INVARIANTS OF QUADRATIC DIFFERENTIAL FORMS

INTRODUCTION

1. In order to discuss in detail the geometry of a plane it is convenient to introduce coordinates. A point in the plane has two degrees of freedom, and therefore to determine it two independent conditions must be satisfied. These conditions may be that two independent quantities* (*e.g.* the distances from two fixed points) take the values u, v at the point, and we then say that the coordinates of that point are u, v. If we suppose one coordinate given, the locus of the point will be a certain curve, *e.g.* $u=$ const., and, generally, a curve in the plane is given by a functional relation $\phi(u, v)=0$.

For the metrical geometry of the plane we need an expression for the distance between any two points in terms of the coordinates of the points. Theoretically this can be calculated if the distance between an arbitrary point (u, v) and any neighbouring point $(u+du, v+dv)$ is known. Suppose that (x, y) and $(x+dx, y+dy)$ are rectangular cartesian coordinates of the two points, then x, y are functions of u, v; also if ds denote the distance between the two points,

$$ds^2 = dx^2 + dy^2,$$

or

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

where

$$E = \left|\frac{\partial x}{\partial u}\right|^2 + \left|\frac{\partial y}{\partial u}\right|^2, \quad F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v}, \quad G = \left|\frac{\partial x}{\partial v}\right|^2 + \left|\frac{\partial y}{\partial v}\right|^2.$$

We have in fact a quadratic form in the variables du, dv, for ds^2, and the coefficients of this form are functions of u, v. If E, F, and G are

* These quantities have not necessarily any obvious geometrical significance.

given it is possible to determine the equations of the straight lines of the plane in terms of u, v, to find the angle between any two of its curves, and, generally, to develope its metrical geometry. Now if the system of coordinates is given, E, F, G can be determined, and the converse question at once arises, namely, if three functions E, F, G are given as three arbitrary functions of u, v, is it possible to take u, v as coordinates of the points in the plane so that the element of length shall be given by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2\,?$$

It appears that this is not possible unless a certain relation

$$K \equiv \frac{1}{2\sqrt{EG-F^2}}\left\{\frac{\partial}{\partial u}\left[\frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial v} - \frac{1}{\sqrt{EG-F^2}}\frac{\partial G}{\partial u}\right]\right.$$
$$\left. + \frac{\partial}{\partial v}\left[\frac{2}{\sqrt{EG-F^2}}\frac{\partial F}{\partial u} - \frac{1}{\sqrt{EG-F^2}}\frac{\partial E}{\partial v} - \frac{F}{E\sqrt{EG-F^2}}\frac{\partial E}{\partial u}\right]\right\} = 0$$

is satisfied for all values of u, v. This condition is also sufficient.

If, instead of limiting ourselves to a plane, we consider any surface in three dimensional space, we have again two coordinates for any point; the element of length is given as before by

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

and there is a surface corresponding to any arbitrary functions E, F, G of u, v. (The particular case $EG = F^2$ is excluded.) It appears however that for a given surface the expression K has the same value at any given point on it, whatever coordinates u, v are chosen. Let u, v and u', v' denote any two sets of point coordinates on the surface, then $u' = f(u, v)$, $v' = \phi(u, v)$, where f and ϕ are arbitrary functions of u, v, and we have the theorem:

If by any transformation

$$u' = f(u, v),\ v' = \phi(u, v),$$

$$Edu^2 + 2Fdudv + Gdv^2 \ \textit{becomes}\ E'du'^2 + 2F'du'dv' + G'dv'^2,$$

then $K = K'$, *where* K' *is* K *in the accented variables.*

2. Definition of a differential invariant.

Any function of E, F, G and their derivatives satisfying this condition is called a *differential invariant* of the form

$$Edu^2 + 2Fdudv + Gdv^2.$$

The idea of a differential invariant may be extended by taking account of any families of curves on the surface, say $\phi(u, v) = \text{const.}$,

$\psi(u, v) = \text{const.}$, etc. When we transform to new variables u', v' we have

$$Edu^2 + 2Fdudv + Gdv^2 = E'du'^2 + 2F'du'dv' + G'dv'^2,$$
$$\phi(u, v) = \phi'(u', v'),$$
$$\psi(u, v) = \psi'(u', v'), \text{ etc.},$$

and a differential invariant is defined as a function of u, v, E, F, G, ϕ, ψ, and their derivatives (u, v being regarded as independent variables) that has the same value whether written in the original or in the transformed variables.

Invariants which involve only u, v, E, F, G and their derivatives are called *Gaussian* invariants, while those which involve also derivatives of ϕ, ψ, etc. are called differential parameters. Thus for example K is a Gaussian invariant and

$$\Delta\phi \equiv \frac{1}{EG - F^2}\left\{E\left(\frac{\partial\phi}{\partial v}\right)^2 - 2F\frac{\partial\phi}{\partial u}\frac{\partial\phi}{\partial v} + G\left(\frac{\partial\phi}{\partial v}\right)^2\right\}$$

is a differential parameter of the quadratic differential form

$$Edu^2 + 2Fdudv + Gdv^2.$$

If the quadratic form is interpreted as the square of the element of length of a surface in space, K and $\Delta\phi$ have also geometrical interpretations. K is the Gaussian or total curvature of the surface, and if $\Delta\phi = 1$, the curves $\phi = \text{const.}$ are the orthogonal trajectories of a family of geodesics on the surface.

The extension of these ideas from two to m variables is immediate, and the quadratic differential form in m variables may be regarded as the square of the element of length in the most general m dimensional manifold. The main point is that invariants are independent of the particular choice of coordinates, in other words they are intrinsically connected with the manifold itself. The course of ideas is as follows. We start with a given manifold, which possesses certain properties. Some of these may be independent of each other, some may be consequences of certain others, and there are relations connecting these. We may develope the discussion on the lines of pure geometry, but we are compelled, sooner or later, to appeal to algebraic methods. These methods involve the introduction of coordinates, and properties of the manifold are then expressed by means of algebraic equations. An algebraic expression has some interpretation in the manifold taken together with the coordinate frame used, and a complication has been introduced, for the discussion will now involve those additional properties which are not intrinsic to the manifold, but arise out of the particular coordinate frame chosen. If however we work only with invariants, we avoid this latter class of properties and are able at the same time to use the powerful methods of analysis. The geometry of the manifold thus breaks up into two parts :

(i) The determination of all invariants and all relations connecting them.

(ii) The geometrical interpretation of all these invariants in the manifold.

3. So far there have been considered only invariants arising through the quadratic form that is equal to ds^2. These are all the invariants when we consider the manifold in itself, but if we suppose it existing in, say, Euclidean space of higher dimensions we introduce other invariants connected with the relation of that space to the manifold. For example, in the case of a surface in space, the totality of invariants is only given when *two* quadratic forms are taken account of, the additional one being that which determines the normal curvature at any point of the surface. The surface, in fact, is not intrinsically determinate by means of the single form, but may be bent provided there is no tearing or stretching, or as we say, it may be *deformed*, without alteration to ds^2. (For instance any developable may be deformed into a plane.) The discussion when there are two or more forms is similar to that when there is only one. The invariants arising from the single form are called deformation invariants.

Thus far it is suggested that the invariants are essentially connected with differential geometry. This is by no means the case. They are connected with a certain form, and any interpretation of this form leads to a corresponding interpretation for the invariants.

Consider in fact any dynamical configuration with Lagrange coordinates $u_1, u_2, \ldots, u_n$. The kinetic energy of this system is $\sum_{r,s=1}^{n} a_{rs}\dot{u}_r\dot{u}_s$, where a_{rs} is a function of the variables u, and dots denote derivatives with regard to the time. By a new choice of coordinates we effect a transformation of exactly the same type as that already considered, and again we have a series of invariants of a quadratic form, and these are those quantities which are dependent on the configuration itself as distinct from the particular system of coordinates.

CHAPTER I

HISTORICAL

4. Group.

Invariance necessarily carries with it the idea of a transformation. Suppose we have a set of transformations in any variables whatever, and suppose that each of the set leaves a certain function of these variables invariant, then any transformation compounded of two or more of the set will also leave that function invariant. If any such transformation as this is not one of the original set we add it to that set, and we may thus continue adding new transformations until we reach a closed set, that is one such that if you apply in turn any two of its transformations the result is another of its transformations. Such a set is called a GROUP, and it is clear that any invariant whatever is invariant under a *group* of transformations.

5. In the case considered in the preceding pages there are a certain number of quadratic differential forms

$$\sum_{r,s=1}^{n} a_{rs}dx_r dx_s,$$

together with a certain number of functions $\phi(x_1, \ldots, x_n)$, and the group of transformations $x_i = x_i(y_1, \ldots, y_n)$, $(i = 1, \ldots, n)$, and we suppose that under a member of this group $\sum_{r,s=1}^{n} a_{rs}dx_r dx_s$ becomes $\sum_{r,s=1}^{n} a'_{rs}dy_r dy_s$, and that ϕ becomes ϕ'. Then there are deducible relations for a'_{rs}, ϕ', and their various derivatives with respect to the y's, and for $dx_1, \ldots, dx_n$ in terms of the original magnitudes a_{rs}, ϕ, etc. In other words there exists a set of transformations for all the variables mentioned. It may be proved that this set is a group, and this group is said to be *extended* from the original group. Our problem is the determination of all the invariants of this extended group.

6. Christoffel.

There have been three main methods of attack. The first, historically, is by comparison of the original and transformed forms, and in this way invariants are obtained by direct processes. The fundamental work in this direction is due to Christoffel* (1869), though the first example of an invariant, the quantity K, was given by Gauss† in 1827. Invariants which involve the derivatives of the functions are called *differential parameters*. Lamé‡, using the linear element in space given by $ds^2 = dx^2 + dy^2 + dz^2$, gave this name to the two invariants

$$(\Delta_1 \phi)^2 \equiv \left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2$$

$$\Delta_2 \phi \equiv \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2},$$

and Beltrami§ adopted it for the invariants that he discovered, those involving first and second derivatives of a function ϕ, taken with a form in two variables.

In the course of Christoffel's work there arise certain functions $(ikrs)$; these were originally found by Riemann in 1861 in his investigations on the curvature of hypersurfaces. For a surface in space they reduce to the one quantity K.

7. Ricci and Levi-Civita.

To Christoffel is due a method whereby from invariants involving derivatives of the fundamental form and of the functions ϕ may be derived invariants involving higher derivatives. This process has been called by Ricci and Levi-Civita *covariant derivation*, and they have made it the base of their researches in this subject. These researches have been collected and given by them in complete form in the *Mathematische Annalen*‖, and on their work they have based a calculus which they call *Absolute differential calculus*. They give a complete solution of the problem, and show that in order to determine all differential invariants of order μ, it is sufficient to determine the algebraic invariants of the system:

(1) The fundamental differential quantic,

* *Crelle*, Vol. 70 (1869) p. 46.

† *Disquisitiones generales circa superficies curvas.*

‡ *Leçons sur les coordonnées curvilignes* (1859).

§ Darboux, *Théorie générale des surfaces*, Vol. III. pp. 193 *sqq*. gives an account of Beltrami's work together with a bibliography.

‖ *Math. Ann.* Vol. 54 (1901) pp. 125 *sqq*.

(2) The covariant derivatives of the arbitrary functions ϕ up to the order μ,

(3) A certain quadrilinear form G_4 and its covariant derivatives up to the order $\mu-2$*.

8. Lie.

The second method is founded on the theory of groups of Lie, and is a direct application of the theory given in his paper *Ueber Differentialinvarianten* †. This theory involves the use of infinitesimal transformations, and the invariants are obtained as solutions of a complete system of linear partial differential equations. Our problem is discussed shortly by Lie ‡ himself, for the case $n=2$. Żorawski § considered this case in detail, and gave the invariants of orders one and two. He also treated the question of the number of functionally independent invariants of any order. C. N. Haskins || has determined the number of functionally independent invariants of any order. Forsyth ¶ has obtained the invariants of orders one two and three for a quadratic form in three variables, and of genus zero, that is to say, for ordinary Euclidean space. He has also obtained the invariants of the first three orders for any surface in space **, that is, for two quadratic forms in two variables, one perfectly general, and the other connected with it by certain differential relations. The problem for the differential parameters has been solved by this method by J. E. Wright ††.

9. Maschke.

The third method is due to Maschke ‡‡, who has introduced a symbolism similar to that for algebraic invariants. He developes processes similar to that of transvection, whereby an endless series of invariants may be constructed.

* *loc. cit.* p. 162.

† *Math. Ann.* Vol. 24 (1884) pp. 537 *sqq.*

‡ *loc. cit.*

§ *Ueber Biegungsinvarianten* in *Acta Math.* Vol. XVI (1892–1893) pp. 1–64.

|| *Trans. Amer. Math. Soc.* Vol. III (1902) pp. 71, 91; also *ib.* Vol. V. (1904) pp. 167, 192.

¶ *Phil. Trans.* Series A, Vol. 202 (1903) pp. 277–333.

** *Phil. Trans.* Series A, Vol. 201 (1903) pp. 329–402.

†† *Amer. Journ. of Math.* Vol. XXVII (1905) pp. 323–342.

‡‡ *Trans. Amer. Math. Soc.* Vol. I (1900) pp. 197, 204; and Vol. IV (1903) pp. 445–469.

The geometrical interpretation of the invariants has been discussed at length by Forsyth*, and a considerable part of the work of Ricci and Levi-Civita deals with geometrical applications.

A general account of the whole subject was given by Maschke† at the St Louis Exposition, 1904.

An account will now be given of these three methods.

* See his two papers already quoted, and *Rendiconti del Circolo Matematico di Palermo*, Vol. 21 (1906) pp. 115–125.

† *St Louis Congress of Arts and Sciences*, Vol. I. pp. 519, 530.

CHAPTER II

THE METHOD OF CHRISTOFFEL

10. The quadratic form in two variables.

Let there be two quadratic forms $F \equiv adx^2 + 2bdxdy + cdy^2$ and $F' \equiv AdX^2 + 2BdXdY + CdY^2$, and suppose that x, y may be expressed as functions of X, Y so that when these values are substituted in F it becomes F'. We have then

$$adx^2 + 2bdxdy + cdy^2 = AdX^2 + 2BdXdY + CdY^2.$$

In this we write $dx = \frac{\partial x}{\partial X} dX + \frac{\partial x}{\partial Y} dY$, with a similar expression for dy, and the equation takes the form

$$PdX^2 + 2QdXdY + RdY^2 = 0$$

where P, Q, R are certain functions of x, y, X, Y and the derivatives of x, y with regard to X and Y. Now X, Y are independent variables and therefore there exists no relation among the differentials dX, dY, and hence P, Q, R are all zero. Thus the necessary and sufficient conditions in order that F shall be transformable into F' are $P = 0$, $Q = 0$, $R = 0$, or written at length.

$$a\left(\frac{\partial x}{\partial X}\right)^2 + 2b\frac{\partial x}{\partial X}\frac{\partial y}{\partial X} + c\left(\frac{\partial y}{\partial X}\right)^2 = A,$$

$$a\frac{\partial x}{\partial X}\frac{\partial x}{\partial Y} + b\left(\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} + \frac{\partial x}{\partial Y}\frac{\partial y}{\partial X}\right) + c\frac{\partial y}{\partial X}\frac{\partial y}{\partial Y} = B,$$

$$a\left(\frac{\partial x}{\partial Y}\right)^2 + 2b\frac{\partial x}{\partial Y}\frac{\partial y}{\partial Y} + c\left(\frac{\partial y}{\partial Y}\right)^2 = C.$$

These three are differential equations of the first order for x, y as functions of X, Y. If they can be solved their solution gives the transformation whereby F is changed into F'. Now there are three equations, and they involve only two dependent variables x, y; hence they cannot in general co-exist unless there be relations between

a, b, c, A, B, C and their derivatives. Our first problem is to find the conditions in order that they may co-exist.

By differentiation we obtain six equations in the six second derivatives of x, y, and these may be solved for the second derivatives in question. If the original three are differentiated twice there are obtained nine equations involving third derivatives, and by means of the equations for second derivatives these may be reduced to a form in which they involve first and third derivatives only. There are only eight third derivatives of two functions x, y, each of two variables X, Y, and therefore by eliminating them from this last set of equations we get a new equation, which, since it involves first derivatives only, must be added to the original three equations. It happens that from these four equations the first derivatives can be eliminated and thus there is given a relation between a, b, c, A, B, C and their first and second derivatives. This relation is precisely $K=K'$.

We can now proceed step by step to find the equations involving higher derivatives of x, y, and then by elimination to find other relations among the coefficients a, b, c, A, B, C and their derivatives. In the case considered, that of two independent variables, these relations all follow from the equivalence of K and K'.

11. The quadratic form in n variables.

The general quadratic in n variables may be treated in exactly the same manner; the statement of the work is much simplified by the use of certain abbreviations which we proceed to define.

The form F itself is written $\sum\limits_{r,s} a_{rs}dx_r dx_s$ and the form F' is $\sum\limits_{r,s} a'_{rs}dy_r dy_s$, the summation being always from 1 to n for each of the letters under the sign of summation. The y's are taken as the independent variables, and the x's are assumed to be functions of these. The determinant of n rows and columns, whose elements are the quantities a_{rs}, is called a. The cofactor of the rth row and sth column in a is written Δ_{rs}. The quantity

$$\tfrac{1}{2}\left[\frac{\partial}{\partial x_h} a_{gk} + \frac{\partial}{\partial x_g} a_{hk} - \frac{\partial}{\partial x_k} a_{gh}\right] \text{ is written } [gh, k],$$

and $\sum\limits_{k} [il, k]\,\Delta_{rk}/a$ is written $\{il, r\}$. Δ_{pq}/a is $a^{(pq)}$ (see p. 20).

$[gh, k]$ and $\{gh, k\}$ are called Christoffel's three-index symbols of the first and second kinds respectively. The expression

$$\frac{\partial}{\partial x_i}[gh, k] - \frac{\partial}{\partial x_h}[gi, k] + \sum_p (\{gi, p\}[hk, h] - \{gh, p\}[ik, p])$$