $x(t_0+\delta)>x(t_0)$,

$$\phi(t_0)=(2n_1+1)\pi, \qquad \phi(t_0+\delta)>(2n_1+1)\pi,$$
$$\phi(t_0+\omega-\delta)<(2n_2+1)\pi, \qquad \phi(t_0+\omega)=(2n_2+1)\pi.$$

[15] If a *positive* number r_1 did not exist, by the corollary of the Lemma, P_0 would coincide with a point on the remainder of the curve; i.e. the complete curve would have a double point, and would not be a simple curve.

[16] If $x(t_0+\delta)<x(t_0)$, the inequalities involving ϕ have to be reversed.

© in this web service Cambridge University Press www.cambridge.org